A. L. Levshin and Z. A. Yanson
Surface waves in vertically and radially inhomogeneous media.
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INTRODUCTION

In recent years, the data from surface waves generated by earthquakes and explosions have been used for a detailed investigation of the properties of the earth and of the source of the disturbance (for example, to discern zones of reduced velocity or large velocity gradients, to estimate the distribution of absorption with depth, to determine the mechanism and time function of sources, etc.). In order to solve this relatively difficult mathematical problem, simplifications are usually made in the model of the medium (layered homogeneous medium) and of the source (point source) (5,12,16-20); these simplifications can be shown to be insufficient. In our work, based on results obtained in (2,4,15), we will expound the basic elements of a more general theory of surface waves, valid for only minimal conditions on the model and source.

We will consider vertically (radially) inhomogeneous media with arbitrary rules for the variation of the elastic constants and density with depth (radius); to find the solution of the equation of motion in such a medium, the spectral theory of operators will to used (6,9). The seismic sources will be treated as fields of volume forces, localized spatially and temporally; the only restrictions imposed on the properties of these fields are those of physical existance. An exact solution is constructed for such a source; asymptotic expansions are made at large distances from the source where the field of the disturbance will separate into propagating Love and Rayleigh waves. Then formulae are obtained for some elementary force fields.

(Some results of Saito's work (22) are used here, but that work is marked by approaches to the final solution which seem quick to us. Hence, a different treatment of the seismic source and the asymptotic expansion is given).

1. Displacement field in an elastic halfspace

<u>Statement of the problem</u>. We will consider an elastic halfspace with coordinates z, r, ϕ $(0 \le z < \infty,\ 0 \le r < \infty,\ 0 \le \phi < 2\pi)$. The equations of motion are (7):

$$\frac{\partial \widehat{rz}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\phi z}}{\partial \phi} + \frac{\partial \widehat{zz}}{\partial z} + \frac{\widehat{rz}}{r} = \rho \frac{\partial^2 u_z}{\partial t^2} - F_z ,$$

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{r\phi}}{\partial \phi} + \frac{\partial \widehat{rz}}{\partial z} + \frac{\widehat{rr} - \widehat{\phi\phi}}{r} = \rho \frac{\partial^2 u_r}{\partial t^2} - F_r , \qquad (1.1)$$

$$\frac{\partial \widehat{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\phi\phi}}{\partial \phi} + \frac{\partial \widehat{\phi z}}{\partial z} + \frac{2\widehat{r\phi}}{r} = \rho \frac{\partial^2 u_\phi}{\partial t^2} - F_\phi .$$

Here $\hat{rz}$, $\hat{rr}$, $\hat{r\phi}$, $\hat{\phi r}$, $\hat{\phi\phi}$, $\hat{zz}$ are components of the stress tensor; u_z u_r, u_ϕ are components of the displacement vector $u(t, z, r, \phi)$ in the directions a_z, a_r, a_ϕ respectively; F_z, F_r, F_ϕ are components of the vector volume force $F(t, z, r, \phi)$ acting at the source along the same directions; t is the time.

For the components of stress we have the following relations:

$$\hat{rz} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \qquad \hat{\phi z} = \mu\left(\frac{1}{r}\frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z}\right)$$

$$\hat{rr} = \lambda\Delta + 2\mu\frac{\partial u_r}{\partial r} \qquad \hat{\phi\phi} = \lambda\Delta + \frac{2\mu}{r}\left(\frac{\partial u_\phi}{\partial \phi} + u_r\right) \tag{1.2}$$

$$\hat{r\phi} = \mu\left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \phi}\right) \qquad \hat{zz} = \lambda\Delta + 2\mu\frac{\partial u_z}{\partial z}$$

where Δ - dilatation,

$$\Delta = \frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \tag{1.3}$$

The Lamé constants λ and μ and the density ρ are piecewise continuous positive functions of a single coordinate z; for $z > Z$, λ, μ, ρ are constant, and the velocity of transverse waves $b = \sqrt{\mu/\rho}$ and of compressional waves $a = \sqrt{(\lambda+2\mu)/\rho}$ are maximal (this is necessary since Z can be as large as desired):

$$b(Z + 0) = \max b(z), \quad a(Z + 0) = \max a(z).$$

The components of displacement and stress are continuous and bounded everywhere in the region $0 \le z < \infty$; the surface $z = 0$ is free of stress, i.e.

$$\hat{rz} = \hat{\phi z} = \hat{zz} = 0 \quad \text{for } z = 0 \tag{1.4}$$

Initial conditions:

$$\vec{u} = \frac{\partial \vec{u}}{\partial t} = 0 \quad \text{for } t < 0 \tag{1.4a}$$

Source. The force field $F(t, z, r, \phi)$ is described as a real source localized in space and time. The following conditions are imposed on F:

1) $F(t,z,r,\phi) = 0$ for $t < 0$;

2) $F(t,z,r,\phi)$ is absolutely integrable and satisfies the Dirichlet conditions for all arguments. Then we are permitted the following representation:

$$\vec{F}(t,z,r,\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipt}\int_0^\infty \left[\sum_{m=-\infty}^{\infty}\sum_{i=1}^{3} f_m^{(i)}\vec{A}_m^{(i)}\right]\xi\, d\xi\, dp \tag{1.5}$$

where

$$\vec{A}_m^{(1)} = \vec{a}_z Y_m, \quad \vec{A}_m^{(2)} = \vec{a}_r\frac{\partial Y_m}{\partial r}\frac{1}{\xi} + \vec{a}_\phi\frac{\partial Y_m}{\partial \phi}\frac{1}{\xi r} \tag{1.6}$$

$$\vec{A}_m^{(3)} = \vec{a}_r\frac{\partial Y_m}{\partial \phi}\frac{1}{\xi r} - \vec{a}_\phi\frac{\partial Y_m}{\partial r}\frac{1}{\xi}, \qquad Y_m = e^{im\phi}J_m(\xi r).$$

Here J_m is a Bessel function of the first kind of order m. The system of vector functions $A_m^{(i)}$ are fields and mutually orthogonal. The coefficients

$f_m^{(i)}(z, \xi\, p)$ are found using the orthogonality relation

$$\int_0^\infty \int_0^{2\pi} [\bar{\vec{A}}_m^{(i)}(\lambda r), \vec{A}_l^{(j)}(\gamma r)]\, r\, d\phi\, dr = 2\pi \delta_{ij} \delta_{ml} \frac{\delta(\gamma-\lambda)}{\sqrt{\gamma\lambda}}$$

(δ_{ij} is the Kronecker delta, $\delta(\lambda-\gamma)$ is the Dirac delta function)[1] and equals

$$f_m^{(i)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipt} \int_0^\infty \int_0^{2\pi} (\vec{F}, \bar{\vec{A}}_m^{(i)})\, r\, d\phi\, dr\, dt$$

Specifically for $i = 1, 2, 3$ we have:

$$f_m^{(1)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipt} \int_0^\infty \int_0^{2\pi} [F_z \bar{Y}_m(\xi r)]\, r\, d\phi\, dr\, dt,$$

$$f_m^{(2)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipt} \int_0^\infty \int_0^{2\pi} \left[F_r \frac{\partial \bar{Y}_m}{\partial r} + F_\phi \frac{\partial \bar{Y}_m}{\partial \phi} \frac{1}{r}\right] \frac{r}{\xi}\, d\phi\, dr\, dt \qquad (1.7)$$

$$f_m^{(3)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipt} \int_0^\infty \int_0^{2\pi} \left[F_r \frac{\partial \bar{Y}_m}{\partial \phi} \frac{1}{r} - F_\phi \frac{\partial \bar{Y}_m}{\partial r}\right] \frac{r}{\xi}\, d\phi\, dr\, dt$$

$$\bar{Y}_m = e^{-im\phi} J_m(\xi r).$$

For the force components F_z, F_r, F_ϕ we have using (1.5), (1.6):

$$F_z = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipt} \int_0^\infty \left[\sum_{m=-\infty}^{\infty} f_m^{(1)} Y_m\right] \xi\, d\xi\, dp,$$

$$F_r = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipt} \int_0^\infty \left[\sum_{m=-\infty}^{\infty} \left(f_m^{(2)} \frac{\partial Y_m}{\partial r} + f_m^{(3)} \frac{\partial Y_m}{\partial \phi} \frac{1}{r}\right)\right] d\xi\, dp \qquad (1.8)$$

$$F_\phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipt} \int_0^\infty \left[\sum_{m=-\infty}^{\infty} \left(f_m^{(2)} \frac{\partial Y_m}{\partial \phi} \frac{1}{r} - f_m^{(3)} \frac{\partial Y_m}{\partial r}\right)\right] d\xi\, dp$$

<u>Formulae for displacement.</u> The solutions arising in non-stationary problems of the theory of elasticity have the form

$$\vec{u}(t, z, r, \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipt}\, \vec{U}\, dp \qquad (1.9)$$

where

$$\vec{U}(p, z, r, \phi) = v.\, [\int_0^\infty \sum_{m=-\infty}^{\infty} \sum_{i=1}^{3} V_m^{(i)}(z, \xi, p) \vec{A}_m^{(i)}\, \xi\, d\xi$$

with a solution similar to the stationary problem of elasticity theory, which is derived in the same way with conditions that V_m (z, ξ, p) be square integrable on the interval $z \in (0, \infty)$. Here and later on, v. indicates that the integration contour goes along the real axis, and around the poles of the integrand with small semicircles above the pole (for integration with ξ) and below the pole (with integration with p). Hence for the components of displacement along the directions e_z, e_r, e_ϕ we obtain:

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1 $\bar{A}_l^{(j)}$ is complex conjugate of $A_l^{(j)}$

$$u_z = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipt}\,\bar{v}_0 \int_0^{\infty}\left[\sum_{m=-\infty}^{\infty} V_m^{(1)} Y_m\right]\xi\, d\xi\, dp$$

$$u_r = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipt}\,\bar{v}_0 \int_0^{\infty}\left[\sum_{m=-\infty}^{\infty}\left(V_m^{(2)}\frac{\partial Y_m}{\partial r} + \frac{V_m^{(3)}}{r}\frac{\partial Y_m}{\partial \phi}\right)\right] d\xi\, dp \qquad (1.10)$$

$$u_\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipt}\,\bar{v}_0 \int_0^{\infty}\left[\sum_{m=-\infty}^{\infty}\left(\frac{V_m^{(2)}}{r}\frac{\partial Y_m}{\partial \phi} - V_m^{(3)}\frac{\partial Y_m}{\partial r}\right)\right] d\xi\, dp$$

Placing (1.8), (1.10) in equation (1.1) and in the boundary condition (1.4), and accepting the permissibility of moving the double differentiation under the integral sign, we obtain the following equations:

1. For $V_m^{(1)}$, $V_m^{(2)}$:

$$L_1(V_m^{(1)}, V_m^{(2)}) \equiv \frac{d}{dz}\left[(\lambda+2\mu)\frac{dV_m^{(1)}}{dz} - \xi\lambda V_m^{(2)}\right] - \xi\mu\frac{dV_m^{(2)}}{dz} + V_m^{(1)}(p^2\rho - \xi^2\mu) = -f_m^{(1)} \qquad (1.11)$$

$$L_2(V_m^{(1)}, V_m^{(2)}) \equiv \frac{d}{dz}\left[\mu\frac{dV_m^{(2)}}{dz} + \xi\mu V_m^{(1)}\right] + \xi\lambda\frac{dV_m^{(1)}}{dz} + V_m^{(2)}(p^2\rho - \xi^2\lambda - 2\xi^2\mu) = -f_m^{(2)}$$

with boundary conditions

$$\sigma_{zz} \equiv (\lambda+2\mu)\frac{dV_m^{(1)}}{dz} - \xi\lambda V_m^{(2)} = 0$$
$$\tau_{rz} \equiv \mu\frac{dV_m^{(2)}}{dz} + \xi\mu V_m^{(1)} = 0 \qquad \text{for } z=0 \qquad (1.12)$$

The functions $V_m^{(1)}$, $V_m^{(2)}$, σ_{zz}, and τ_{rz} are continuous and bounded for all z.

2. For $V_m^{(3)}$

$$L_3(V_m^{(3)}) \equiv \frac{d}{dz}\left(\mu\frac{dV_m^{(3)}}{dz}\right) + V_m^{(3)}(p^2\rho - \xi^2\mu) = -f_m^{(3)} \qquad (1.13)$$

with boundary condition

$$\tau_{\phi z} \equiv \mu\frac{dV_m^{(3)}}{dz} = 0 \qquad \text{for } z=0 \qquad (1.14)$$

$V_m^{(3)}$ and $\tau_{\phi z}$ are continuous and bounded for all z. The left side of equation (1.11) and the boundary condition (1.12) define a self-adjoint operator &

in the region of integration with the square integrable vector function $\begin{vmatrix} V_m^{(1)} \\ V_m^{(2)} \end{vmatrix}$

The left side of (1.13) and the boundary condition (1.14) defines a self-adjoint operator L_3 in the region of integration with the function $V_m^{(3)}$. If the vector function $\begin{vmatrix} f_m^{(1)} \\ f_m^{(2)} \end{vmatrix}$ and the function $f_m^{(3)}$ are also square integrable on the interval $z \in (0,\infty)$ (which follows from the conditions imposed on the function $F(t,z,r,\beta)$, then then the following relations between the functions $V_m^{(1)}$, $V_m^{(2)}$, $V_m^{(3)}$ and the eigenfunctions of the above described operators are valid (2, 6, 10).

<u>Expression of V_m in terms of eigenfunctions</u>. $V_m^{(i)}$ $(i = 1,2)$ can be expressed in the following manner:

$$V_m^{(i)} = \sum_{k=1}^{k_R(\xi)} c_{km}^R \tilde{V}_k^{(i)} + \int_{p_0^2}^{\infty} c_m^R(\beta)\, \tilde{V}^{(i)}(\beta, z)\, d\beta \qquad (1.15)$$

Here, for the coefficients c_{km}^R, c_m^R we have:

$$c_{km}^R = \frac{1}{p_{kR}^2(\xi) - p^2} \frac{D_{km}^R}{I_{kR}} \qquad c_m^R = \frac{D_m^R}{\beta(\xi) - p^2} \qquad (1.16)$$

$$D_{km}^R = \int_0^\infty \left(f_m^{(1)} \tilde{V}_k^{(1)} + f_m^{(2)} \tilde{V}_k^{(2)}\right) dz \qquad D_m^R = \int_0^\infty \left(f_m^{(1)} \tilde{\tilde{V}}^{(1)} + f_m^{(2)} \tilde{\tilde{V}}^{(2)}\right) dz$$

$$I_{kR} = \int_0^\infty \rho \left[(\tilde{V}_k^{(1)})^2 + (\tilde{V}_k^{(2)})^2 \right] dz \qquad p_0^2 = \xi^2 b^2 (z+0)$$

Here $\tilde{V}_k^{(1)}$ and $\tilde{V}_k^{(2)}$, $\tilde{V}^{(1)}$ and $\tilde{V}^{(2)}$ are the eigenfunctions of the operators constituting the left hand sides of (1.11) with the boundary conditions (1.12).

The first part of the function corresponds to the discrete spectra of eigenvalues p_{kR}^2 $(k = 1, 2, \dots k_R(\xi))$; $\xi^2 v_R^2 < p_{kR}^2 < p_0^2$, where v_R is the minimal velocity of Rayleigh waves in a halfspace with constants equal $a(z)$, $b(z)$, $\rho(z)$ for some value of depth (1,2)). The second part corresponds to the continuous spectra of eigenvalues β $(p_0^2 \leq \beta < \infty)$. Here the wave number ξ plays the role of a free parameter. Formula (1.16) is obtained using the orthogonality relations:

$$\int_0^\infty \rho \left[\tilde{V}_i^{(1)} \tilde{V}_j^{(1)} + \tilde{V}_i^{(2)} \tilde{V}_j^{(2)} \right] dz = 0 \qquad i \neq j$$

$$\int_0^\infty \rho \left[\tilde{V}^{(1)}(\beta) \tilde{\tilde{V}}^{(1)}(p^2) + \tilde{V}^{(2)}(\beta) \tilde{\tilde{V}}^{(2)}(p^2) \right] dz = \delta(p^2 - \beta)$$

where $\tilde{\tilde{V}}^{(i)}$ is the complex conjugate of $\tilde{V}^{(i)}$.

Similarly $V_m^{(3)}$ can be expressed as

$$V_m^{(3)} = \sum_{k=1}^{k_L(\xi)} c_{km}^L \tilde{V}_k^{(3)} + \int_{p_0^2}^{\infty} c_m^L(\beta)\, \tilde{V}^{(3)}(z, \beta)\, d\beta, \qquad (1.17)$$

where

$$c_{km}^L = \frac{1}{p_{kL}^2(\xi) - p^2} \frac{D_{km}^L}{I_{kL}}, \qquad c_m^L = \frac{D_m^L}{\beta(\xi) - p^2};$$

$$D_{km}^L = \int_0^\infty f_m^{(3)} \tilde{V}_k^{(3)} dz \qquad D_m^L = \int_0^\infty f_m^{(3)} \tilde{\tilde{V}}^{(3)} dz \qquad (1.18)$$

$$I_{kL} = \int_0^\infty \rho \left(V_k^{(3)}\right)^2 dz$$

this forces inclusion of ρ in definition of $\tilde{V}\ \tilde{\tilde{V}}$

For this derivation the following orthogonality conditions were used:

$$\int_0^\infty \rho \tilde{V}_i^{(3)} \tilde{V}_j^{(3)} dz = 0, \; i \neq j \qquad \int_0^\infty \rho \tilde{V}^{(3)}(p^2) \hat{\tilde{V}}^{(3)}(\beta) dz = \delta(p^2 - \beta),$$

Placing these expressions for V_m into (1.10), we obtain the complete precise formulaes for displacements:

$$u_z = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipt} \left\{ V. \int_0^\infty \left[\sum_{m=-\infty}^{\infty} \left(\sum_{k=1}^{K_R(\xi)} c_{km}^R \tilde{V}_k^{(1)} + \int_{p_0^2}^{\infty} c_m^R(\beta) \tilde{V}^{(1)} d\beta \right) Y_m \right] \xi \, d\xi \right\} dp$$

$$u_r = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipt} \left\{ V. \int_0^\infty \left[\sum_{m=-\infty}^{\infty} \left(\left(\sum_{k=1}^{K_R(\xi)} c_{km}^R \tilde{V}_k^{(2)} + \int_{p_0^2}^{\infty} c_m^R(\beta) \tilde{V}^{(2)} d\beta \right) \frac{\partial Y_m}{\partial r} + \right.\right.\right.$$

$$\left.\left.\left. + \frac{1}{r} \left(\sum_{k=1}^{K_L(\xi)} c_{km}^L \tilde{V}_k^{(3)} + \int_{p_0^2}^{\infty} c_m^L(\beta) \tilde{V}^{(3)} d\beta \right) \frac{\partial Y_m}{\partial \varphi} \right] d\xi \right\} dp \qquad (1.19)$$

$$u_\varphi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipt} \left\{ V. \int_0^\infty \left[\sum_{m=-\infty}^{\infty} \left(\frac{1}{r} \left(\sum_{k=1}^{K_R(\xi)} c_{km}^R \tilde{V}_k^{(2)} + \int_{p_0^2}^{\infty} c_m^R(\beta) \tilde{V}^{(2)} d\beta \right) \frac{\partial Y_m}{\partial \varphi} \right.\right.\right.$$

$$\left.\left.\left. - \left(\sum_{k=1}^{K_L(\xi)} c_{km}^L \tilde{V}_k^{(3)} + \int_{p_0^2}^{\infty} c_m^L(\beta) \hat{V}^{(3)} d\beta \right) \frac{\partial Y_m}{\partial r} \right) \right] d\xi \right\} dp$$

It can be shown that the derived solution for non-stationary problem satisfies the null initial conditions (1.4a). Thus for fixed ξ the functions of p in (1.19), multiplied by exp(ipt), are only the coefficients c_{km}^Q, c_m^Q which are analytic everywhere except at a finite number of points for Im p ≥ 0, decreasing as $p \to \infty$ not slower than $O(1 / p^{1+|\varepsilon|})$. Changing the order of integration among p and ξ taking the integration contour with respect to p in the lower halfspace, we obtain the null condition on u(t) and du/dt for t ≤ 0.

Asymptotic expressions for large r. At large distances r, which are not commensurable with the length dimensions of the seismic source, the main part of the disturbance given by formula (1.19) becomes the Rayleigh and Love surface waves. Their contribution equals the sum of residues of the poles in the integrand, standing under the summation sign $\left[\sum_{k=1}^{K_Q(\xi)} \right]$. Keeping only parts, decreasing not faster than r^{-1}, we obtain the following asymptotic formula for displacements at large r (5):

$$u_z(t,z,r,\varphi) = \frac{1}{\pi} \operatorname{Re} \int_{\bar{p}}^{\infty} \exp i(pt - \pi/4) \times$$

$$\times \left[\sum_{k=1}^{K_R(p)} \frac{U_{kR}(p,\varphi)\tilde{V}^{(1)}(p,z)}{2 v_{kR} C_{kR} I_{kR}} \pi \sqrt{\frac{2}{\pi \xi_{kR} r}} \exp(-i \xi_{kR} r) \right] dp,$$

$$u_r(t,z,r,\varphi) = \frac{1}{\pi} \operatorname{Re} \int_{\bar{p}}^{\infty} \exp i(pt - \tfrac{3\pi}{4})$$

$$\times \left[\sum_{k=1}^{K_R(p)} \frac{U_{kR}(p,\varphi)\tilde{V}^{(2)}(p,z)}{2 v_{kR} C_{kR} I_{kR}} \pi \sqrt{\frac{2}{\pi \xi_{kR} r}} \exp(-i \xi_{kR} r) \right] dp, \qquad (1.20)$$

$$u_\varphi(t,z,r,\varphi) = \frac{1}{\pi} \int_{\bar{p}}^{\infty} \exp i(pt + \pi/4) \times$$

$$\times \left[\sum_{k=1}^{K_L(p)} \frac{U_{kL}(p,\varphi)\tilde{V}_k^{(3)}(p,z)}{2 v_{kL} C_{kL} I_{kL}} \pi \sqrt{\frac{2}{\pi \xi_{kL} r}} \exp(-i \xi_{kL} r) \right] dp$$

Here for $Q = R, L$ $K_Q(p)$ is the maximum number of harmonics of Rayleigh (R) and Love (L) waves which exist for a given p.

$$U_{kQ}(p,\varphi) = \sum_{m=-\infty}^{\infty} D_{km}^{Q} \exp im(\varphi + \pi/2), \qquad (1.21)$$

ξ_{kQ} is the wave number, a root of the equation $p_{kQ}^2(\xi) - p^2 = 0$. The phase and group velocities of the k'th harmonic v_{kQ} and C_{kQ} are related to ξ_{kQ} by

$$v_{kQ} = \frac{p}{\xi_{kQ}}, \qquad C_{kQ} = \frac{dp_{kQ}}{d\xi} \quad (\xi = \xi_{kQ}) \qquad (1.22)$$

$\bar{p}$ is the limiting frequency; oscillations with frequencies less than $\bar{p}$ make up the quasi-static part of the disturbance and are not of interest to us.

For the derivation of formula (1.20) the following asymptotic relations were used:

$$V_1 \int_0^{\infty} \frac{\Phi(\xi) J_m(\xi r) \xi d\xi}{p_{kQ}^2(\xi) - p^2} \approx \sqrt{\frac{\pi \xi_{kQ}}{2r}} \frac{\Phi(\xi_{kQ})}{p C_{kQ}} \times$$

$$\times \exp\left\{-i\left[\xi_{kQ} r - (m - \tfrac{1}{2})\tfrac{\pi}{2}\right]\right\},$$

$$r \cdot \int_0^\infty \frac{\Phi(\xi)\dfrac{d J_m(\xi r)}{dr}}{p^2_{kQ}(\xi) - p^2}\, d\xi \approx \sqrt{\frac{\pi \xi_{kQ}}{2r}}\; \frac{\Phi(\xi_{kQ})}{p C_{kQ}} \times$$

$$\times \exp\left\{-i\left[\xi_{kQ} r - (m - 3/2)\pi/2\right]\right\}.$$

The vertical u_z and radial u_r components of displacement make up the Rayleigh wave; the tangential component u_φ makes up the Love wave.

Expressions for p_{kQ}, C_{kQ} by integrals of eigenfunctions. The calculus of variations can be used to obtain formulae for p_{kQ} and C_{kQ} (2, 15, 23):

$$p^2_{kQ} = (\xi^2 G^Q_{1k} + 2\xi G^Q_{2k} + G^Q_{3k})/I_{kQ}\ , \qquad C_{kQ} = (\xi G^Q_{1k} + G^Q_{2k})/p_{kQ} I_{kQ}$$

where G^Q_{jk} are the following integrals:

$$\begin{aligned} G^R_{1k} &= \int_0^\infty \left[(\lambda+2\mu)(\tilde{V}^{(2)}_k)^2 + \mu(\tilde{V}^{(1)}_k)^2\right] dz \\ G^R_{2k} &= \int_0^\infty \left[\mu \frac{d\tilde{V}^{(2)}}{dz}\tilde{V}^{(1)}_k - \lambda \frac{d\tilde{V}^{(1)}_k}{dz}\tilde{V}^{(2)}_k\right] dz \\ G^R_{3k} &= \int_0^\infty \left[(\lambda+2\mu)\left(\frac{d\tilde{V}^{(1)}_k}{dz}\right)^2 + \mu\left(\frac{d\tilde{V}^{(2)}_k}{dz}\right)^2\right] dz \end{aligned} \qquad (1.24)$$

$$\begin{aligned} G^L_{1k} &= \int_0^\infty \mu\,(\tilde{V}^{(3)}_k)^2\, dz \\ G^L_{2k} &= 0 \\ G^L_{3k} &= \int_0^\infty \mu\left(\frac{d\tilde{V}^{(3)}_k}{dz}\right)^2 dz. \end{aligned} \qquad (1.25)$$

Methods for calculating p_{kR}, $\tilde{V}^{(1)}_k$ and $\tilde{V}^{(2)}_k$ are described in (11, 21), and the methods for p_{kL}, $\tilde{V}^{(3)}_k$ in (2, 15).

SURFACE WAVES FROM ELEMENTARY SOURCES. We will consider what form equation (1.20) takes for some elementary sources: axially symmetric vertical and radial impacts , torsional impact, field of horizontal forces, dipoles, center of compression. The fields of many more complex sources can be obtained by adding the fields of these elementary sources with respective constants of proportionality. Because the function U_{kQ} in equation (1.20) is the only term which is affected by the source, it is only necessary to consider the expressions for U_{kQ} for various

I. Vertical axially symmetric impact. Let

$$\vec{F} = F_z(t, z, r)\, \hat{a}_z.$$

In this case we get from (1.7):

$$f^{(1)}_0(z, \xi, p) = \int_{-\infty}^{\infty} e^{-ipt} \int_0^\infty F_z\, J_0(\xi r)\, r\, dr\, dt\ ;$$

$$f^{(1)}_m = 0 \quad \text{for } m \neq 0\ ; \quad f^{(2)}_m \equiv f^{(3)}_m \equiv 0\ ;$$

$$\mathcal{D}_{k0}^{R} = \int_0^\infty f_0^{(1)}(z,\xi_{kR})\tilde{V}_k^{(1)}(z,\xi_{kR})\,dz \;;$$
$$\mathcal{D}_{km}^{R} = 0 \quad \text{for} \quad m \neq 0 \;; \quad \mathcal{D}_{km}^{L} \equiv 0 \,.$$

In summary,

$$\mathcal{U}_{kR} = \int_0^\infty f_0^{(1)}\tilde{V}_k^{(1)}dz \,, \quad \mathcal{U}_{kL} = 0 . \qquad (1.25)$$

In particular, for an ideally concentrated vertical force at the point $z = h$, $r = 0$, $F = \delta(z-h)\,\delta(r)\varphi(t)e_z/r$ *

$$U_{kR} = V_k(h,p)\;S(p) \qquad (1.26)$$

Here and later on $S(p) = \int_{-\infty}^{\infty}\varphi(t)\exp(-ipt)dt$ — the source time spectrum.

II. Radial axia ly symmetric impact . Let

$$F = F_r(t,z,r)e_r .$$

In this case we obtain from (1.7):

$$f_0^{(2)} = -\int_{-\infty}^{\infty} e^{-ipt}\int_0^\infty F_r\,J_1(\xi r)\,r\,dr\,dt; \; f_m^{(2)} = 0 \text{ for } m \neq 0;$$
$$f_m^{(1)} \equiv f_m^{(3)} \equiv 0 \,;$$

$$\mathcal{D}_{k0}^{R} = -\int_0^\infty f_0^{(2)}\tilde{V}_k^{(2)}dz; \; \mathcal{D}_{km}^{R} = 0 \text{ for } m \neq 0; \; \mathcal{D}_{km}^{L} \equiv 0 .$$

Hence,

$$\mathcal{U}_{kR} = -\int_0^\infty f_0^{(2)}\tilde{V}_k^{(2)}dz \,, \quad \mathcal{U}_{kL} = 0 . \qquad (1.27)$$

In the idel case of a radial source concentrated at the point $z = h$, $r = 0$, $F = 2\delta(z-h)\,\delta(r)\varphi(t)e_r/r^2$ *

$$U_{kR} = -\xi_{kR}\tilde{V}_k^{(2)}(h,p)S(p) \qquad (1.28)$$

III. Rotational impact . Let

$$F = F_\varphi(t,z,r)e_\varphi .$$

From (1.7) we get

$$f_0^{(3)} = \int_{-\infty}^{\infty} e^{-ipt}\int_0^\infty F_\varphi\,J_1(\xi r)\,r\,dr\,dt; \quad f_m^{(3)} = 0 \quad \text{for } m \neq 0$$
$$f_m^{(1)} \equiv f_m^{(2)} \equiv 0 \;; \quad \mathcal{D}_{km}^{R} \equiv 0 \;; \quad \mathcal{D}_{k0}^{L} = \int_0^\infty f_0^{(3)}\tilde{V}_k^{(3)}dz;$$
$$\mathcal{D}_{km}^{L} = 0 \quad \text{for } m \neq 0$$

* Translator's note: these concentrated forces are not normalized by the factor $1/2\pi$ problem not normalization but in not carrying 1-7 through ?

The final result is

$$U_{kR} = 0 \,, \qquad U_{kL} = \int_0^\infty f_0^{(3)} \tilde{V}_k^{(3)} dz \qquad (1.29)$$

In the case of a concentrated at $z = h$, $r = 0$ and $F = 2\delta(z-h)\delta(r)\varphi(t)e_\phi / r^2$

$$U_{kL} = \delta_{kL} V_k(h,p) S(p) \qquad (1.30)$$

IV. Field of a horizontal force with fixed orientation. We will consider only the particular case of this force field

$$F = F_T(t,z,r)\bar{a}_T$$

where a_T is a horizontal unit vector of fixed orientation:

$$(\bar{a}_T,\bar{a}_z) = 0 \,, \; (\bar{a}_T,\bar{a}_r) = \cos(\delta - \phi),$$
$$(\bar{a}_T,\bar{a}_\phi) = \sin(\delta - \phi).$$

Then,

$$F = F_T\left[\cos(\delta - \phi)\bar{a}_r + \sin(\delta - \phi)\bar{a}_\phi\right]$$

and from (1.7) we have

$$f_m^{(1)} \equiv 0 \;; \quad f_m^{(2)} = f_m^{(3)} = 0 \quad \text{for } m \neq \pm 1 \;;$$
$$f_1^{(2)} = \frac{e^{-i\delta}}{2} f_T \;; \quad f_{-1}^{(2)} = -\frac{e^{+i\delta}}{2} f_T \;;$$
$$f_1^{(3)} = \frac{e^{-i\delta}}{2i} f_T \;; \quad f_{-1}^{(3)} = \frac{e^{i\delta}}{2i} f_T \;;$$
$$f_T = \int_{-\infty}^{\infty} e^{-ipt} \int_0^\infty F_T \, J_0(\xi r) \, r \, dr \, dt$$

Summing over m, we obtain the following relation:

$$U_{kR} = i\cos(\delta-\varphi)\int_0^\infty f_T \tilde{V}_k^{(2)} dz \,,$$
$$U_{kL} = -i\sin(\delta-\varphi)\int_0^\infty f_T \tilde{V}_k^{(3)} dz. \qquad (1.31)$$

In the case of an ideal concentrated force acting at the point $r = 0$, $z = h$, $F = \delta(z-h)\delta(r)\varphi(t)a_T / r$;

$$U_{kR} = i\cos(\delta-\varphi)\tilde{V}_k^{(2)}(h,p)\,S(p) \qquad (1.32)$$
$$U_{kL} = -i\sin(\delta-\varphi)V_k^{(3)}(h,p)\,S(p).$$

V. An arbitrarily oriented concentrated force. Let

$$F = \delta(z-h)\frac{\delta(r)}{r}\left[a_z\cos\beta + a_T\sin\beta\right]\phi(t)$$

Combining (1.25) and (1.32), we get

$$U_{kR} = [\cos\beta\, V_k^{(1)}(h,p) + i\sin\beta\cos(\delta-\phi)\,\hat{V}_k^{(2)}(h,p)]\, S(p)$$
$$U_{kL} = -i\sin\beta\sin(\delta-\phi)\,\hat{V}_k^{(3)}(h,p)]\, S(p) \qquad (1.33)$$

VI. Dipole without moment. The field of a dipole is made up of a pair of forces without moment

$$F_{\pm} = \pm\delta(z-h)\,\frac{\delta(r)}{r}\,\delta(t)\,[\vec{a}_z\cos\beta + \vec{a}_r\sin\beta]\,\phi(t)$$

we obtain the result by using the following operator on U_{kQ} IN (1.33) (keeping only those parts decreasing slower than r^{-1}):

$$\Big[\cos\beta\frac{d}{dh} + i\xi_{kQ}\cos(\delta-\phi)\sin\beta\Big].$$

We obtain the results:

$$U_{kR} = \Big[\cos^2\beta\,\frac{d\hat{V}_k^{(1)}(h,p)}{dh} - \xi_{kR}\sin^2\beta\cos^2(\delta-\phi)\,\tilde{V}_k^{(2)}(h,p) + $$
$$+ \frac{i}{2}\sin 2\beta\cos(\delta-\phi)\Big(\xi_{kR}\hat{V}_k^{(1)}(h,p) + \frac{d\tilde{V}_k^{(2)}}{dh}(h,p)\Big)\Big]\, S(p) \qquad (1.34)$$

$$U_{kL} = -i\sin\beta\sin(\delta-\phi)\,\Big[\cos\beta\,\frac{d\tilde{V}_k^{(3)}}{dh}(h,p) + $$
$$+ i\xi_{kL}\cos(\delta-\phi)\sin\beta\,\tilde{V}_k^{(3)}(h,p)\Big]\, S(p).$$

VII. Dipole with moment. Let a pair of forces act in the same direction as in VI, but allow the force to have a moment; the axis of the dipole, i.e., the line associated with the points where the force acts, is given by the vector a_n, where

$$(a_n, a_z) = \cos\gamma\ , \quad (a_n, a_r) = \sin\gamma\cos(\alpha-\phi)\ ;$$
$$(a_n, a_\phi) = \sin\gamma\sin(\alpha-\phi).$$

The angles $\gamma, \delta, \beta, \alpha$ are not independent, and are related by the relation $\operatorname{ctg}\gamma\operatorname{ctg}\beta = -\cos(\delta-\alpha)$. The displacement field of this dipole valid for large distances r is obtained by using the following operator on U_{kQ} in (1.33):

$$\Big[\cos\gamma\frac{d}{dh} + i\xi_{kQ}\cos(\alpha-\phi)\sin\gamma\Big]$$

$$U_{kR} = \Big[\cos\gamma\cos\beta\,\frac{d\hat{V}_k^{(1)}}{dh}(h,p) - \xi_{kR}\sin\beta\sin\gamma\cos(\delta-\phi)\cos(\alpha-\phi)$$
$$\times\tilde{V}_k^{(2)}(h,p) + i\xi_{kR}\sin\gamma\cos\beta\cos(\alpha-\phi)\,\tilde{V}_k^{(1)}(h,p)$$
$$+ i\cos\gamma\sin\beta\cos(\delta-\phi)\,\frac{d\tilde{V}_k^{(2)}}{dh}(h,p)\Big]\times S(p)$$

$$U_{kL} = -i\sin\beta\sin(\delta-\phi)\,\Big[\cos\gamma\,\frac{d\hat{V}_k^{(3)}}{dh}(h,p) + i\xi_{kL}\sin\gamma\cos(\alpha-\phi)\,\hat{V}_k^{(3)}(h,p)\Big]\times S(p)$$

VIII. Center of expension. The center of expansion can be considered to be equivalent to a source composed of three orthogonal dipoles without moment. We will take one dipole to be vertical ($\beta = 0$), and the other two horizontal ($\beta = \pi/2$, $\delta = 0$ and $\pi/2$). The resultant field is

$$\mathcal{U}_{kR} = \left[\frac{d\hat{V}_k^{(1)}}{dh}(h,p) - \xi_{kR} \hat{V}_k^{(2)} \right] \times S(p) \;, \quad \mathcal{U}_{kL} = 0 \qquad (1.35)$$

2. Displacement fields in an elastic sphere

Statement of the problem. We will consider an elastic sphere with coordinates R, Θ, ϕ ($0 \le R \le R_0$, $0 \le \Theta \le \pi$, $0 \le \phi < 2\pi$). The euqations of motion in this coordinate system are (7):

$$\frac{1}{R}\frac{\partial \hat{\Theta R}}{\partial \Theta} + \frac{1}{R\sin\Theta}\frac{\partial \hat{\phi R}}{\partial \phi} + \frac{\partial \hat{RR}}{\partial R} + \frac{1}{R}\left(2\hat{RR} - \hat{\Theta\Theta} - \hat{\phi\phi} + \hat{\Theta R}\,\mathrm{ctg}\,\Theta\right) = \rho \frac{\partial^2 u_R}{\partial t^2} - F_R$$

$$\frac{1}{R}\frac{\partial \hat{\Theta\Theta}}{\partial \Theta} + \frac{1}{R\sin\Theta}\frac{\partial \hat{\Theta\phi}}{\partial \phi} + \frac{\partial \hat{\Theta R}}{\partial R} + \frac{1}{R}\left[3\hat{\Theta R} + (\hat{\Theta\Theta} - \hat{\phi\phi})\,\mathrm{ctg}\,\Theta\right] = \rho \frac{\partial^2 u_\Theta}{\partial t^2} - F_\Theta \qquad (2.1)$$

$$\frac{1}{R}\frac{\partial \hat{\Theta\phi}}{\partial \Theta} + \frac{1}{R\sin\Theta}\frac{\partial \hat{\phi\phi}}{\partial \phi} + \frac{\partial \hat{\phi R}}{\partial R} + \frac{1}{R}\left[3\hat{\phi R} + 2\hat{\Theta\phi}\,\mathrm{ctg}\,\Theta\right] = \rho \frac{\partial^2 u_\phi}{\partial t^2} - F_\phi$$

Here ΘR, $\Theta\Theta$, $\Theta\phi$, ϕR, $\phi\phi$, RR are components of the stress tensor; u_R, u_Θ, u_ϕ are components of the displacement vector u along the directions e_R, e_Θ, e_ϕ ; F_R, F_Θ, F_ϕ are components of the vector volume force acting at the source along the same directions.

For the components of stress we have the following relations

$$\hat{\Theta R} = \mu\left(\frac{\partial u_\Theta}{\partial R} - \frac{u_\Theta}{R} + \frac{1}{R}\frac{\partial u_R}{\partial \Theta}\right),$$

$$\hat{\Theta\Theta} = \lambda\Delta + 2\frac{\mu}{R}\left(\frac{\partial u_\Theta}{\partial \Theta} + u_R\right),$$

$$\hat{\Theta\phi} = \frac{\mu}{R}\left(\frac{\partial u_\phi}{\partial \Theta} - u_\phi\,\mathrm{ctg}\,\Theta + \frac{1}{\sin\Theta}\frac{\partial u_R}{\partial \phi}\right), \qquad (2.2)$$

$$\hat{\phi R} = \mu\left(\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R} + \frac{1}{R\sin\Theta}\frac{\partial u_R}{\partial \phi}\right),$$

$$\hat{\phi\phi} = \lambda\Delta + 2\frac{\mu}{R}\left(u_R + u_\Theta\,\mathrm{ctg}\,\Theta + \frac{1}{\sin\Theta}\frac{\partial u_\phi}{\partial \phi}\right),$$

$$\hat{RR} = \lambda\Delta + 2\mu\frac{\partial u_R}{\partial R}.$$

Here the dilatation has the form

$$\Delta = \frac{\partial u_R}{\partial R} + \frac{2u_R}{R} + \frac{1}{R\sin\Theta}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{R}\frac{\partial u_\Theta}{\partial \Theta} + \frac{u_\Theta}{R}\,\mathrm{ctg}\,\Theta \qquad (2.3)$$

The Lamé constants λ, μ and the density ρ are piecewise continuous positive functions of a single coordinate R; the components of displacement and stress are continuous and bounded for all points in $[0, R_0]$. The surface of the sphere is free from stress, i.e.

$$\hat{\Theta R} = \hat{\phi R} = \hat{RR} = 0 \text{ for } R = R_0 \qquad (2.4)$$

The initial conditions are

$$\vec{u} = \frac{\partial \vec{u}}{\partial t} = 0 \quad \text{for } t < 0. \qquad (2.4a)$$

The force field $\vec{F}(t,R,\theta,\phi)$ is described as a real source localized in space and time. The following limitations are laid upon F:

1) $\vec{F}(t,R,\theta,\phi) \equiv \emptyset$ for $t < 0$;
2) $\vec{F}(t,R,\theta,\phi)$

is absolutely integrable with respect to t and satisfies the Dirichlet conditi ns for all arguments. Then we are permitted the following representation:

$$\vec{F} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipt}\Big[\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{i=1}^{3} f_{mn}^{(i)}(R,p)\,\vec{A}_{m,n}^{(i)}\Big]dp \qquad (2.5)$$

where

$$\vec{A}_{mn}^{(1)} = \vec{a}_R\, Y_{mn},$$

$$A_{mn}^{(2)} = \Big(\vec{a}_\theta \frac{\partial Y_{mn}}{\partial\theta} + \vec{a}_\phi \frac{1}{\sin\theta}\frac{\partial Y_{mn}}{\partial\phi}\Big)\frac{1}{N}, \qquad (2.6')$$

$$A_{mn}^{(3)} = \Big(\vec{a}_\theta \frac{\partial Y_{mn}}{\partial\phi}\frac{1}{\sin\theta} - \vec{a}_\phi \frac{\partial Y_{mn}}{\partial\theta}\Big)\frac{1}{N},$$

$$Y_{mn}(\theta,\phi) = e^{im\phi} P_n^m(\cos\theta), \qquad N = \sqrt{n(n+1)}$$

$P_n^m(\cos\theta)$ is the associated Legendre polynomial, defined by the following formulae (8):

$$P_n^m(x) = \frac{(1-x^2)^{m/2}}{2^n \cdot n!}\,\frac{d^{n+m}(x^2-1)^n}{dx^{n+m}} \quad \text{for } m \geq 0.$$

$$P_n^m(x) = (-1)^m \frac{(n-|m|)!}{(n+|m|)!} P_n^{|m|}(x) \quad \text{for } m < 0.$$

The system of spherical vector functions $\vec{A}_{mn}^{(i)}$ are a field of a system of vectors satisfying the following orthogonality conditions on a unit sphere:

$$\int_0^{2\pi}\int_0^{\pi} \big(\vec{A}_{mn}^{(i)}, \vec{A}_{lq}^{(j)}\big)\sin\theta\, d\theta\, d\phi = 4\pi\,\delta_{ij}\,\delta_{ml}\,\delta_{nq}\,\frac{(n+m)!}{(n-m)!\,(2n+1)}.$$

Another system of vector functions was suggested for the solution of a similar problem in elastic wave theory in the work of G. I. Petrashin (11), which are linear combinations of the system $A_{mn}^{(i)}$. In distinction with (11) where the wave field in the sphere is given in the form of a summation of potentials and solenoidal fields, the analysis of the wave fields according to the system $A_{mn}^{(i)}$ permits a separate study of the spheroidal and torsional osci lations of a sphere (22). All proofs on the orthogonality and completeness of the system, studied in (11) are easily applied to the system $A_{mn}^{(i)}$.

The coefficients $f_{mn}^{(i)}$ making up the impact F in the system $A_{mn}^{(i)}$ are

$$f_{mn}^{(i)}(R,p) = \frac{2n+1}{4\pi}\,\frac{(n-m)!}{(n+m)!}\int_{-\infty}^{+\infty} e^{-ipt}\int_0^{2\pi}\int_0^{\pi}\big(\vec{F}, \vec{A}_{mn}^{(i)}\big)\sin\theta\, d\theta\, d\phi\, dt.$$

[1] For $n = 0$ $N = 0$, $Y_{mn} = \text{const}$ and (2.6) loses meaning. We will consider that $A_{00}^{(2)} = A_{00}^{(3)} = 0$.

Specifically for $i = 1, 2, 3$ we have:

$$f_{mn}^{(1)} = \frac{(n-m)!}{(n+m)!}\frac{2n+1}{4\pi}\int_{-\infty}^{\infty} e^{-ipt}\int_0^{2\pi}\int_0^{\pi} F_R \bar{Y}_{mn} \sin\theta \, d\theta \, d\phi \, dt ,$$

$$f_{mn}^{(2)} = \frac{(n-m)!}{(n+m)!}\frac{2n+1}{4\pi N}\int_{-\infty}^{\infty} e^{-ipt}\int_0^{2\pi}\int_0^{\pi}\left[F_\theta \frac{\partial \bar{Y}_{mn}}{\partial\theta} + F_\phi \frac{1}{\sin\theta}\frac{\partial \bar{Y}_{mn}}{\partial\phi}\right]\sin\theta \, d\theta \, d\phi \, dt , \qquad (2.7)$$

$$f_{mn}^{(3)} = \frac{(n-m)!}{(n+m)!}\frac{2n+1}{4\pi N}\int_{-\infty}^{\infty} e^{-ipt}\int_0^{2\pi}\int_0^{\pi}\left[F_\theta \frac{1}{\sin\theta}\frac{\partial \bar{Y}_{mn}}{\partial\phi} - F_\phi \frac{\partial \bar{Y}_{mn}}{\partial\theta}\right]\sin\theta \, d\theta \, d\phi \, dt ,$$

$$\bar{Y}_{mn}(\theta,\phi) = e^{-im\phi} P_n^m(\cos\theta).$$

For the force components F_R, F_θ, F_ϕ we obtain from (2.5), (2.6):

$$F_R = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipt}\left[\sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{mn}^{(1)} Y_{mn}\right] dp,$$

$$F_\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipt}\left[\sum_{n=1}^{\infty}\frac{1}{N}\sum_{m=-n}^{n}\left(f_{mn}^{(2)}\frac{\partial Y_{mn}}{\partial\theta} + f_{mn}^{(3)}\frac{1}{\sin\theta}\frac{\partial Y_{mn}}{\partial\phi}\right)\right] dp \qquad (2.8)$$

$$F_\phi = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipt}\left[\sum_{n=1}^{\infty}\frac{1}{N}\sum_{m=-n}^{n}\left(f_{mn}^{(2)}\frac{1}{\sin\theta}\frac{\partial Y_{mn}}{\partial\phi} - f_{mn}^{(3)}\frac{\partial Y_{mn}}{\partial\theta}\right)\right] dp$$

<u>Formulae for displacements</u>. We will seek displacements in the form

$$\vec{u}(t, R, \theta, \phi) = \frac{1}{2\pi}\, \text{v.}\int_{-\infty}^{\infty} e^{ipt}\left[\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{i=1}^{3} V_{mn}^{(i)} A_{mn}^{(i)}\right] dp \qquad (2.9)$$

where $V_{mn}^{(i)} \equiv V_{mn}^{(i)}(R, p)$. Hence for the projection of displacements along the directions a_R, a_θ, a_ϕ, we obtain:

$$u_R = \frac{1}{2\pi}\, \text{v.}\int_{-\infty}^{\infty} e^{ipt}\left[\sum_{n=0}^{\infty}\sum_{m=-n}^{n} V_{mn}^{(1)} Y_{mn}\right] dp ,$$

$$u_\theta = \frac{1}{2\pi}\, \text{v.}\int_{-\infty}^{\infty} e^{ipt}\left[\sum_{n=1}^{\infty}\frac{1}{N}\sum_{m=-n}^{n}\left(V_{mn}^{(2)}\frac{\partial Y_{mn}}{\partial\theta} + V_{mn}^{(3)}\frac{1}{\sin\theta}\frac{\partial Y_{mn}}{\partial\phi}\right)\right] dp \qquad (2.10)$$

$$u_\phi = \frac{1}{2\pi}\, \text{v.}\int_{-\infty}^{\infty} e^{ipt}\left[\sum_{n=1}^{\infty}\frac{1}{N}\sum_{m=-n}^{n}\left(V_{mn}^{(2)}\frac{1}{\sin\theta}\frac{\partial Y_{mn}}{\partial\phi} - V_{mn}^{(3)}\frac{\partial Y_{mn}}{\partial\theta}\right)\right] dp .$$

Placing (2.8), (2.10) into equation (2.1) and the boundary conditions, we arrive at the following equation for $V_{mn}^{(i)}$.

1. For $V_{mn}^{(1)}$, $V_{mn}^{(2)}$:

$$L_1(V_{mn}^{(1)}, V_{mn}^{(2)}) \equiv \frac{d}{dR}\left[(\lambda+2\mu)\frac{dV_{mn}^{(1)}}{dR} + \frac{2\lambda}{R}V_{mn}^{(1)} - \frac{\lambda N}{R}V_{mn}^{(2)}\right] +$$

$$+ \frac{\mu}{R^2}\left[4\frac{dV_{mn}^{(1)}}{dR}R - 4V_{mn}^{(1)} + N\left(3V_{mn}^{(2)} - R\frac{dV_{mn}^{(2)}}{dR} - NV_{mn}^{(1)}\right)\right]$$

$$+ p^2\rho V_{mn}^{(1)} = -f_m^{(1)} \quad - \frac{4\rho g V_{mn}^{(1)}}{r} \qquad (2.11)$$

V^1 = radial
$V^2 = \theta$
$V^3 = \phi$

$$L_2(V_{mn}^{(1)}, V_{mn}^{(2)}) \equiv \frac{d}{dR}\Big[\mu\Big(\frac{dV_{mn}^{(2)}}{dR} - \frac{V_{mn}^{(2)}}{R} + \frac{NV_{mn}^{(1)}}{R}\Big)\Big]$$
$$+ \frac{\lambda N}{R}\Big(\frac{dV_{mn}^{(1)}}{dR} + \frac{2}{R}V_{mn}^{(1)} - \frac{N}{R}V_{mn}^{(2)}\Big) + \frac{\mu}{R^2}\Big(5NV_{mn}^{(1)} + 3R\frac{dV_{mn}^{(2)}}{dR} - V_{mn}^{(2)} - 2N^2V_{mn}^{(2)}\Big)$$
$$+ \quad p^2\rho V_{mn}^{(2)} = -f_{mn}^{(2)}$$

with the boundary conditions:

$$\sigma_{RR} \equiv (\lambda+2\mu)\frac{dV_{mn}^{(1)}}{dR} + \frac{2\lambda}{R}V_{mn}^{(1)} - \frac{\lambda N}{R}V_{mn}^{(2)} = 0,$$
$$\tau_{\theta R} \equiv \mu\Big(\frac{dV_{mn}^{(2)}}{dR} - \frac{V_{mn}^{(2)}}{R} + \frac{NV_{mn}^{(1)}}{R}\Big) = 0 \qquad \text{for } R=R_0 \qquad (2.12)$$
$$V_{mn}^{(1)} = V_{mn}^{(2)} = 0 \qquad \text{for } R=0.$$

The functions $V_{mn}^{(1)}$, $V_{mn}^{(1)}$, σ_{RR} and $\tau_{\theta R}$ are continuous and bounded everywhere on $[0, R_0]$.

2. For V_{mn} we have

$$L_3(V_{mn}^{(3)}) \equiv \frac{d}{dR}\Big[\mu\Big(\frac{dV_{mn}^{(3)}}{dR} - \frac{V_{mn}^{(3)}}{R}\Big)\Big] + \frac{3\mu}{R}\frac{dV_{mn}^{(3)}}{dR} \qquad (2.13)$$
$$-\frac{\mu}{R^2}(N^2+1)V_{mn}^{(3)} + p^2\rho V_{mn}^{(3)} = -f_{mn}^{(3)}$$

and the boundary condition:

$$\tau_{\varphi R} \equiv \mu\Big(\frac{dV_{mn}^{(3)}}{dR} - \frac{V_{mn}^{(3)}}{R}\Big) = 0 \qquad \text{for } R=R_0$$
$$(2.14)$$
$$V_{mn}^{(3)} = 0 \qquad \text{for } \quad R=0$$

The functions $V_{mn}^{(3)}$, $\tau_{\varphi R}$ are continuous and bounded everywhere on $[0, R_0]$.

<u>Expressions for V_{mn} in terms of eigenfunctions.</u> $V_{mn}^{(i)}$ $(i = 1, 2)$ can be expressed in the following manner:

$$V_{mn}^{(i)} = \frac{1}{R_0^2}\sum_{k=1}^{\infty} c_{kmn}^{s}\tilde{V}_{kn}^{(i)} \qquad (2.15)$$

where for the coefficients c_{kmn}^{s} we have:

$$c_{kmn}^{s} = \frac{1}{p_{kns}^2 - p^2}\frac{D_{kmn}^{s}}{I_{kns}} \qquad (2.16)$$
$$D_{kmn}^{s} = \int_0^{R_0}\big(f_{mn}^{(1)}\tilde{V}_{kn}^{(1)} + f_{mn}^{(2)}\tilde{V}_{kn}^{(2)}\big)R^2\,dR$$
$$I_{kn}^{s} = \frac{1}{R_0^3}\int_0^{R_0}\rho R^2\Big[\big(\tilde{V}_{kn}^{(1)}\big)^2 + \big(\tilde{V}_{kn}^{(2)}\big)^2\Big]dR$$

Here $\tilde{V}_{kn}^{(i)}(R)$, $i = 1, 2$ is the eigenfunction, p_{kns}^2 is the eigenvalue of the operator consisting of the left hand side of (2.11) together with the boundary conditions (2.12), for a given integer value of the parameter n.

Similarly

$$V_{mn}^{(3)} = \frac{1}{R_o^2}\sum_{k=1}^{\infty} c_{kmn}^{T}\ \tilde{V}_{kn}^{(3)} \qquad (2.17)$$

where

$$c_{kmn}^{T} = \frac{1}{p_{knT}^2 - p^2}\ \frac{D_{kmn}^{T}}{I_{knT}}$$

$$D_{kmn}^{T} = \int_0^{R_o} f_{mn}^{(3)}\ \tilde{V}_{kn}^{(3)}\ R^2\, dR \qquad (2.18)$$

$$I_{knT} = \frac{1}{R_o^2}\int_0^{R_o} \rho R^2 (\tilde{V}_{kn}^{(3)})^2 dR.$$

where $\tilde{V}_{kn}^{(3)}$ is the eigenfunction, p_{knT}^2 is the eigenvalue of the operator consisting of the left hand side of (2.13) and the boundary conditions (2.14).

To derive (2.15) - (2.18) these orthogonality conditions were used:

$$\int_0^{R_o} \rho R^2 (\tilde{V}_{kn}^{(1)} \tilde{V}_{ln}^{(1)} + \tilde{V}_{kn}^{(2)} \tilde{V}_{ln}^{(2)})\, dR = 0$$

$$\int_0^{R_o} \rho R^2 (\tilde{V}_{kn}^{(3)} \tilde{V}_{ln}^{(3)})\, dR = 0 \qquad \text{for } k \neq l$$

Placing the expressions developed for $V_{mn}^{(i)}(R,p)$ into (2.10), the following formulae for displacements are obtained:

$$u_R = \frac{1}{2\pi R_o^2} V_o \int_{-\infty}^{+\infty} e^{ipt}\Big[\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{k=1}^{\infty} c_{kmn}^{S}(p)\ \tilde{V}_{kn}^{(1)}(R)\ Y_{mn}\Big]\, dp,$$

$$u_\theta = \frac{1}{2\pi R_o^2} V_o \int_{-\infty}^{\infty} e^{ipt}\Big[\sum_{n=1}^{\infty}\frac{1}{N}\sum_{m=-n}^{n}\sum_{k=1}^{\infty}\Big(c_{kmn}^{S}(p)\ \tilde{V}_{kn}^{(2)}(R)\frac{\partial Y_{mn}}{\partial\theta}$$

$$+ c_{kmn}^{T}(p)\ \tilde{V}_{kn}^{(3)}(R)\frac{1}{\sin\theta}\frac{\partial Y_{mn}}{\partial\varphi}\Big)\Big]\, dp, \qquad (2.19)$$

$$u_\varphi = \frac{1}{2\pi R_o^2} V_o \int_{-\infty}^{\infty} e^{ipt}\Big[\sum_{n=1}^{\infty}\frac{1}{N}\sum_{m=-n}^{n}\sum_{k=1}^{\infty}\Big(c_{kmn}^{S}(p)\ \tilde{V}_{kn}^{(2)}(R)\frac{1}{\sin\theta}\frac{\partial Y_{mn}}{\partial\varphi}$$

$$- c_{kmn}^{T}(p)\ \tilde{V}_{kn}^{(3)}(R)\frac{\partial Y_{mn}}{\partial\theta}\Big)\Big]\, dp.$$

The proof that this constructed solution (2.19) satisfies the initial conditions (2.4a) procedes analogously with that of the plane case for each term in the series $\sum_{m,n}$

__Expression of displacements in the form of a sum of proper oscillations of a sphere.__ In formula (2.19) only products of the type $c_{mn}^{Q}\exp(ipt)$ depend on p. Thus calculation of integrals with respect to p reduces to evaluating a number of integrals of the type

$$V_o \int_{-\infty}^{\infty} \frac{e^{ipt}\,\psi(p)}{\alpha^2 - p^2}\, dp$$

By introducing the conditions on the source, $\psi(p)$ can have singularities for Im $p \geq 0$ and ~~...~~ must be analytic in the lower halfspace. As $p \to \infty$ $\psi(p) \to 0$. Besides this $\psi(-p) = \psi(p)$. Hence it is not difficult to obtain

Mal Seismometer.

$$V_0\int_{-\infty}^{\infty} e^{ipt}\,\frac{\psi(p)}{\alpha^2-p^2}\,dp = 2\,\mathrm{Re}\left[V_0\int_{0}^{\infty} e^{ipt}\,\frac{\psi(p)}{\alpha^2-p^2}\,dp\right] =$$

$$= \frac{2\pi}{\alpha}\,\mathrm{Re}\left[\psi(\alpha)\,e^{i(\alpha t-\pi/2)}\right] + \alpha(t)$$

The function $\alpha(t)$ is associated with the particular source function $f_{mn}^{(1)}$ in the upper halfplane for $p = 0$. Its form, either exponential decay with time or static impact, is not of interest to us.

Placing this evaluation into (2.19) and ignoring $\alpha(t)$, we get

$$u_R = \frac{1}{R_0^2}\,\mathrm{Re}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{k=1}^{\infty}\exp\left[i\,(p_{knS}t-\pi/2)\right]D_{kmn}^{S}\,\frac{\tilde V_{kn}^{(1)}(R)}{p_{knS}}\,\frac{Y_{mn}}{I_{knS}}$$

$$u_\theta = \frac{1}{R_0^2}\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{1}{N}\sum_{m=-n}^{n}\sum_{k=1}^{\infty}\Big\{\exp\left[i(p_{knS}t-\pi/2)\right]D_{kmn}^{S}\,\frac{\tilde V_{kn}^{(2)}(R)}{p_{knS}I_{knS}}\,\frac{\partial Y_{mn}}{\partial\theta}$$

$$+\exp\left[i(p_{knT}t-\pi/2)\right]D_{kmn}^{T}\,\frac{\tilde V_{kn}^{(3)}(R)}{p_{knT}I_{knT}\sin\theta}\,\frac{\partial Y_{mn}}{\partial\varphi}\Big\} \qquad (2.19a)$$

$$u_\varphi = \frac{1}{R_0^2}\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{1}{N}\sum_{m=-n}^{n}\sum_{k=1}^{\infty}\Big\{\exp\left[i\,(p_{knS}t-\pi/2)\right]D_{kmn}^{S}\,\frac{\tilde V^{(2)}(R)}{p_{knS}I_{knS}\sin\theta}\,\frac{\partial Y_{mn}}{\partial\varphi}$$

$$-\exp\left[i\,(p_{knT}t-\pi/2)\right]D_{kmn}^{T}\,\frac{\tilde V_{kn}^{(3)}(R)}{I_{knT}p_{knT}}\,\frac{\partial Y_{mn}}{\partial\theta}\Big\}.$$

Thus we have expressed the displacements in the form of proper S spheroidal and toroidal T oscillations of a sphere with discrete frequencies p_{knS} and p_{knT}. The radial component u_R is composed only of spheroidal oscillations; the components u_θ and u_φ are made up of both spheroidal and torsional components.

Asymptotic values of displacement for $n\sin\theta \gg |m|$. We will follow (13) for separating the propagating surface waves from the displacements described in (2.19a). We will change the order of summation (i.e., change the sum $\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{k=1}^{\infty}$ to $\sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}$) and we will change the summation with respect to n to an integration along a contour L_0 enclosing positive part of the real axis in the plane of the complex variable ν, where n = Integer ($\mathrm{Re}\,\nu$):

$$\sum_{n=0}^{\infty} f_n P_n^m(\cos\theta) = -\frac{1}{2i}\oint_{L_0}\frac{f(\nu)\,P_\nu^m(\cos(\pi-\theta))\,(-1)^m\,d\nu}{\cos\pi(-\nu+\tfrac12)}$$

Expressing $(\cos\pi(\nu+\tfrac12))^{-1}$ for $\mathrm{Im}\,\nu \geq 0$ by the series

$2\sum_{l=0}^{\infty}(-1)^l \exp[i(2l+1)(\nu+\frac{1}{2})\pi]$, and for $\operatorname{Im}\nu<0$ the series

$2\sum_{l=0}^{\infty}(-1)^l \exp[-i(2l+1)(\nu+\frac{1}{2})\pi]$, fixing the contour of integration to the real axis and introducing a new variable $p = p_{k0}(\nu)$, we have

$$\sum_{n=0}^{\infty} f_n P_n^m(\cos\theta) \approx 2\sum_{l=0}^{\infty}(-1)^{l+|m|}\int_0^{\infty} f(p) P_\nu^m(\cos(\pi-\theta))$$

$$\times \sin[(2l+1)(\nu+\tfrac{1}{2})\pi]\frac{d\nu}{dp}\,dp.$$

Excluding from consideration the low frequency part of the field with $p < \bar{p}$ and applying the asymptotic expansion of the associated Legendre polynomial for large values $n\sin\theta \gg |m|$, we arrive at

$$\sum_{\mathrm{ENT}[\nu(\bar{p})]}^{\infty} f_n \frac{dP_n^m(\cos\theta)}{d\theta} \approx \frac{2\sqrt{2}}{\sqrt{\pi\sin\theta}}\sum_{l=0}^{\infty}(-1)^l\int_{\bar{p}}^{\infty}\nu^{m+1/2} f(p)\frac{d\nu}{dp}$$

$$\times \cos[(\nu+\tfrac{1}{2})(\pi-\theta)-\tfrac{3\pi}{4}+\tfrac{m\pi}{2}]\sin[(2l+1)(\nu+\tfrac{1}{2})\pi]\,dp$$

$$\sum_{\mathrm{INT}[\nu(\bar{p})]}^{\infty} f_n P_n^m(\cos\theta) \approx \frac{2\sqrt{2}}{\sqrt{\pi\sin\theta}}\sum_{l=0}^{\infty}(-1)^l\int_{\bar{p}}^{\infty}\nu^{m-1/2} f(p)\frac{d\nu}{dp}$$

$$\times \cos[(\nu+\tfrac{1}{2})(\pi-\theta)-\tfrac{\pi}{4}+\tfrac{m\pi}{2}]\sin[(2l+1)(\nu+\tfrac{1}{2}\pi)]\,dp$$

Using these formulae for transforming (2.19a), we can obtain the following asymptotic expression for the displacements in surface waves, going around the sphere l times and passing through the pole $\theta = 0$ and the antipodes $\theta = \pi$ g times:

$$u_R(t,R,\theta,\varphi) = \frac{1}{\sqrt{2\pi R_0\sin\theta}}\operatorname{Re}\int_{\bar{p}}^{\infty}\frac{1}{p}\exp[i(pt+\tfrac{\pi}{4}(2g-1))]$$

$$\times\left\{\sum_{k=1}^{K_s(p)}\frac{U_{ks}(p,\varphi)}{I_{kvs}\,c_{ks}}\sqrt{\frac{v_{ks}}{R_0}}\,\tilde{V}_{kv}^{(1)}\exp[-i(v_{ks}+\tfrac{1}{2})\tilde{\theta}]\right\}dp$$

$$u_\theta(t,R,\theta,\varphi) = \frac{1}{\sqrt{2\pi R_0\sin\theta}}\operatorname{Re}\int_{\bar{p}}^{\infty}\frac{1}{p}\exp[i(pt-\tfrac{\pi}{4}(2g+3))] \tag{2.20}$$

$$\times\left\{\sum_{k=1}^{K_s(p)}\frac{U_{ks}(p,\varphi)}{I_{kvs}\,c_{ks}}\sqrt{\frac{v_{ks}}{R_0}}\,\tilde{V}_{kv}^{(2)}\exp[-i(v_{ks}+\tfrac{1}{2})\tilde{\theta}]\right\}dp$$

$$u_\varphi(t,R,\theta,\varphi) = \frac{1}{\sqrt{2\pi R_0 \sin\theta}} \, Re \int_{\bar p}^{\infty} \frac{1}{p} \exp\left[i\left(pt - \frac{\pi}{4}(2g-1)\right)\right]$$

$$\times \left\{ \sum_{k=1}^{K_T(p)} \frac{u_{kT}(p,\varphi)}{I_{kvT}\, C_{kT}} \, \tilde V_{kv}^{(3)} \exp\left[-i\,(v_{kT}+1/2)\tilde\theta\right]\right\} dp$$

Here for $Q \equiv S$, T, $K_Q(p)$ is the maximum number of harmonics in Rayleigh waves ($Q \equiv S$) and Love waves ($Q \equiv T$) for a given p:

$$u_{kQ} = \sum_{m=-\infty}^{\infty} v_{kQ}^{m-1} D_{kmn}^{Q} \exp\left[i m\left(\varphi + (-1)^g \pi/2\right)\right] \qquad (2.21)$$

where v_{kQ} is the analog of the wave number, a root of the equation $p_{kQ}^2(v) - p^2 = 0$.

The phase and group velocities of the k-th harmonic v_{kQ} and C_{kQ} along the surface of the sphere are expressed through v_{kQ} :

$$v_{kQ} = \frac{pR_0}{v_{kQ}+1/2}\,, \quad C_{kQ} = R_0 \frac{dp_{kQ}}{dv}\,(v = v_{kQ}) \qquad (2.22)$$

The multiple passages of the waves through the poles $\tilde\theta\,(0 \le \tilde\theta < \infty)$ are related to the coordinate of the observation point $\theta\,(0 \le \theta < \pi)$:

$$\tilde\theta = (-1)^g \theta + 2\pi(g-\ell).$$

The components u_R and u_θ form the Rayleigh wave; u_φ, the Love wave.

<u>Expressions for p_{kQ}, C_{kQ} in terms of integrals of the eigenfunctions.</u>
The method of variational analysis can be used (23) for $Q \equiv S$, T to obtain

$$p_{kvQ}^2 = \frac{N^2 G_{1k}^Q + 2NG_{2k}^Q + G_{3k}^Q}{R_0^2 I_{kvQ}}\,, \quad C_{kQ} = \frac{NG_{1k}^Q + G_{2k}^Q}{R_0\, p_{kQ}\, I_{kvQ}} \qquad (2.23)$$

Here the integrals G_{1k}^Q–G_{3k}^Q have the form

$$G_{1k}^S = \int_0^{R_0} \left[(\lambda+2\mu)(\tilde V_{kv}^{(2)})^2 + \mu(\tilde V_{kv}^{(1)})^2\right] dR,$$

$$G_{2k}^S = \int_0^{R_0} \left[\left(\mu \tilde V_{kv}^{(1)} \frac{d\tilde V_{kv}^{(2)}}{dR} - \lambda \tilde V_{kv}^{(2)} \frac{d\tilde V_{kv}^{(1)}}{dR}\right)R - (3\lambda+2\mu)\tilde V_{kv}^{(1)} \tilde V_{kv}^{(2)}\right] dR$$

$$G_{3k}^S = \int_0^{R_0} \left\{ \left[(\lambda+2\mu)\left(\frac{d\tilde V_{kv}^{(1)}}{dR}\right)^2 + \mu\left(\frac{d\tilde V_{kv}^{(2)}}{dR}\right)^2\right] R^2\right.$$

$$\left. + 2\left(2\lambda \tilde V_{kv}^{(1)} \frac{d\tilde V_{kv}^{(2)}}{dR} - \mu \tilde V_{kv}^{(2)} \frac{d\tilde V_{kv}^{(1)}}{dR}\right)R + 4(\lambda+\mu)(\tilde V_{kv}^{(1)})^2 - \mu(\tilde V_{kv}^{(2)})^2\right\} dR \qquad (2.24)$$

$$G_{1k}^T = \int_0^{R_0} \mu (\tilde V_{kv}^{(3)})^2 dR\,, \qquad G_{2k}^T \equiv 0$$

$$G_{3k}^T = \int_0^{R_0} \mu \left[\left(\frac{d\tilde V_{kv}^{(3)}}{dR}\right)^2 R^2 - 2\tilde V_{kv}^{(3)} \frac{d\tilde V_{kv}^{(3)}}{dR} R - (\tilde V_{kv}^{(3)})^2\right] dR$$

$$N = \sqrt{n(n+1)}$$

Methods for calculation of $\tilde{V}^{(i)}_{k\nu}$, $P^{i}_{k\nu q}$ are given in (3, 4, 14).

SURFACE WAVES FROM ELEMENTARY SOURCES. The source type influences only the terms U_{kQ} in (2.20). We will consider expressions for U_{kQ} for several elementary impacts, noting that $\nu \gg |m|$.

I. Radial axially symmetric force. Let

$$\vec{F} = F_R(t, R, \theta)\,\vec{z}_R$$

From (2.7) we obtain:

$$f^{(1)}_{o\nu} = (\nu_{ks} + 1/2)\int_{-\infty}^{\infty} e^{-ipt}\int_0^{\pi} F_R P_\nu(\cos\theta)\sin\theta\, d\theta\, dt\,,$$

$$f^{(1)}_{m\nu} = 0 \text{ for } m \neq 0\,, \qquad f^{(2)}_{m\nu} \equiv f^{(3)}_{m\nu} \equiv 0 \qquad (2.25)$$

$$U_{ks} = \frac{1}{\nu_{ks}}\int_0^{R_0} f^{(1)}_{o\nu}\,\tilde{V}^{(1)}_{k\nu}(R)\,R^2\,dR$$

$$U_{kT} = 0.$$

In particular, for an ideally concentrated radial force at the point $R = H$, $\theta = 0$

$$\vec{F} = \frac{\delta(R-H)\,\delta(\theta)\,\phi(t)}{H^2\sin\theta}\vec{z}_R$$

$$U_{ks} = \tilde{V}^{(1)}_{k\nu}(H) \times S(p)\,, \qquad (2.26)$$

where $S(p) = \int_{-\infty}^{\infty} e^{-ipt}\phi(t)\,dt$ - time spectra of the source.

II. Axially symmetric force directed along a meridian. Let

$$\vec{F} = F_\theta(t, R, \theta)\,\vec{z}_\theta$$

In this case we have from (2.7):

$$f^{(1)}_{m\nu} \equiv f^{(3)}_{m\nu} \equiv 0\,; \quad f^{(2)}_{m\nu} = 0 \quad \text{for } m \neq 0$$

$$f^{(2)}_{o\nu} = \int_{-\infty}^{\infty} e^{-ipt}\int_0^{\pi} F_\theta \frac{dP_\nu(\cos\theta)}{d\theta}\sin\theta\, d\theta\, dt \quad \ldots (2.27)$$

$$U_{ks} = \frac{1}{\nu_{ks}}\int_0^{\infty} f^{(2)}_{o\nu}\,\tilde{V}^{(2)}_{k\nu}\,R^2\,dR$$

$$U_{kT} = 0.$$

For an ideally concentrated meridianal source acting at $R = H$, $\theta = 0$

$$\vec{F} = \frac{2\,\delta(R-H)\,\delta(\theta)\,\delta(t)}{H^3\sin^2\theta}\vec{z}_\theta \qquad U_{ks} = -\frac{\nu_{ks}}{H}\tilde{V}^{(2)}_{k\nu}(H)\,S(p) \qquad (2.28)$$

III. Rotational impact. Let

$$\vec{F} = F_\varphi(t, R, \theta)\vec{a}_\varphi$$

From (2.7) we obtain

$$f^{(1)}_{m\nu} = f^{(2)}_{m\nu} = 0\ ; \qquad f^{(3)}_{m\nu} = 0 \quad \text{for } m \neq 0$$

$$f^{(3)}_{o\nu} = -\int_{-\infty}^{\infty} e^{-ipt}\int_0^\pi F_\varphi \frac{dP_\nu(\cos\theta)}{d\theta}\sin\theta\, d\theta\, dt\ ;$$

$$U_{kS} = 0\ ; \qquad U_{kT} = -\frac{1}{\nu_{kT}}\int_0^{R_0} f^{(3)}_{o\nu}\, \tilde{V}^{(3)}_{k\nu}\, R^2 dR\ . \qquad (2.29)$$

In the special case of an ideally concentrated rotational impact acting at the point $R = H$, $\theta = 0$,

$$F = 2\delta(R-H)\frac{\delta(\theta)\phi(t)}{H^3 \sin^2\theta}\vec{a}_\varphi$$

$$U_{kT} = \frac{\nu_{kT}}{H}\tilde{V}^{(3)}_{k\nu}(H)\cdot S(p) \qquad (2.30)$$

IV. Tangential force field with fixed azimuth. We will consider only a particular case of this force field:

$$\vec{F} = F_T(t, R, \theta)\vec{a}_T$$

where $\vec{a}_T$ is a unit horizontal vector of fixed azimuth:

$$(\vec{a}_T, \vec{a}_R) = 0\ , \quad (\vec{a}_T, \vec{a}_\theta) = \cos(\delta - \phi), \quad (\vec{a}_T, \vec{a}_\varphi) = \sin(\delta - \phi).$$

Then,

$$\vec{F} = F_T[\vec{a}_\theta \cos(\delta-\phi) + \vec{a}_\varphi \sin(\delta-\phi)]$$

and we obtain from (2.7):

$$f^{(1)}_{m\nu} = 0\ ; \quad f^{(2)}_{m\nu} = f^{(3)}_{m\nu} \quad \text{for } m \neq \pm 1\ ;$$

$$f^{(2)}_{1\nu} = \frac{e^{-i\delta}f_{T\nu}}{2\nu(\nu+1)}\ ; \quad f^{(2)}_{-1\nu} = \frac{-e^{+i\delta}}{2}f_{T\nu}$$

$$f^{(3)}_{1\nu} = \frac{e^{-i\delta}f_{T\nu}}{2i\nu(\nu+1)}\ ; \quad f^{(3)}_{-1\nu} = \frac{e^{i\delta}}{2i}f_{T\nu}$$

$$f_{T\nu} = \int_{-\infty}^{\infty} e^{-ipt}\int_0^\pi F_T\left(\frac{dP^1_\nu}{d\theta} + \frac{P^1_\nu}{\sin\theta}\right)\sin\theta\, d\theta\, dt.$$

Summing with respect to m,

$$U_{kS} = \frac{(-1)^g}{v^2_{kS}}\, i \cos(\delta-\varphi)\int_0^{R_0} f_{T\nu}\, \tilde{V}^{(2)}_{k\nu}\, R^2\, dR ,$$
$$U_{kT} = \frac{-(-1)^g}{v^2_{kT}}\, i \sin(\delta-\varphi)\int_0^{R_0} f_{T\nu}\, \tilde{V}^{(3)}_{k\nu}\, R^2\, dR . \qquad (2.31)$$

In the case of a tangential concentrated force at $R = H$, $\theta = 0$,

$$\vec{F} = \frac{\delta(R-H)\,\delta(\theta)\,\varphi(t)\,\vec{e}_T}{H^2 \sin\theta}$$

$$U_{kS} = (-1)^g\, i \cos(\delta-\varphi)\, \tilde{V}^{(2)}_{k\nu}(H)\, S(p)$$
$$U_{kT} = -(-1)^g\, i \sin(\delta-\varphi)\, \tilde{V}^{(3)}_{k\nu}(H)\, S(p) \qquad (2.32)$$

V - VIII. <u>Other elementary sources.</u> The formulas for the remaining elementary sources considered in section I can be obtained from equations (1.33) - (1.36) upon replacing U_{kR} by U_{kS}, U_{kL} by U_{kT}, ξ_{kR} by ν_{kS}/H, ξ_{kT} by ν_{kT}/H, $\tilde{V}^{(1)}_k$ by $\tilde{V}^{(1)}_{k\nu}$, $\tilde{V}^{(2)}_k$ by $(-1)^g\, \tilde{V}^{(2)}_{k\nu}$, $\tilde{V}^{(3)}_k$ by $(-1)^g\, \tilde{V}^{(3)}_{k\nu}$.

<u>Asymptotic formulae for</u> (2.19) - (2.32) <u>as</u> $pR_0/b(R_0) \to \infty$. Let $pR_0/b(R_0) \to \infty$, and let the quantity $p(R_0 - R)/b(R_0) = pz/b(R_0)$ under these conditions. Then it is not difficult to show that equations (2.19) - (2.32) tend toward the corresponding formulae (1.19) - (1.32), if it is noted that as $pR_0/b(R_0) \to \infty$ and

$$g = 0 , \quad \ell = 0 \quad H = R_0 - h ; \quad R_0\theta \to r ;$$

$$U_{kR} \to -U_{kz} ; \quad U_{k\theta} \to U_{kr} ; \quad \frac{\nu_{kQ}}{R} \approx \frac{\nu_{kQ}}{H} \to \xi_{kQ}$$

$$\tilde{V}^{(i)}_{k\nu}(R_1)\, \tilde{V}^{(i)}_{k\nu}(R_2) \to V^{(i)}_k(z_1)\, V^{(i)}_k(z_2) , \qquad (2.33)$$

where $z_1 = R_0 - R_1$, $z_2 = R_0 - R_2$, $i = 1, 2, 3$;

$$\tilde{V}^{(1)}_{k\nu}(R_1)\, \tilde{V}^{(2)}_{k\nu}(R_2) \to -\tilde{V}^{(1)}_k(z_1)\, \tilde{V}^{(2)}_k(z_2) .$$

3. Some properties of the derived solutions

We will consider some properties of the derived equations (1.20), (2.20), which are essential for understanding the excitation and propagation of surface waves. Let us consider that the force field F, describing the seismic source, is localized in some zone, situated in the halfspace near the initial coordinates $z = 0$, $r = 0$, and for the sphere near the pole $R = R_0$, $\theta = 0$. We will locate a non-disturbing receiver at the point z, r, ϕ (R, θ, ϕ), which registers the q'th component of displacement at this point ($q = z$, r, ϕ in a halfspace and $q = R$, θ, ϕ for a sphere). The quantity r (or θ) has the meaning of epicentral distance, ϕ - azimuth at the epicenter to the station, z (or R) -- depth of the receiver, measured from the free surface (or center of sphere).

The theoretical seismogram $u_q(t)$ of the surface wave, recorded by such a receiver, is described by equations (1.20) or (2.20) for sufficiently large r (or θ). The Rayleigh surface wave (index $Q = R$ or S) will be recorded only for $q = z$, r or R, θ; it is polarized in a vertical plane (section of great circle), passing through the epicenter and receiver. The Love surface wave (wave index $Q = L$ or T) will be recorded only for $q = \phi$; it is linearly polarized, the displacement vector normal to the polarization plane of Rayleigh waves.

Each seismogram can be considered as a superposition of an infinite number of harmonics (in other terminology -- normal waves, operators, or modes); the index k ($k = 1, 2, \ldots, \infty$) designates the number of the harmonic. Let $u_{kq}(t)$ be the contribution of the k'th harmonic to the q'th component of the seismogram. Then

$$u_q(t) = \sum_{k=1}^{\infty} u_{kq}(t). \qquad (3.1)$$

We will consider further not the seismogram itself, $u_q(t)$ or $u_{kq}(t)$ but their spectra, i.e., the Fourier transform with respect to time of the form $\int_0^\infty \Phi(t) e^{-ipt} dt$; we designate these spectra as $\Phi_q(p)$ and $\Phi_{kq}(p)$ respectively. It is evident that

$$\Phi_q = \sum_{k=1}^{\infty} \Phi_{kq} \qquad (3.2)$$

$$u_{kq}(t) = \frac{1}{\pi} \, Re \int_0^\infty \Phi_{kq}(p) \, e^{ipt} dp \qquad (3.3)$$

<u>Surface wave spectra</u>. The spectral density Φ_{kq} differs from zero only for $p > \bar{p}_{kQ}$, where $\bar{p}_{kQ}$ is the limiting frequency in the spectra of the k'th harmonic of the Q'th wave. The boundary frequencies are related as

$$\bar{p}_{k+1,Q} > \bar{p}_{kQ} > \bar{p}_{k-1,Q} > \cdots > \bar{p}_{1Q}$$

In a halfspace $p_{1L} = p_{1R} = 0$, in a sphere $p_{1S} > 0$, $p_{1T} > 0$. However since we are not interested in the low frequency part of the spectra of the disturbance with $p < \bar{p}$, which is impossible to express in the form of a propagating wave, it will be accepted that for surface waves $\Phi_{kq} = 0$ for $p < \max(\bar{p}, \bar{p}_{kQ})$.

The spectral density can be expressed in the form of a product

$$\Phi_{kq} = \prod_{i=0}^{4} B_{kq}^{(i)} \qquad (3.4)$$

For the terms $B_{kq}^{(i)}$ we have the following formulaes:

a) in a halfspace (where $Q = R$ for $q = z, r$; $Q = L$ for $q = \phi$):

$$B_{kq}^{(0)} = B^{(0)} = \sqrt{\frac{\pi}{2}}\; e^{-i\pi/4} \tag{3.5}$$

$$B_{kq}^{(1)} = \left[v_{kQ}(p)\, C_{kQ}(p)\, I_{kQ}(p) \right]^{-1} \tag{3.6}$$

$$B_{kq}^{(2)} = \frac{\exp[-ir\xi_{kQ}(p)]}{\sqrt{r\xi_{kQ}(p)}} \tag{3.7}$$

$$B_{kq}^{(3)} = U_{kQ}(p, \phi) \tag{3.8}$$

$$B_{kq}^{(4)} = \alpha_q \tilde{V}_k^{(j_q)}(p) \, , \quad j_z = 1 \, , \; j_r = 2 \, , \quad j_\phi = 3 \tag{3.9}$$

$$\alpha_z = 1 \, , \quad \alpha_r = -i \, , \quad \alpha_\phi = i$$

b) in a sphere (where $Q = S$ for $q = R, \Theta$, $Q = T$ for $q = \phi$):

$$B_{kq}^{(0)} = B^{(0)} = \sqrt{\frac{\pi}{2}}\; e^{-i\pi/4} \tag{3.5a}$$

$$B_{kq}^{(1)} = \left[v_{kQ}(p)\, C_{kQ}(p)\, I_{kvQ} \right]^{-1} \tag{3.6a}$$

$$B_{kq}^{(2)} = \frac{\exp\{-i[(v_{kQ}(p)+\frac{1}{2})\tilde{\Theta} - \frac{\pi}{2}g]\}}{\sqrt{v_{kQ}(p)\sin\Theta}} \tag{3.7a}$$

$$B_{kq}^{(3)} = U_{kQ}(p, \phi) \tag{3.8a}$$

$$B_{kq}^{(4)} = \alpha_q \tilde{V}_{kv}^{(j_q)} \, , \quad j_R = 1 \, , \quad j_\Theta = 2 \, , \; j_\phi = 3 \tag{3.9a}$$

$$\alpha_R = 1 \, , \quad \alpha_\Theta = i(-1)^{g+1} \, , \quad \alpha_\phi = i(-1)^{g}$$

Here $\tilde{\Theta}$ is the total great circle path of the wave leaving the source, g is the number of passages through the epicenter and anti-epicenter.

The multiplier $B_{kq}^{(1)}$ depends only the properties of the medium (variation of velocity and density with depth or radius) and the type of waves recorded.

The multiplier $B_{kq}^{(2)}$ describes the propagation effect of the waves: the numerator defines the phase delay of the oscillations for a given frequency, the denominator--the decrease of amplitude due to geometric spreading along the path r (or θ). The additional phase contribution of $\pi g/2$ in a sphere arises on account of g-passages through the epicenter and anti-epicenter.

The multiplier $B_{kq}^{(3)}$ for a given wave depends only on the source mechanism, and also on the azimuth of the recording station with respect to the source. For an axially symmetric source $B_{kq}^{(3)}$ does not depend on ϕ.

The multiplier $B_{kq}^{(4)}$ depends on the depth of burial of the receiver and its orientation (i.e., for which component of displacement it is set up to measure).

Polarization of Rayleigh waves. The ratio of spectral densities Φ_{kr}/Φ_{kz} or $\Phi_{k\theta}/\Phi_{kR}$ defines the polarization of Rayleigh waves. Since $B_{kr}^{(j)} = B_{kz}^{(j)}$ and $B_{kR}^{(j)} = B_{k\theta}^{(j)}$ for $j < 4$, then

$$\frac{\Phi_{kr}}{\Phi_{kz}} = \frac{B_{kr}^{(4)}}{B_{kz}^{(4)}} = -i\left[\frac{\tilde{V}^{(2)}_k(p,z)}{\tilde{V}^{(1)}_k(p,z)}\right] \qquad (3.10)$$

$$\frac{\Phi_{k\theta}}{\Phi_{kR}} = \frac{B_{k\theta}^{(4)}}{B_{kR}^{(4)}} = i(-1)^{g+1}\left[\frac{\tilde{V}^{(2)}_{kr}(R)}{\tilde{V}^{(1)}_{kr}(R)}\right] \qquad (3.10a)$$

Thus the eigenfunctions $\tilde{V}_k^{(2)}$, $\tilde{V}_k^{(1)}$ define the Rayleigh wave polarization ellipse; the direction of rotation and ratio of the ellipse axes depends on the depth of burial of the receiver; the apparent change in the direction of rotation for a sphere for changes in g is associated with the fixed choice of direction in designating u_θ (+ for motion away from the epicenter).

Reciprocity principle. For simple sources of the concentrated force type, it is easy establish the principle of reciprocity--the invariance of spectral density Φ_{kq} for change in the placement of the source and receiver with preservation of the source orientation with respect to the receiver and vice versa. In fact for a simple vertical force at depth we have according to (1.26):

$$B_{kq}^{(3)} = \tilde{V}_k^{(1)}(p,h)\,S(p)$$

$$\Phi_{kz}^{\downarrow}(h,z) = \left[\prod_{i=0}^{2} B_{kz}^{(i)}\right] S(p)\,\tilde{V}_k^{(1)}(p,h)\,\tilde{V}_k^{(1)}(p,z) \qquad (3.11)$$

$$\phi^{\downarrow}_{kr}(h,z) = \left[\prod_{i=0}^{2} B^{(i)}_{kz}\right] S(p)\,\tilde{V}^{(1)}_{k}(p,h)(-1)\tilde{V}^{(2)}_{k}(p,z) .$$

For a simple horizontal force at depth h, oriented toward the receiver, according to (1.32) we have

$$B^{(3)}_{kq} = i\tilde{V}^{(2)}_{k}(p,h)\,S(p) ,$$

$$\phi^{\rightarrow}_{kz}(h,z) = \left[\prod_{i=0}^{2} B^{(i)}_{kz}\right] S(p)\,i\tilde{V}^{(2)}_{k}(p,h)\,\tilde{V}^{(1)}_{k}(p,z) , \qquad (3,12)$$

$$\phi^{\rightarrow}_{kr}(h,z) = \left[\prod_{i=0}^{2} B^{(i)}_{kz}\right] S(p)\,\tilde{V}^{(2)}_{k}(p,h)\,\tilde{V}^{(2)}_{k}(p,z) .$$

For a simple horizontal force at depth h perpendicular to the epicenter-receiver direction, from (1.32),

$$B^{(3)}_{kq} = -i\tilde{V}^{(3)}_{k}(p,h)\,S(p) ,$$

$$\phi^{\rightarrow}_{k\varphi}(h,z) = \left[\prod_{i=0}^{2} B^{(i)}_{k\varphi}\right] S(p)\,\tilde{V}^{(3)}_{k}(p,h)\,\tilde{V}^{(3)}_{k}(p,z) . \qquad (3,13)$$

From this the reciprocity relations follow:

$$\phi^{\downarrow}_{kz}(h,z) = \phi^{\downarrow}_{kz}(z,h)$$

$$\cancel{\phi^{\rightarrow}_{kr}(h,z) = \phi^{\rightarrow}_{kr}(z,h)}$$

$$\phi^{\rightarrow}_{k\varphi}(h,z) = \phi^{\rightarrow}_{k\varphi}(z,h) \qquad (3,14)$$

$$\cancel{\phi^{\downarrow}_{kr}(h,z) = -\phi^{\rightarrow}_{kz}(z,h)}$$

$$\cancel{\phi^{\rightarrow}_{kz}(h,z) = -\phi^{\downarrow}_{kr}(z,h)}$$

Similar relations for a sphere are not difficult to obtain.

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__Additional force systems.__ These forces have been normalized to represent unit forces of couples acting at the source. They are suitable for direct substitution in to (1.20); The resultant (1.20) agrees with the theoretical equations of Saito and Tsai and Aki (1970).

This differs from Section V on page 10 by factor $(2\pi)^{-1}$. $(2\pi)^{-1}$ needed for normalization of unit Force.

ua 1.7

(a) Arbitrarily directed force. (δ, β). The force is represented as

$$\vec{F} = \frac{\delta(z-h)\,\delta(r)}{2\pi r}\left[\cos(\delta-\phi)\sin\beta\,\vec{a}_r + \sin(\delta-\phi)\sin\beta\,\vec{a}_\phi + \cos\beta\,\vec{a}_z\right] s(t)$$

where δ is the azimuth of the force from north and β is the angle that the force makes with the positive z axis, which is taken to be downward. Note that $0 \le \delta \le 2\pi$ and $0 \le \beta \le \pi$. The $\hat{a}_q$ are the unit vectors in the q'th direction. $s(t)$ is the source time function. The U_{kQ} become

$$U_{kR} = \left[\cos\beta\, V_k^{(1)}(h,p) - i\sin\beta\cos(\delta-\phi)\, V_k^{(2)}(h,p)\right]\cdot S(p)/2\pi$$

$$U_{kL} = -i\sin\beta\sin(\delta-\phi)\, V_k^{(3)}(h,p)\cdot S(p)/2\pi$$

where $S(p) = \int_{-\infty}^{\infty} \exp(-ipt)\,s(t)\,dt$;

(b) Dipole without moment. (δ, β).

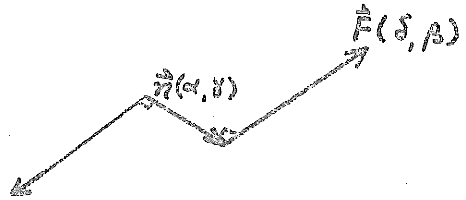
$$U_{kR} = \Big[\cos^2\beta\,\frac{dV_k^{(1)}}{dh}(h,p) - k_R\sin^2\beta\cos^2(\delta-\phi)\,V_k^{(2)}(h,p) + \frac{i}{2}\sin 2\beta\cos(\delta-\phi)\Big(k_R V_k^{(1)}(h,p) + \frac{dV_k^{(2)}}{dh}(h,p)\Big)\Big]\cdot S(p)/2\pi$$

$$U_{kL} = -i\sin\beta\sin(\delta-\phi)\left[\cos\beta\,\frac{dV_k^{(3)}}{dh}(h,p) + ik_L\cos(\delta-\phi)\sin\beta\,V_k^{(3)}(h,p)\right]\cdot S(p)/2\pi$$

(c) Couple $(\delta, \beta), (\alpha, \gamma)$. The positive force in the direction (δ, β) is converted into a couple by the application of an operator to (a):

$$\left[\cos\gamma\,\frac{d}{dh} + ik_Q\cos(\alpha-\phi)\sin\gamma\right],\quad Q = L \text{ or } R\ ;$$

graphically, the couple appears as



The U_{kQ} become

$$U_{kR} = \Big\{ \cos\gamma \cos\beta \frac{dV_k^{(1)}}{dh}(h,p) - k_R \sin\beta \sin\gamma \cos(\delta - \phi)\cos(\alpha - \phi) V_k^{(2)}(h,p)$$
$$+ ik_R \sin\gamma \cos\beta \cos(\alpha - \phi) V_k^{(1)}(h,p)$$
$$+ i\cos\gamma \sin\beta \cos(\delta - \phi) \frac{dV_k^{(2)}}{dh}(h,p) \Big\} \cdot \frac{S(p)}{2\pi}$$

$$U_{kL} = -i\sin\beta \sin(\delta - \phi) \Big[\cos\gamma \frac{dV_k^{(3)}}{dh}(h,p)$$
$$+ ik_L \sin\gamma \cos(\alpha - \phi) V_k^{(3)}(h,p) \Big] \cdot S(p)/2\pi$$

where (δ , β) and (α , γ) are related by the condition of perpendicularity

$$\operatorname{ctg}\gamma \operatorname{ctg}\beta = -\cos(\delta - \alpha).$$

(d) Two perpendicular dipoles without moment of equal magnitude and opposite sign (δ , β), (α , γ). For this set of dipoles, the angles define the directions of the tensional T and compressional P axes by

$$(\delta, \beta) = T \qquad (\alpha, \gamma) = P.$$

$$U_{kR} = \Big\{ (\cos^2\beta - \cos^2\gamma) \frac{dV_k^{(1)}}{dh}(h,p)$$
$$- k_R \big[\sin^2\beta \cos^2(\delta - \phi) - \sin^2\gamma \cos^2(\alpha - \phi) \big] V_k^{(2)}(h,p)$$
$$+ \frac{i}{2} \cdot \big[\sin 2\beta \cos(\delta - \phi) - \sin 2\gamma \cos(\alpha - \phi) \big] \cdot$$
$$\cdot \Big[k_R V_k^{(1)}(h,p) + \frac{dV_k^{(2)}}{dh}(h,p) \Big] \Big\} \frac{S(p)}{2\pi}$$

$$U_{kL} = \Big\{ -\frac{i}{2} \big[\sin 2\beta \sin(\delta - \phi) - \sin 2\gamma \sin(\alpha - \phi) \big] \frac{dV_k^{(3)}}{dh}(h,p)$$
$$+ \frac{i}{2} k_L V_k^{(3)}(h,p) \big[\sin^2\beta \sin(2\delta - 2\phi) - \sin^2\gamma \sin(2\alpha - 2\phi) \big] \Big\} \frac{S(p)}{2\pi}$$

with the necessary perpendicularity relation

$$\operatorname{ctg}\gamma \operatorname{ctg}\beta = -\cos(\delta - \alpha).$$

(e) Double couple without moment (δ , β), (α , γ). The angles are defined as in (c).

$$U_{kR} = \Big\{ 2\cos\gamma \cos\beta \frac{dV_k^{(1)}}{dh}(h,p) - 2k_R V_k^{(2)} \sin\beta \sin\gamma \cos(\delta - \phi)\cos(\alpha - \phi)$$
$$+ i\Big[k_R V_k^{(1)}(h,p) + \frac{dV_k^{(2)}}{dh}(h,p) \Big] \cdot$$
$$\cdot \big[\sin\gamma \cos\beta \cos(\alpha - \phi) + \sin\beta \cos\gamma \cos(\delta - \phi) \Big\} \cdot S(p)/2\pi$$

$$U_{kL} = \Big\{ -i \left[\sin\beta \sin(\delta - \phi)\cos\gamma + \sin\gamma \cos\beta \sin(\alpha - \phi)\right] \frac{dV_k^{(3)}}{dh}(h,p)$$

$$+ k_L V_k^{(1)}(h,p)\sin\beta \sin\delta \sin(\alpha + \delta - 2\phi)\Big\} \; S(p)/2\pi$$

and

$$\operatorname{ctg}\gamma \operatorname{ctg}\beta = -\cos(\delta - \alpha)$$

(f) Explosive source.

$$U_{kR} = \left[\frac{dV_k^{(1)}}{dh}(h,p) - k_R V_k^{(2)}(h,p)\right] S(p)/2\pi$$

$$U_{kL} = 0.$$

(g) Double couple mechanism in terms of strike, dip, and slip of motion on the fault plane. The orthogonality conditions on the couples or dipoles of (d) or (e) imply that only three quantities are required to define motion on the fault plane. These quantities are the fault plane dip d, the strike θ, and the slip s. These parameters are shown in Figure A.1. The dip is measured from the horizontal in a downward direction and vaires between 0° and 90°. The strike is measured clockwise from north and varies from 0° through 360°. The slip angle gives the direction of motion of the hanging wall with respect to the foot wall; the slip angle varies from 0° to 360° and is measured counterclockwise from the strike.

With this convention, the U_{kQ} become

$$U_{kR} = \frac{S(p)}{2\pi}\Big\{\sin s \sin 2d \left[\frac{dV_k^{(1)}}{dh}(h,p) + \tfrac{1}{2}\xi_{kR}V_k^{(2)}(h,p)\right]$$

$$+ \xi_{kR}V_k^{(1)}(h,p)\left[-\cos s \sin d \sin 2(\phi-\theta) - \tfrac{1}{2}\sin s \sin 2d \cos 2(\phi-\theta)\right]$$

$$+ i\left[\xi_{kR}V_k^{(1)}(h,p) + \frac{dV^{(2)}}{dh}(h,p)\right]\cdot$$

$$\left[-\cos d \cos s \cos(\phi-\theta) + \cos 2d \sin s \sin(\phi-\theta)\right]\Big\}$$

$$U_{kL} = \frac{S(p)}{2\pi}\Big\{- i\left[\sin s \cos 2d \cos(\phi-\theta) + \cos s \cos d \sin(\phi-\theta)\right]\frac{dV_k^{(3)}}{dh}(h,p)$$

$$+ \xi_{kL}V_k^{(3)}(h,p)\left[\cos s \sin d \cos 2(\phi-\theta) - \tfrac{1}{2}\sin s \sin 2d \sin 2(\phi-\theta)\right]\Big\}.$$

A reverse fault striking north and dipping 45° to the east is given by $\theta = 0°$, $d = 45°$, and $s = 90°$. A right lateral vertical strike slip fault which strikes north is given by $\theta = 0°$, $d = 90°$, and $s = 180°$.

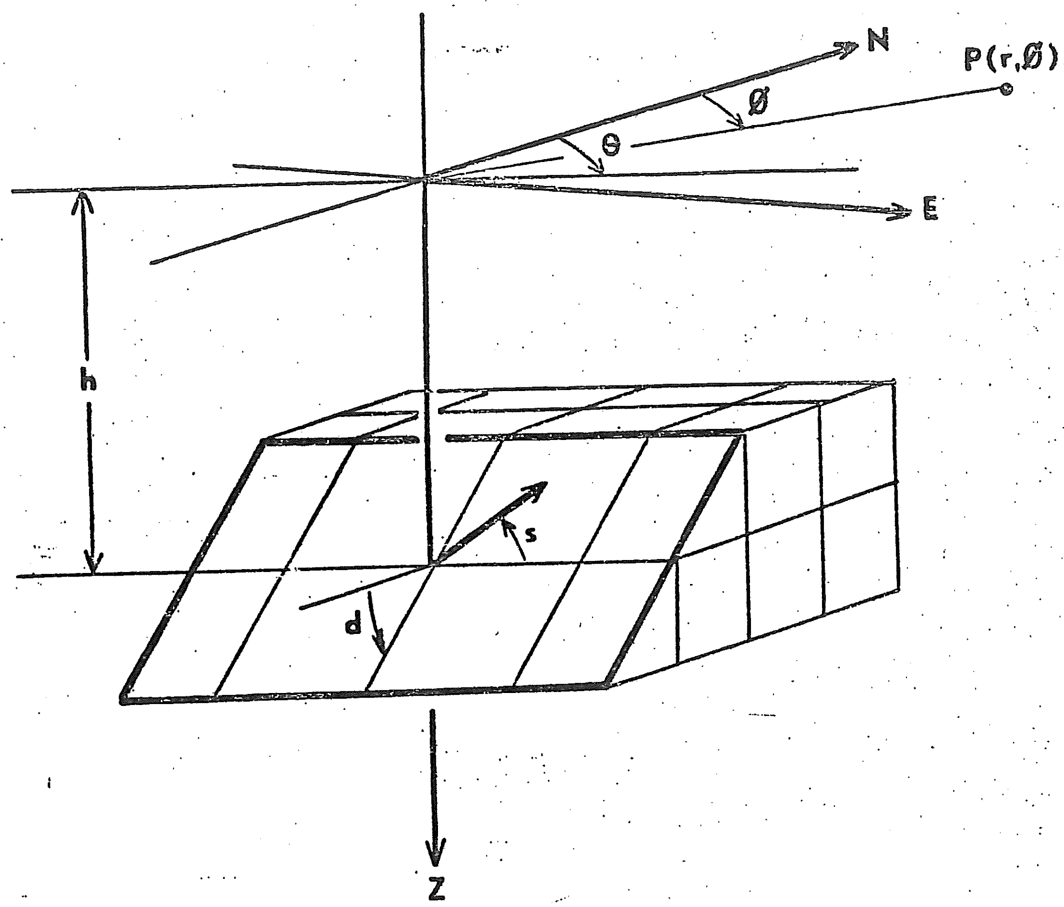


Fig. A.1. Coordinate and fault plane geometry