Theoretical Seismology

Earth Science 247

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Freeman Gilbert
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1. Review of Continuum Mechanics

This review is abstracted from lecture notes prepared by Professor George E. Backus.

The Eulerian form of the equation for the conservation of mass is

(1.1) $$\partial\rho/\partial t + \underline{\nabla}\cdot(\rho\underline{v}) = 0$$

where ρ is density and $\underline{v}$ is "particle" velocity. In subscript notation (1.1) becomes

(1.2) $$\rho_{,t} + (\rho v_i)_{,i} = 0$$

The equation for the conservation of linear momentum is

(1.3) $$\rho\, d\underline{v}/dt = \underline{f} + \underline{\nabla}\cdot\underline{\underline{T}}$$

where $\underline{f}$ is the body force per unit volume and $\underline{\underline{T}}$ is the stress dyadic. Alternative forms of (1.3) are

$$\partial(\rho\underline{v})/\partial t + \underline{\nabla}\cdot(\rho\underline{v}\underline{v} - \underline{\underline{T}}) = \underline{f}$$

(1.4) $$\rho\,\dot{v}_i = f_i + T_{ij,j}$$

$$(\rho v_i)_{,t} + (\rho v_i v_j - T_{ij})_{,j} = f_i$$

1.4) cont. $(\rho v_i)_{,t} + (\rho v_i v_j - T_{ij})_{,j} = f_i$

The equation for the conservation of angular momentum is

1.5) $\underline{\underline{T}} = \underline{\underline{T}}^T$ or $T_{ij} = T_{ji}$

under the almost universal assumption that the angular momentum density, torque density, and torque stress tensor all vanish.

The equation for the conservation of internal energy is

1.6) $\rho\, dU/dt + \underline{\nabla} \cdot \underline{H} = \underline{\underline{T}} : (\underline{\nabla}\, \underline{v}) + h$

where U is the internal energy per unit mass, h is the rate of appearance of U per unit volume, and $\underline{H}$ is the heat flux. Alternative forms of (1.6) are

$\rho \dot{U} + H_{i,i} = T_{ij} v_{i,j} + h$

1.7) $\partial(\rho U)/\partial t + \underline{\nabla} \cdot (\rho U \underline{v} + \underline{H}) = \underline{\underline{T}} : (\underline{\nabla}\, \underline{v}) + h$

$(\rho U)_{,t} + (\rho U v_i + H_i)_{,i} = T_{ij} v_{i,j} + h$

The equation for the conservation

of total energy, E, where

$$E = U + \underline{v}\cdot\underline{v}/2$$

can be written in the form

$$1.8) \quad \partial(\rho E)/\partial t + \underline{\nabla}\cdot(\rho E\underline{v} + \underline{H} - \underline{v}\cdot\underline{\underline{T}}) = h + \underline{v}\cdot\underline{f}$$

or

$$1.9) \quad (\rho E)_{,t} + (\rho E v_j + H_j - v_i T_{ij})_{,j} = h + v_i f_i$$

There are many situations in continuum mechanics where the body force, $\underline{f}$, is derivable from a potential

$$1.10) \quad \underline{f} = -\rho\underline{\nabla}\psi$$

and $\partial\psi/\partial t = 0$. Then the mechanical potential energy is defined

$$1.11) \quad \Psi = \int_V dv\, \rho\psi$$

If we define $E = U + \underline{v}\cdot\underline{v}/2 + \Psi$ then the equation for the conservation of $\mathcal{E}$ is

1.12) $\partial(\rho \mathcal{E})/\partial t + \nabla\cdot(\rho \mathcal{E} \underline{v} + \underline{H} - \underline{v}\cdot \underline{\underline{T}}) = h$

or

1.13) $(\rho \mathcal{E})_{,t} + (\rho \mathcal{E} v_j + H_j - v_i T_{ij})_{,j} = h$

All of the conservation equations have the form

1.14) $\partial \phi/\partial t + \nabla\cdot \underline{K} = k$

where ϕ is the conserved quantity. We say that (1.14) is the equation for the conservation of phi; k is the rate of production of phi per unit time per unit volume; $\underline{K}$ is the current density of phi per unit time per unit area and is often called the phi flux vector; and $\underline{v} = \underline{K}/\phi$ is the instantaneous velocity of phi transport. The conservation equations are summarized in Table 1.1

$\underline{V}$	ϕ	$\underline{K}$	k	1.14)
$\underline{v}$	ρ	$\rho\underline{v}$	0	(mass) 1.1)
$\underline{v} - \underline{v}\cdot\underline{\underline{T}}/\rho v^2$	$\rho\underline{v}$	$\rho\underline{v}\underline{v} - \underline{\underline{T}}$	$\underline{f}$	(momentum) 1.4)
$\underline{v} + H/\rho U$	ρU	$\rho U\underline{v} + H$	$\underline{\underline{T}}:(\underline{\nabla}\,\underline{v}) + h$	1.7) (internal energy)
$\underline{v} + H/\rho E - \underline{v}\cdot\underline{\underline{T}}/\rho E$	ρE	$\rho E\underline{v} + H - \underline{v}\cdot\underline{\underline{T}}$	$\underline{v}\cdot\underline{f} + h$	1.8) (Total energy, non cons. forces)
$\underline{v} + H/\rho\mathcal{E} - \underline{v}\cdot\underline{\underline{T}}/\rho\mathcal{E}$	$\rho\mathcal{E}$	$\rho\mathcal{E}\underline{v} + H - \underline{v}\cdot\underline{\underline{T}}$	h	1.19) (Total energy conservative forces)

Table 1.1 Summary of Conservation Equations

The conservation equations summarized in Table 1.1 are valid inside a continuum where the indicated partial derivatives exist. If some of the variables are discontinuous we must supplement the conservation equations with boundary (interface) conditions.

Suppose the boundary between two continua is given by the equation

$$\psi(\underline{r}, t) = 0 \tag{1.15}$$

Then, on the boundary,

$$d\psi/dt = \partial\psi/\partial t + (d\underline{r}/dt)\cdot\underline{\nabla}\psi = 0 \tag{1.16}$$

Now $d\underline{r}/dt$ is the velocity $\underline{v}$. If $\underline{v}^{(1)}$ and $\underline{v}^{(2)}$ are the velocities in the first and second continua, respectively, we have from (1.16)

$$\partial\psi/\partial t + \underline{v}^{(1)}\cdot\underline{\nabla}\psi = \partial\psi/\partial t + \underline{v}^{(2)}\cdot\underline{\nabla}\psi = 0 \tag{1.17}$$

on the boundary. The vector $\underline{\nabla}\psi$ is normal to the boundary. Therefore it follows from (1.17) that

$$\underline{v}^{(1)}\cdot\hat{n} = \underline{v}^{(2)}\cdot\hat{n} \tag{1.18}$$

on the boundary, where $\hat{n}$ is the unit normal to the boundary.

Thus, on the boundary between two continua, that always remain in contact without interpenetration, there must be continuity of the normal component of velocity. If the contact between the two continua is "welded" so that there is no slippage, there must be continuity of the tangential components of velocity. These boundary conditions are called kinematical conditions:

1.19) $\underline{v}^{(1)} \cdot \hat{n} = \underline{v}^{(2)} \cdot \hat{n}$ on a simple boundary

$\underline{v}^{(1)} = \underline{v}^{(2)}$ on a simple welded boundary

To get the dynamical boundary conditions we integrate (1.4a) over a volume cleft by the boundary and use Gauss' theorem to transform the volume integral of $\underline{\nabla} \cdot \underline{\underline{T}}$ to a surface integral

$$1.20) \quad \int_V dv \left[\frac{\partial}{\partial t} (\rho \underline{v}) + \underline{\nabla} \cdot (\rho \underline{v}\underline{v}) - \underline{f} \right] = \int_A dA \; \hat{n} \cdot \underline{\underline{T}}$$

We assume that the bracketed term in (1.20) is bounded. Keeping the area of the boundary enclosed by the volume fixed, we collapse the volume integral to get

1.21) $\int_A dA \left(\underline{\underline{T}}^{(1)} \cdot \hat{n}^{(1)} + \underline{\underline{T}}^{(2)} \cdot \hat{n}^{(2)} \right) = 0$

Since $\hat{n}^{(1)}$ and $\hat{n}^{(2)}$ point in opposite directions

1.22) $\int_A dA \left(\underline{\underline{T}}^{(1)} - \underline{\underline{T}}^{(2)} \right) \cdot \hat{n} = 0$

and since (1.22) is true for any area on the boundary

1.23) $\hat{n} \cdot \underline{\underline{T}}^{(1)} = \hat{n} \cdot \underline{\underline{T}}^{(2)}$

Thus the dynamical boundary condition requires continuity of $\hat{n} \cdot \underline{\underline{T}}$ across the boundary. Notice that the whole stress tensor need not be continuous.

The tensor $\partial s_i/\partial x_j$ is called the distortion tensor. s_i is the displacement vector and x_j is the Lagrangian coordinate vector. Suppose that initially two "particles" in a continuum are separated by a vector distance ξ_i. Following a small distortion, $|\partial s_i/\partial x_j| << 1$, the two particles will be separated by a vector distance Ξ_i and

1.24 $$\Xi_i = (\delta_{ij} + \partial s_i/\partial x_j)\,\xi_j + \text{higher order terms}$$

When $|\partial s_i/\partial x_j| << 1$ we neglect the higher order terms in 1.24. If we write

1.25 $$\sigma_{ij} = \tfrac{1}{2}(\partial s_i/\partial x_j + \partial s_j/\partial x_i)$$

$$\alpha_{ij} = \tfrac{1}{2}(\partial s_i/\partial x_j - \partial s_j/\partial x_i)$$

then we can write 1.24 in the form

1.26 $$\Xi_i = (\delta_{ij} + \sigma_{ij})(\delta_{jk} + \alpha_{jk})\,\xi_k$$

correct to first order. The interpretation of 1.26 is classical. A small sphere around material point $\underline{x}$ has undergone a displacement $\underline{s}$, a rotation $\tfrac{1}{2}\nabla\times\underline{s}$ (that is a rotation through the angle $|\tfrac{1}{2}\nabla\times\underline{s}|$ about the axis $\tfrac{1}{2}\nabla\times\underline{s}$), and a stretching $\delta_{ij} + \sigma_{ij}$. The eigenvectors $\underline{\sigma}_k$

of the matrix pertor σ_{ij} are called the principal directions of strain and their eigen values $\sigma^{(k)}$ are called the principal strains.

Consider a continuum whose states of thermostatic equilibrium are completely specified by a knowledge of its configuration and its absolute temperature θ. Consider a state in which the stress tensor is isotropic

1.27 $$T_{ij}^{\circ} = - p^{\circ} \delta_{ij} \quad , \quad \theta = \theta^{\circ}$$

The continuum is carried into a nearby equilibrium state by a slight deformation and by a small change $\Delta\theta$ in θ. A particle initially at $\underline{x}$ is moved to

1.28 $$\underline{r}(\underline{x}) = \underline{x} + \underline{s}(\underline{x})$$

where $|\partial s_i / \partial x_j| << 1$. Suppose that $\partial s_i / \partial x_j$ is independent of $\underline{x}$. Using 1.25 we write

1.29 $$\partial s_i / \partial x_j = \sigma_{ij} + \alpha_{ij}$$

The rotation of the material given by α_{ij} will produce no changes, but the temperature change, $\Delta\theta$, and the strain, σ_{ij}, may produce changes. The stress may alter by an amount ΔT_{ij} and the entropy, S, per unit mass may change by an amount

(No confusion should arise from using S for entropy and $\underline{S}$ or S_i for displacement). If we write ΔT_{ij} as a Taylor series in the small quantities σ_{ij} and $\Delta\Theta$ we get

1.30 $$\Delta T_{ij} = \underset{\uparrow \text{isothermal}}{C_{ijkl}}\, \sigma_{kl} + W_{ij}\, \Delta\Theta + \text{higher order terms.}$$ Taylor series

If we know σ_{ij} and S we can solve $S(\sigma_{ij}, \Theta) = S$ for Θ so we can write another series Eqn. of state

1.31 $$\Delta T_{ij} = \underset{\uparrow \text{adiabatic}}{\tilde{C}_{ijkl}}\, \sigma_{kl} + \tilde{W}_{ij}\, \Delta S + \text{higher order terms}$$

In 1.30 the C_{ijkl} are called the isothermal ($\Delta\Theta = 0$) elastic coefficients and in 1.31 the $\tilde{C}_{ijkl}$ are called the isentropic or adiabatic ($\Delta S = 0$) elastic coefficients. Clearly there are at most 81 of the C_{ijkl} and 9 of the W_{ij}.

When $\sigma_{kl} = 0$ we have $W_{ij} = W_{ji}$ since the stress tensor is symmetric as a consequence of the equations for the conservation of angular momentum 1.5. Thus only 6 of the W_{ij} are independent. Since σ_{kl} is symmetric (see 1.25) we see that $C_{ijkl} = C_{ijlk}$. For a fixed kl and $\Delta\Theta = 0$ in 1.30 we have that $C_{ijkl} = C_{jikl}$. Thus only 36 of the C_{ijkl} are independent

When a system in thermostatic equilibrium experiences a small reversible change to another equilibrium state

1.32 $$\Theta dS = dU - \Delta W$$

where dS is the entropy change, dU is the internal energy change and ΔW is the work done on the system by external forces. The heat added to the system to produce the change is ΘdS

If, during the change, the centre of mass of the system is accelerated by some of the external work so that the kinetic energy is increased by ΔT, then 1.32 is

1.33 $$\Theta dS = dU - \Delta W + \Delta T$$

In the absence of body forces and in the presence of a constant stress T_{ij} causing a strain $\Delta \sigma_{ij}$ the work done in time Δt is

1.33½ $$\Delta W = \Delta t \int_A dA \; v_i T_{ij} n_j$$

Using Gauss's theorem we can write 1.33½ as

1.34 $$\Delta W = \int_V dv \, (\Delta s_i T_{ij})_{,j}$$

Similarly $\rightarrow$

$$\frac{dT}{dt} = \frac{1}{2}\int_V \rho \frac{d}{dt}(v_j v_j)\, dv = \int_V \rho v_j \frac{dv_j}{dt}\, dv \; ; \; \rho \frac{dv_j}{dt} = T_{kj,k}$$

$$v_j = ds_j/\cancel{dt}$$

1.35 $$\Delta T = \int_V dv\, \Delta s_i\, T_{ij,j}$$

Thus

1.36 $$\Delta W - \Delta T = \int_V dv\, T_{ij}\, \Delta\sigma_{ij}$$

For a unit mass of material we write 1.36 as

1.37 $$\Delta W - \Delta T = \tau\, T_{ij}\, \Delta\sigma_{ij} \quad \text{per unit mass}$$

where $\tau = 1/\rho$ is the specific volume.

Substituting 1.37 into 1.33 we get the second law of thermostatics in the form

1.38 $$\Theta\, ds = dU - \tau\, T_{ij}\, \Delta\sigma_{ij} \quad \text{per unit mass}$$

We rewrite 1.38 as

1.39 $$dU = \Theta\, ds + \tau\, T_{ij}\, \Delta\sigma_{ij}$$

Since U is a function of s and $\underline{\sigma}$: Θ and $\underline{\sigma}$ 1.37 implies

1.40 $$\left(\frac{\partial U}{\partial \sigma_{ij}}\right)_s = \tau\, T_{ij} \; ; \; \left(\frac{\partial U}{\partial s}\right)_\sigma = \Theta$$

Introducing the free energy, $F = U - \Theta S$,

1.41 $dF = -S d\Theta + \tau T_{ij} \Delta\sigma_{ij}$

we get

1.42 $$\left(\frac{\partial F}{\partial \Theta}\right)_\sigma = -S \; ; \; \left(\frac{\partial F}{\partial \sigma_{ij}}\right)_\Theta = \tau T_{ij}$$

For a change keeping Θ fixed

1.43 $T_{ij} = -p^0 \delta_{ij} + C_{ijkl}\,\sigma_{kl}$ (see 1.30 & 1.27)

$T_{ij} = -p^0 \delta_{ij} + \Delta T_{ij}$

Therefore

1.44 $$\left(\frac{\partial T_{ij}}{\partial \sigma_{kl}}\right)_\Theta = C_{ijkl}$$

From the second of 1.42 ~~and~~ from 1.44 we get

1.45 $$\left(\frac{\partial T_{ij}}{\partial \sigma_{kl}}\right)_\Theta = \frac{1}{\tau}\left(\frac{\partial^2 F}{\partial \sigma_{kl}\,\partial \sigma_{ij}}\right)_\Theta - \frac{1}{\tau^2}\left(\frac{\partial F}{\partial \sigma_{ij}}\right)_\Theta \left(\frac{\partial \tau}{\partial \sigma_{kl}}\right)_\Theta$$ trace?

Now $\partial\tau/\partial\sigma_{kl} = \tau\delta_{kl}$ since $\Delta\tau/\tau = |\sigma_{ij}|$. Thus

1.46 $$\left(\frac{\partial T_{ij}}{\partial \sigma_{kl}}\right)_\Theta = \frac{1}{\tau}\left(\frac{\partial^2 F}{\partial \sigma_{kl}\,\partial \sigma_{ij}}\right)_\Theta - T_{ij}\,\delta_{kl} = C_{ijkl}$$

In the initial state $T_{ij} = -p^0 \delta_{ij}$ so 1.46 is

1.47 $\quad C_{ijkl} = P^0 \delta_{ij} \delta_{kl} + \frac{1}{2}\left(\frac{\partial^2 F}{\partial \sigma_{kl} \partial \sigma_{ij}}\right)_{\Theta}$

From 1.47 we see that

1.48 $\quad C_{ijkl} = C_{klij}$

Only 21 of the C_{ijkl} are independent. No further reduction is possible in general.
The elastic coefficients are the elements of a fourth order tensor. An isotropic material is one whose C_{ijkl} is an isotropic fourth order tensor. The most general 4th order isotropic tensor is

1.49 $\quad C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \nu \delta_{il} \delta_{jk}$

(Jeffreys, 1957, p. 66)

Harold Jeffreys, 1957, Cartesian Tensors: Cambridge University Press.

Since $C_{ijkl} = C_{jikl}$ for the elastic coefficient tensor we have from 1.49

1.50 $\quad (\mu - \nu)(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) = 0$

For $i = k = 1$, $j = l = 2$ in 1.50 we see that $\mu = \nu$

Thus 1.49 becomes

1.51 $$C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

which is the most general form taken by the elastic coefficient tensor in an isotropic material. The scalars λ and μ are called Lamé parameters.

In an isotropic material W_{ij} is written $W_{ij} = \kappa\,\delta_{ij}$.

The relation between the isothermal and isentropic coefficients is easy to derive. Since S is a function of σ_{ij} and θ

1.52 $$dS = \left(\frac{\partial S}{\partial \theta}\right)_{\sigma} d\theta + \left(\frac{\partial S}{\partial \sigma_{ij}}\right)_{\theta} d\sigma_{ij}$$

From 1.38 we derive

1.53 $$\left(\frac{\partial S}{\partial \theta}\right)_{\sigma} = \frac{1}{\theta}\left(\frac{\partial U}{\partial \theta}\right)_{\sigma} = \frac{C_{\sigma}}{\theta}$$

where C_σ is called the specific heat at constant strain. From 1.42

1.54 $$S = -\left(\frac{\partial F}{\partial \theta}\right)_{\sigma}$$

Thus

1.55 $$\left(\frac{\partial S}{\partial \sigma_{ij}}\right)_\theta = -\left(\frac{\partial}{\partial \theta}\right)_\sigma \left(\frac{\partial}{\partial \sigma_{ij}}\right)_\theta F = -\tau \left(\frac{\partial T_{ij}}{\partial \theta}\right)_\sigma$$

From 1.30 $(\partial T_{ij}/\partial \theta)_\sigma = W_{ij}$ so

1.56 $$\left(\frac{\partial S}{\partial \sigma_{ij}}\right)_\theta = -\tau W_{ij}$$

Using 1.53 and 1.56 in 1.52

1.57 $$dS = \frac{C_\sigma}{\theta} d\theta - \tau W_{ij} d\sigma_{ij}$$

Using 1.57 in 1.30

1.58 $$dT_{ij} = \left(C_{ijkl} + \frac{\theta \tau}{C_\sigma} W_{ij} W_{kl}\right) d\sigma_{kl} + \frac{\theta}{C_\sigma} W_{ij}\, dS$$

Comparing 1.58 with 1.31

1.31 $$dT_{ij} = \tilde{C}_{ijkl}\, d\sigma_{kl} + \tilde{W}_{ij}\, dS$$

we get

1.59 $$\tilde{C}_{ijkl} = C_{ijkl} + \frac{\theta \tau}{C_\sigma} W_{ij} W_{kl} = C_{ijkl} + \frac{\theta}{\rho C_\sigma} W_{ij} W_{kl}$$

$$\tilde{W}_{ij} = \frac{\theta}{C_\sigma} W_{ij}$$

In an isotropic material

$$\tilde{\lambda} = \lambda + \frac{\Theta}{\rho c_\sigma} \kappa^2$$

1.60 $\tilde{\mu} = \mu$

$$\tilde{\kappa} = \frac{\Theta}{c_\sigma} \kappa$$

When the initial state is uniformly hydrostatic

$$T_{ij} = -p^0 \delta_{ij} + \Delta T_{ij} \quad , \quad \nabla p^0 = 0$$

and there are no body forces, $f_i = 0$, then the linearized momentum equation (see 1.3) is

1.61 $$\rho \frac{\partial^2 s_i}{\partial t^2} = \Delta T_{ij,j}$$

For very slow motions we approximate 1.30 by setting $d\Theta = 0$. For very rapid motions we approximate 1.31 by setting $dS = 0$. The heat flow equation

1.62 $$\frac{\partial \Theta}{\partial t} = \nabla \cdot (\overset{\text{diffusivity}}{\kappa} \nabla \Theta)$$

leads to the criterion that the motion be regarded as rapid when $L^2/T \gg \kappa$ where L is an appropriate length scale and T is an appropriate time scale. This criterion is met in seismic again so we take $dS=0$ as a reasonable approximation to 1.81.

We have

1.63
$$2\Delta T_{ij} = \tilde{C}_{ijkl}\left(s_{k,l} + s_{l,k}\right) \quad \text{and}$$

$$\tilde{C}_{ijkl} = \tilde{\lambda}\,\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$ for an isotropic material when $dS=0$. Using 1.63 in 1.61 the linearized momentum equation becomes

1.64
$$\rho\frac{\partial^2 \underline{s}}{\partial t^2} = (\tilde{\lambda}+\mu)\nabla(\nabla\cdot\underline{s}) + \mu\nabla^2\underline{s}$$

From the identity $\nabla^2\underline{F} = \nabla\nabla\cdot\underline{F} - \nabla\times\nabla\times\underline{F}$ on $\underline{s}$ in 1.64 gives

1.65
$$\rho\frac{\partial^2 \underline{s}}{\partial t^2} = (\tilde{\lambda}+2\mu)\nabla\nabla\cdot\underline{s} - \mu\nabla\times\nabla\times\underline{s}$$

Equation 1.65 is called the elastic wave equation. We shall write λ for $\tilde{\lambda}$ from now on.

2. General Solutions to the Elastic Wave Equation

Equation 1.65 has the form of a wave equation. Let us see what kinds of simple waves can be solutions to 1.65. For no body forces we write 1.65 in the form

2.1 $$\rho s_{i,tt} = (\lambda+\mu) s_{j,ji} + \mu s_{i,jj}$$

Consider a plane pulse of displacement propagating in the (1) direction with speed c,

2.2 $$s_i = a_i S(t - x_1/c)$$

where the a_i are unit vectors. Substituting 2.2 into 2.1 gives

2.3 $$\rho a_i S''(t-x_1/c) = (\lambda+\mu) \frac{S''(t-x_1/c)}{c^2} a_1 \delta_{i1} + \mu \frac{S''(t-x_1/c)}{c^2} a_i$$

For $i=1$ in 2.3 we find $c^2 = (\lambda+2\mu)/\rho$. For $i = 2$ or 3 $c^2 = \mu/\rho$. We conclude that a plane pulse of displacement travels with speed $[(\lambda+2\mu)/\rho]^{1/2}$ if the direction of motion is parallel to the direction of propagation. It travels with a speed $(\mu/\rho)^{1/2}$ if the direction of motion is normal to the direction of propagation. (For an elastic material to be stable we must have $(\lambda+2\mu)>0$ and $\mu>0$ (Bullen, 1960). Thus in a stable material $\lambda+2\mu > \mu$.) Pulses waves of the former type are called longitudinal waves and the latter type are called transverse waves.

To obtain a little more information we set $\underline{s} = \underline{s}_l + \underline{s}_t$ where

2.4 $$\nabla \times \underline{s}_l = 0 \; , \; \nabla \cdot \underline{s}_t = 0$$

Substituting 2.4 into 1.65

2.5 $$(\lambda + 2\mu) \nabla^2 \underline{s}_l = \rho \, \partial^2 \underline{s}_l / \partial t^2$$
$$\mu \nabla^2 \underline{s}_t = \rho \, \partial^2 \underline{s}_t / \partial t^2$$

The two wave equations 2.5 show that curl-free waves (waves of compression and rarefaction) travel with a speed $[(\lambda + 2\mu)/\rho]^{1/2}$ the same as longitudinal plane waves & Divergence-free waves (equivoluminal waves) travel with a speed $(\mu/\rho)^{1/2}$ the same as transverse plane waves. The former travel faster than the latter. The faster waves are called P waves (P = primary, or push or pull); the slower waves are called S waves (S = secondary, or shake, or shear)

From 2.4 we see that another useful representation of $\underline{s}$ is

2.6 $$\underline{s} = \nabla \phi + \nabla \times \underline{A}$$

and it is sufficient that

$$(\lambda+2\mu)\nabla^2\varphi = \rho\,\partial^2\phi/\partial t^2$$

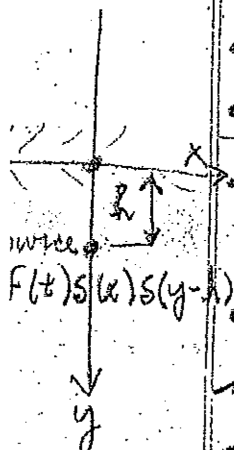
2.7

$$\mu\nabla^2\underline{A} = \rho\,\partial^2\underline{A}/\partial t^2$$

We shall return to 2.6 and 2.7 later.

3. Reflection and Refraction of an SH pulse in two Dimensions

In cartesian coordinates (x, y, z) the plane $y=0$ separates two different elastic solids. Parallel to this plane there is a line of uniform force density acting in the z direction. The only component of motion is s_z which will be denoted by $u(x,y)$. s_z is not a function of z because of the uniformity of the line force. Any vector with only a z component that is not a function of z (other than a constant) is a divergence-free vector. Therefore the linearized momentum equation (eq. 3) becomes for an isotropic material



3.1 $$\mu \nabla^2 u(x,y) - \rho\, \partial^2 u(x,y)/\partial t^2 = -f$$

where f stands for f_z and the time argument of u is suppressed. Imagine the plane $y=0$ to be horizontal. Then s_z is horizontal and is an S pulse from 3.1. A horizontally polarized S pulse is called an SH pulse. For a line source $f = f(t)\,\delta(x)\,\delta(y-h)$, the source is concentrated at $x=0$, $y=h$. Then $f(t)$ is the force per unit length of the line source. We write 3.1 as

3.2 $$\mu \nabla^2 u(x,y,t) - \rho\, \partial^2 u(x,y,t)/\partial t^2 = -f(t)\,\delta(x)\,\delta(y-h)$$

We assume that $f(t) \equiv 0$ for $t < 0$, and that $u = \partial u/\partial t \equiv 0$ for $t \le 0$.

First we seek a solution to 3.2 when the continuum is homogeneous and of infinite extent. We introduce the one-sided (unilateral) Laplace transform with respect to t for some function $g(t)$

3.3 $$\bar{g}(s) = \int_0^\infty g(t)\, e^{-st}\, dt$$

and keep s positive real. The Laplace transform of 3.2 is

3.4 $$\mu \nabla^2 \bar{u}(x,y,s) - \rho s^2 \bar{u}(x,y,s) = -\bar{f}(s)\,\delta(x)\,\delta(y-h)$$

If r is the distance from the line source, $r^2 = x^2 + (y-h)^2$, pulses that have traveled far will locally look like plane waves, $u \propto u(t - r/\beta)$ where $\beta = \sqrt{\mu/\rho}$. Then $\bar{u} \propto \bar{u}(s)\, e^{-sr/\beta}$. That is, far from the line source $\bar{u}$ decays exponentially for real s.

Next we introduce the two-sided (bilateral) Laplace transform with respect to x

3.5 $$G(\xi, s) = \int_{-\infty}^{\infty} \bar{g}(x,s)\, e^{-\xi x}\, dx$$

where ξ may be complex. Applying 3.5 to 3.4 and integrating by parts (the integrated parts vanish due to the exponential decay of $\bar{u}$) yields

3.6 $$\mu\left(\frac{d^2}{dy^2} U(\xi,y,s) - \nu^2 U(\xi,y,s)\right) = -\bar{f}(s)\,\delta(y-h)\ ; \quad \cancel{\beta \le \mathrm{Re}\,\xi \le \beta}\quad -k \le \mathrm{Re}\,\xi \le k$$

where $\nu^2 = k^2 - \xi^2$ and $k^2 = s^2/\beta^2$. Equation 3.6 is an ordinary differential equation with an inhomogeneous ~~term~~ term. Its solution can be obtained by standard methods. We are interested in the particular solution and continue via the Laplace transform method. We apply the bilateral Laplace transform with respect to y

3.7 $$\mathcal{G}(\xi, \zeta, s) = \int_{-\infty}^{\infty} G(\xi, y, s)\, e^{-\zeta y}\, dy$$

to 3.6 and integrate by parts (again the integrated parts vanish by what is left of the exponential decay)

3.8 $$\mu(\zeta^2 - \nu^2)\,\mathcal{U}(\xi, \zeta, s) = -\bar{f}(s)\, e^{-\zeta h} \quad ; \quad -|\mathrm{Re}\,\nu| \le \mathrm{Re}\,\zeta \le |\mathrm{Re}\,\nu|$$

Thus

3.9 $$\mathcal{U}(\xi, \zeta, s) = -\frac{\bar{f}(s)\, e^{-\zeta h}}{\mu(\zeta^2 - \nu^2)}$$

The inversion integral to 3.7 is

3.10 $$G(\xi, y, s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mathcal{G}(\xi, \zeta, s)\, e^{\zeta y}\, d\zeta$$

so that the inversion of 3.9 is

3.11 $$U(\xi, y, s) = -\frac{\bar{f}(s)}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{\zeta(y-h)}}{\mu(\zeta^2 - \nu^2)}\, d\zeta$$

In 3.11 the initial path of integration is such that $-|Re\,\nu| \doteq Re\,\xi \doteq |Re\,\nu|$. The integrand in 3.11 has poles at $\xi = \pm\nu$ but there is an ambiguity since the sign of $\nu = (k^2 - \xi^2)^{1/2}$ is not specified. Without loss of generality we take $Re\,\nu \geq 0$. The ξ plane appears as in Fig 3.1

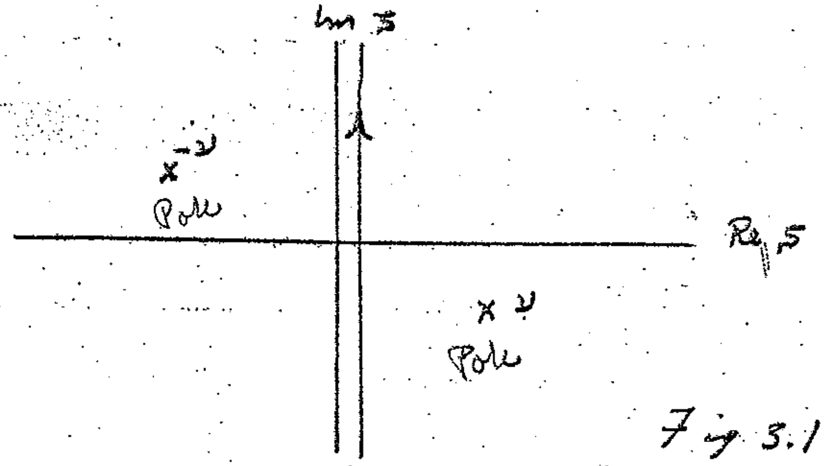


Fig 3.1

Suppose $y > h$. From 3.11 we see that $Re\,\xi < 0$ for the integral to converge. Therefore we distort the path of integration into the left half plane and pick up the pole at $\xi = -\nu$. The infinite arc gives no contribution since the integrand vanishes exponentially on it. The result is

3.12 $$U(\xi, y, s) = \frac{\bar{f}(s)}{2\mu\nu} e^{-\nu(y-h)} \qquad \text{for } y > h$$

For $y < h$ $Re\,\xi > 0$ for convergence in 3.11 so we distort the path into the right half plane to obtain

3.13 $$U(\xi, y, s) = \frac{\bar{f}(s)}{2\mu\nu} e^{\nu(y-h)} \qquad \text{for } y < h$$

We combine 3.12 and 3.13

3.14 $$U(\xi, y, s) = \frac{\bar{f}(s)}{2\mu\nu} e^{-\nu|y-h|} \qquad \text{for all } y$$

The next inversion is

3.15 $$\bar{u}(x,y,s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} U(\xi, y, s)\, e^{\xi x}\, d\xi$$

and the initial path of integration is such that $-k \le \mathrm{Re}\,\xi \le k$. Combining 3.14 and 3.15 we get

3.16 $$\bar{u}(x,y,s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\bar{f}(s)}{2\mu\nu}\, e^{\xi x - \nu|y-h|}\, d\xi \quad ; \; -k \le \mathrm{Re}\,\xi \le k$$

and we must remember that $\mathrm{Re}\,\nu \ge 0$. There are branch points of $\nu = (k^2 - \xi^2)^{1/2}$ in the ξ plane at $\xi = \pm k$. For $\mathrm{Re}\,\nu \ge 0$ over the entire ξ plane there must be branch cuts as shown in Fig 3.2

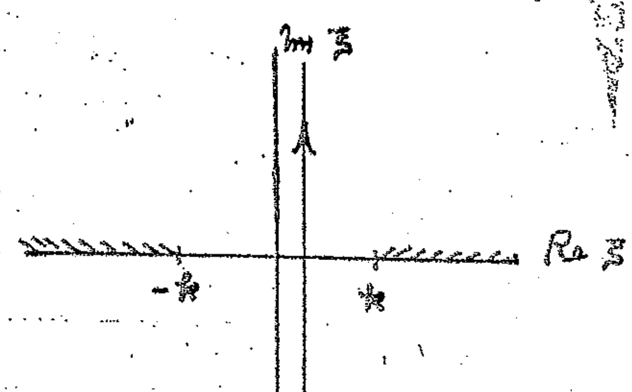


Fig 3.2

The integral in 3.16 is known and represents the function

3.17 $$\bar{u}(x,y,s) = \frac{\bar{f}(s)}{2\pi\mu} K_0(kr)$$

where $r^2 = x^2 + (y-h)^2$ and K_0 is the modified Bessel function of the second kind and zero order (Watson, 1952, p.172).

G. N. Watson, A treatise on the theory of Bessel functions: Cambridge University Press.

Equation 3.17 is the product of two Laplace transforms $\bar{f}(s)$ and $K_0(kr)$ ($k = s/\beta$). Therefore the function $u(x,y,t)$ must be the convolution of two time functions, $f(t)$ and the inverse of $K_0(kr)$. The inverse of $K_0(kr)$ is known to be (Watson, 1952, p17

3.18 $$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K_0(sr/\beta)\, e^{st}\, ds = H(t - r/\beta)\big/(t^2 - r^2/\beta^2)^{1/2}$$

Therefore

3.19 $$u(x,y,t) = \frac{1}{2\pi\mu}\int_{r/\beta}^{t} \frac{f(t-\tau)}{(\tau^2 - r^2/\beta^2)^{1/2}}\, d\tau \qquad t \geq r/\beta$$

and the singularity at $\tau = r/\beta$ is integrable. If f is a delta function of strength F, $f(t) = F\delta(t)$ 3.19 becomes

3.20 $$u(x,y,t) = \frac{F}{2\pi\mu}\,\frac{H(t - r/\beta)}{(t^2 - r^2/\beta^2)^{1/2}}$$

On the wavefront $t = r/\beta$ and 3.20 diverges. This is typical in two dimensional wave propagation problems.

A method whereby 3.20 can be derived without explicitly using the properties of the modified Bessel function is very useful in wave propagation problems. Such a method has been known for 60 years (Lamb, 1904)

H. Lamb, 1904, On the propagation of tremors over the surface of an elastic solid: Phil. Trans. Roy. Soc. London, A, 203, 1-42.

We follow the exposition of de Hoop (1960)

A. T. de Hoop, 1960, a modification of Cagniard's method for solving seismic pulse problems: Appl. Sci. Res., B, 8, 349-356.

Let us return to 3.16 with $\bar{f}(s) = F$, a constant

3.21 $$\bar{u}(x,y,s) = \frac{F}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi x - \nu|y-h|} \frac{d\xi}{2\mu\nu} \quad ; \; -k \leq Re\,\xi \leq k$$

Let $\xi = sp$, $\nu = s\eta$, $\eta = (1/\beta^2 - p^2)^{1/2}$; $Re\,\eta \geq 0$. Remember that s is positive real. Then 3.21 becomes

3.22 $$\bar{u}(x,y,s) = \frac{F}{2\pi i}\int_{-i\infty}^{i\infty} e^{spx - s\eta|y-h|} \frac{dp}{2\mu\eta} \quad ; \; -1/\beta \leq Re\,p \leq 1/\beta$$

Using the principle of reflection (Titchmarsh, 1939, p. 155)

E. C. Titchmarsh, 1939, The theory of functions; 2nd ed., Oxford University Press.

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we can write 3.22 in the form

3.22½ $$\bar{u}(x,y,s) = \frac{F}{2\mu\pi}\,\mathrm{Im}\int_0^{i\infty} e^{spx - s\eta|y-h|}\,\frac{dp}{\eta}$$

The integrand in 3.22½ has branch points where $\eta = \pm 1/\beta$ and branch cuts as shown in Fig 3.3

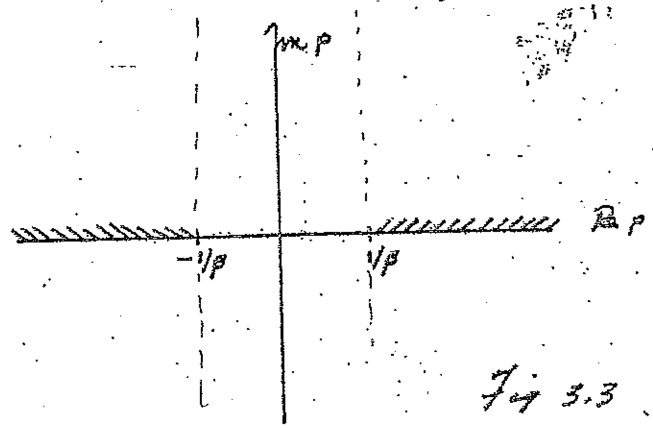

Fig 3.3

From 3.3 we know that

3.23 $$\bar{u}(x,y,s) = \int_0^{\infty} u(x,y,t)\,e^{-st}\,dt$$

Combining 3.22½ and 3.23 we have

3.24 $$\int_0^{\infty} u(x,y,t)\,e^{-st}\,dt = \frac{F}{2\mu\pi}\,\mathrm{Im}\int_0^{i\infty} e^{spx - s\eta|y-h|}\,\frac{dp}{\eta}$$

In 3.24 the Laplace transform variable, s, appears only in the exponent of each integrand. If we can recast the right hand side of 3.24 into a form like the left hand side we should be able to obtain $u(x,y,t)$ by inspection. The first step is to require

3.25 $$px - \eta|y-h| = -t$$

The solution to 3.25 expressing p as a function of t is multivalued. The second quadrant solution is

$$p = \frac{-xt + i\,|y-h|\,(t^2 - r^2/\beta^2)^{1/2}}{r^2}, \quad t > r/\beta$$

3.26

$$p = \frac{-xt + |y-h|\,(r^2/\beta^2 - t^2)^{1/2}}{r^2}, \quad t < r/\beta$$

for $x > 0$ and $r^2 = x^2 + (y-h)^2$. Let $x = r\sin\theta$, $|y-h| = r\cos\theta$, $0 \le \theta \le \pi/2$. Then 3.26 is

$$p = -\frac{t}{r}\sin\theta + i\,(t^2/r^2 - 1/\beta^2)^{1/2}\cos\theta, \quad t > r/\beta$$

3.27

$$p = -\frac{t}{r}\sin\theta + (1/\beta^2 - t^2/r^2)^{1/2}\cos\theta, \quad t < r/\beta$$

For η we have

$$\eta = \frac{t}{r}\cos\theta + i\,(t^2/r^2 - 1/\beta^2)^{1/2}\sin\theta, \quad t > r/\beta$$

3.27½

$$\eta = \frac{t}{r}\cos\theta + |t^2/r^2 - 1/\beta^2|^{1/2}\sin\theta, \quad t < r/\beta$$

In the p-plane the path 3.26 for fixed x, y, h and variable t is shown in Fig 3.4

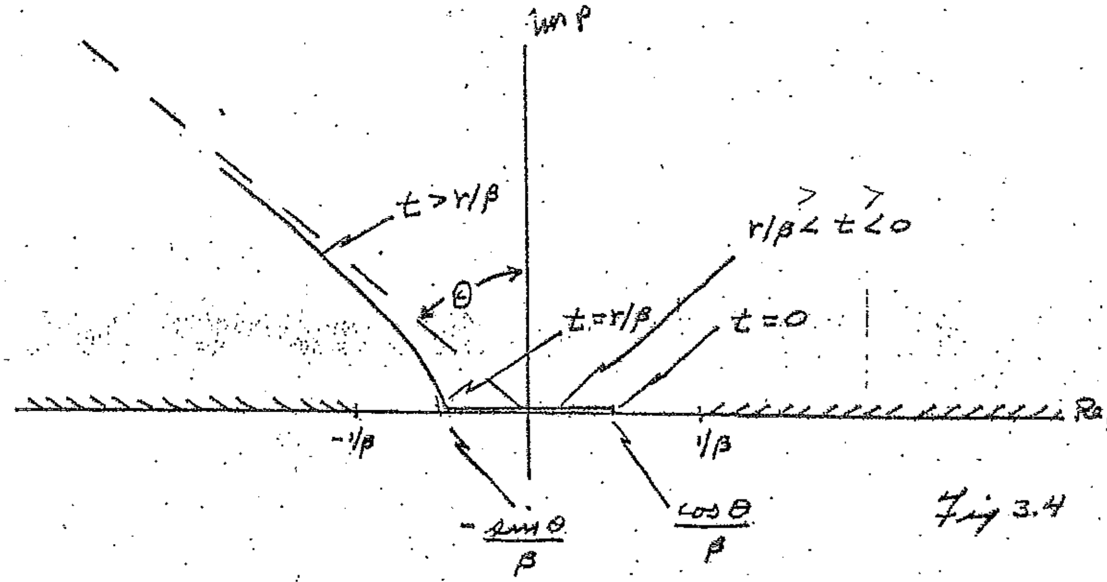

Fig 3.4

In 3.22½ the path of integration is restricted to lie in the strip $-1/\beta \le \operatorname{Re} p \le 1/\beta$. We take the path from $\cos\theta/\beta$ to $\cos\theta/\beta + i\infty$. For $x > 0$ we must deform the path into the left half plane for the integral to converge. We deform the path onto the one shown in Fig 3.4. Call that path W

Then 3.24 becomes

3.28 $$\int_0^\infty u(x,y,t)\, e^{-st}\, dt = \frac{F m}{2\mu\pi} \int_W e^{spx - s\eta|y-h|} \frac{dp}{\eta}$$

since the contribution from the infinite arc vanishes exponentially. On the right hand side of 3.28 we change the variable of integration from p to t. From 3.25 and 3.27 we find

3.29 $$\frac{dp}{dt} = \begin{cases} i\eta/(t^2 - r^2/\beta^2)^{1/2}, & t > r/\beta \\ -\eta/(r^2/\beta^2 - t^2)^{1/2} & t < r/\beta \end{cases}$$

We write 3.28 in the form

3.30 $$\int_0^{\infty} u(x,y,t)\, e^{-st}\, dt = \frac{F}{2\mu\pi}\, \mathrm{Im} \int_0^{\infty} \frac{1}{\eta}\frac{dp}{dt}\, e^{-st}\, dt$$

Since $\frac{dp}{dt}$ and η are real for $t < r/\beta$ the imaginary part of the integrand is zero. Hence

3.30½ $$\int_0^{r/\beta} u(x,y,t)\, e^{-st}\, dt = 0 \quad \text{for any positive real } s$$

Thus $u(x,y,t) \equiv 0$ for $t < r/\beta$. For $t > r/\beta$ we use 3.29 in 3.30 to get

3.30⅔ $$\int_{r/\beta}^{\infty} u(x,y,t)\, e^{-st}\, dt = \frac{F}{2\pi\mu} \int_{r/\beta}^{\infty} \frac{1}{(t^2 - r^2/\beta^2)^{1/2}}\, e^{-st}\, dt \quad \text{for any positive } s$$

Thus

3.30¾ $$u(x,y,t) = \frac{F}{2\pi\mu}\frac{1}{(t^2 - r^2/\beta^2)^{1/2}} \quad \text{for } t > r/\beta$$

Combining 3.30½ and 3.30¾ we get

3.31 $$u(x,y,t) = \frac{F}{2\mu\pi}\,\frac{H(t - r/\beta)}{(t^2 - r^2/\beta^2)^{1/2}}$$

which is the same as 3.20

A slightly different way of applying this trick is as follows. Starting with 3.22½ we recognize that $e^{spx - s\eta|y-h|}$ is the Laplace transform of $\delta(t + px - \eta|y-h|)$ so that the inversion of 3.22½ is

3.32 $$u(x,y,t) = \frac{F}{2\mu\pi} \operatorname{Im} \int_0^{i\infty} \delta(t + px - \eta|y-h|) \frac{dp}{\eta}$$

The integral in 3.32 is evaluated at the zero of the delta function which must be real since t is real. The requirement $t + px - \eta|y-h| = 0$ is the same as 3.25 so we are directly led to 3.31.

Consider now the problem stated at the beginning of this section. Let the medium $y>0$ be denoted by the subscript (1) and use (2) for $y<0$. The line source is in the first medium. Then

3.33
$$\nabla^2 u_1 - \beta_1^{-2} \partial^2 u_1/\partial t^2 = -F\delta(t)\,\delta(x)\,\delta(y-h)/\mu_1$$
$$\nabla^2 u_2 - \beta_2^{-2} \partial^2 u_2/\partial t^2 = 0$$

The doubly transformed equations are

3.34
$$d^2U_1/dy^2 - \nu_1^2 U_1 = -F\,\delta(y-h)/\mu_1$$
$$d^2U_2/dy^2 - \nu_2^2 U_2 = 0$$

The particular solution to the first of 3.34 is given by 3.14. To this we add a homogeneous solution that is bounded for $y>0$. For the second of 3.34 we take the homogeneous solution bounded for $y<0$. Then solutions to 3.34 are

3.35

$$U_1 = \frac{F}{2\mu_1\nu_1}\left[e^{-\nu_1|y-h|} + A\,e^{-\nu_1 y}\right]$$

$$U_2 = \frac{F}{2\mu_1\nu_1}\,B\,e^{\nu_2 y}$$

where A and B are determined from the boundary conditions. These are given by 1.17 and 1.23. The only ~~component~~ nonzero element of the normal component of stress, $\hat{y}\cdot\underline{\underline{T}}$, is $T_{yz} = T_{zy}$

3.36 $T_{yz} = \mu\, dU/dy$

Therefore U and $\mu\, dU/dy$ must be continuous at $y=0$. From 3.35 and 3.36 the boundary conditions are

3.37

$$A - B = -e^{-\nu_1 h}$$

$$\mu_1\nu_1 A + \mu_2\nu_2 B = \mu_1\nu_1 e^{-\nu_1 h}$$

Solving 3.37 we get

3.38

$$A = \frac{\mu_1\nu_1 - \mu_2\nu_2}{\mu_1\nu_1 + \mu_2\nu_2}\,e^{-\nu_1 h}$$

$$B = \frac{2\mu_1\nu_1}{\mu_1\nu_1 + \mu_2\nu_2}\,e^{-\nu_1 h}$$

Substituting 3.38 into 3.35 gives

$$U_1 = \frac{F}{2\mu_1\nu_1}\left[e^{-\nu_1|y-h|} + \frac{\mu_1\nu_1 - \mu_2\nu_2}{\mu_1\nu_1 + \mu_2\nu_2} e^{-\nu_1(y+h)}\right]$$

3.39

$$U_2 = \frac{F}{\mu_1\nu_1 + \mu_2\nu_2} e^{-\nu_1 h + \nu_2 y}$$

The first term in the expression for U_1 in 3.39 represents the direct pulse from the source and we have already discussed it. The pulse response is given by 3.31. Consider now the second term

3.40 $$U_1^r = \frac{F}{2\mu_1\nu_1} \frac{\mu_1\nu_1 - \mu_2\nu_2}{\mu_1\nu_1 + \mu_2\nu_2} e^{-\nu_1(y+h)}$$

The inversion of 3.40 is

3.41 $$\bar{u}_1^r(x,y,s) = \frac{F}{4\pi i}\int_{-i\infty}^{i\infty} \frac{1}{\mu_1\nu_1} \frac{\mu_1\nu_1 - \mu_2\nu_2}{\mu_1\nu_1 + \mu_2\nu_2} e^{\xi x - \nu_1(y+h)} \, d\xi$$

In 3.41 the path of integration lies in the strip ~~$\min(\beta_1^{-1}, \beta_2^{-1}) \leq 1$~~ $-\min(k_1, k_2) \leq Re\,\xi \leq \min(k_1, k_2)$.
We make the change of variable $\xi = sp$, $\nu_1 = s\eta_1$, $\nu_2 = s\eta_2$, $Re\,\eta_1, \eta_2 \geq 0$ where $\eta_1 = (1/\beta_1^2 - p^2)^{1/2}$, $\eta_2 = (1/\beta_2^2 - p^2)^{1/2}$.
Then 3.41 is

3.42 $$\bar{u}_1^r(x,y,s) = \frac{F}{4\pi i}\int_{-i\infty}^{i\infty} \frac{1}{\mu_1\eta_1} \frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2} e^{spx - s\eta_1(y+h)} \, dp$$

In 3.42 the path of integration lies in the strip $-\min(1/\beta_1, 1/\beta_2) \le \mathrm{Re}\, p \le \min(1/\beta_1, 1/\beta_2)$. In the p-plane the integrand of 3.42 has branch cuts as shown in Fig 3.5 for $\beta_1 > \beta_2$

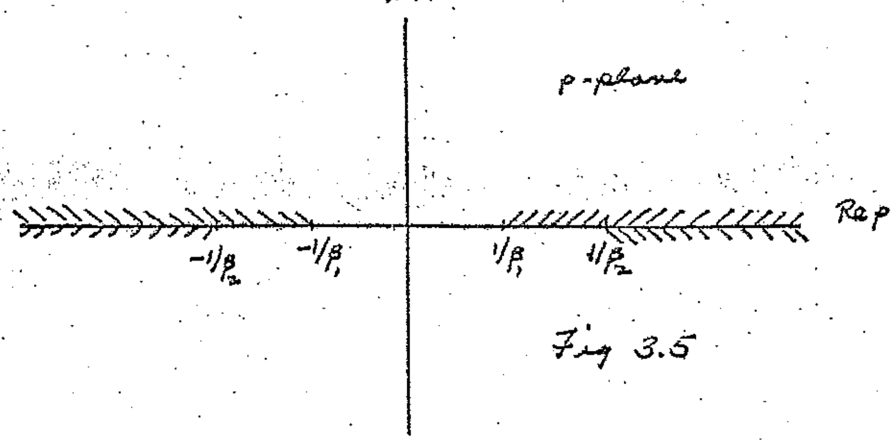

Fig 3.5

In the strip $-1/\beta_1 \le \mathrm{Re}\, p \le 1/\beta_1$, where the path of integration lies, the integrand is real on the real p axis and we may use the principle of reflection to write 3.42 as

3.42½ $$\bar{u}_1^r(x,y,s) = \frac{F}{2\pi}\, \mathrm{Im} \int_0^{i\infty} \frac{1}{\mu_1\eta_1}\, \frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\, e^{spx - s\eta_1(y+h)}\, dp$$

We limit our discussion to $x > 0$ and require that

3.43 $$px - \eta_1(y+h) = -t$$

in 3.42½. Let $R^2 = x^2 + (y+h)^2$, $x = R\sin\theta$, $y+h = R\cos\theta$, $0 \le \theta \le \pi/2$. Then the solution to 3.43 in the second quadrant of the p-plane is

$$p = -\frac{t}{R}\sin\theta + (1/\beta_1^2 - t^2/R^2)^{1/2}\cos\theta \quad ; \quad t < R/\beta_1$$

$$p = -\frac{t}{R}\sin\theta + i\,(t^2/R^2 - 1/\beta_1^2)^{1/2}\cos\theta \quad ; \quad t > R/\beta_1$$

3.44

$$\eta_1 = \frac{t}{R}\cos\theta + (1/\beta_1^2 - t^2/R^2)^{1/2}\sin\theta \quad ; \quad t < R/\beta_1$$

$$\eta_1 = \frac{t}{R}\cos\theta + i\,(t^2/R^2 - 1/\beta_1^2)^{1/2}\sin\theta \quad ; \quad t > R/\beta_1$$

In the p plane the path 3.44 for fixed R, θ and variable t is shown in Fig 3.4 if we replace β by β_1. Call that path W. Shift the path in 3.42½ onto W and change the variable of integration from p to t. Then 3.42½ is

3.45 $$\bar{u}_1^r(x,y,s) = \frac{F}{2\mu_1\pi}\,\mathrm{Im}\int_0^\infty \frac{1}{\eta_1}\,\frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\,\frac{dp}{dt}\,e^{-st}\,dt$$

Thus the time solution must be

3.46 $$u_1^r(x,y,t) = \frac{F}{2\mu_1\pi}\,\mathrm{Im}\left(\frac{1}{\eta_1}\,\frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\,\frac{dp}{dt}\right)$$

The path, $p(t) = W$ ~~touches~~ leaves the real axis in the p plane when $t = R/\beta_1$. From 3.43 and 3.44 we find

3.47 $$\frac{dp}{dt} = \begin{cases} i\eta_1/(t^2 - R^2/\beta_1^2)^{1/2} & ; \; t > R/\beta_1 \\ -\eta_1/(t^2 - R^2/\beta_1^2)^{1/2} & ; \; t < R/\beta_1 \end{cases}$$

derive

When $t < R/\beta_1$ the term in parentheses in 3.46 is real and $u_1^r = 0$. When $t > R/\beta_1$, 3.46 is

3.48 $$u_1^r(x,y,t) = \frac{F}{2\mu_1\pi} \mathrm{Re}\left(\frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\right) \frac{H(t - R/\beta_1)}{(t^2 - R^2/\beta_1^2)^{1/2}}$$

The pulse appears to come from the point $x=0$, $y=-h$ the image of the source. This pulse is called the reflected pulse. It has the same form as the direct pulse 3.31 except for the factor $\mathrm{Re}\left(\frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\right)$. This factor is sometimes called the generalized reflection coefficient. At $t = R/\beta_1$ it is identical to the plane wave reflection coefficient. When $t = R/\beta_1$, $p(t) = -\sin\theta/\beta_1$. For $\beta_1 > \beta_2$ the point $-\sin\theta/\beta_1$ is always to the right of the branch point of η_1 at $p = -1/\beta_1$.

Consider now the case $\beta_1 < \beta_2$. Again the displacement is given by 3.46. When $p(R/\beta_1) = -\sin\theta/\beta_1$ is ~~less~~ greater than $-1/\beta_2$ (the branch point nearer the origin in this case) the displacement is zero for $t < R/\beta_1$. For $t > R/\beta_1$ it is given by 3.48. If $-\sin\theta/\beta_1 < -1/\beta_2$ the point $p(R/\beta_1)$ lies between the two branch points. On the portion of the path $-\sin\theta/\beta_1 \le p(t) \le -1/\beta_2$ the radical η_2 is pure imaginary and 3.46 does not vanish. This condition first arises when $\sin\theta = \beta_1/\beta_2$. We call $\theta_c = \sin^{-1}(\beta_1/\beta_2)$ the critical angle. For $\theta < \theta_c$ the displacement is given by 3.48. For $\theta > \theta_c$ there is an additional term that comes from that part of the path, $p(t)$, from $p = -1/\beta_2$ to $p = -\sin\theta/\beta_1$ as shown in Fig 3.6

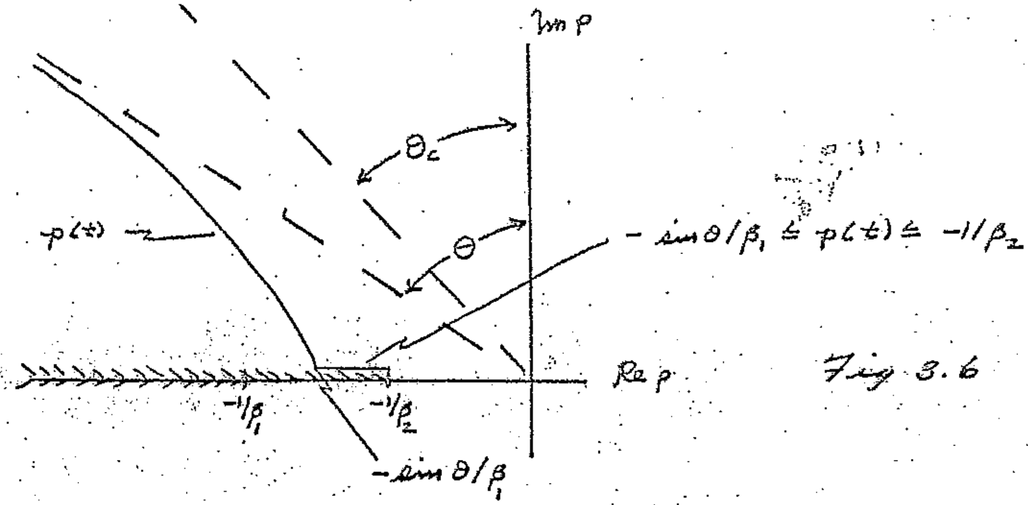

Fig 3.6

The time, t_c, corresponding to $p = -1/\beta_2$ is called the critical time and is the time that the new feature, the critical event, begins to arrive. From 3.43

3.49 $$t_c = X/\beta_2 + (y+h)(1/\beta_1^2 - 1/\beta_2^2)^{1/2} \quad ; \ \theta > \theta_c$$

or

3.50 $$t_c = (R/\beta_2)\left(\sin\theta + \cos\theta\,(\beta_2^2/\beta_1^2 - 1)^{1/2}\right)$$

or

3.51 $$t_c = (R/\beta_1)\cos(\theta - \theta_c)$$

From 3.29 and 3.46 we find the expression for the displacement to be

$$u_1^r(x,y,t) = \frac{F}{2\pi\mu_1} \operatorname{Re}\left(\frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\right) \frac{H(t - R/\beta_1)}{(t^2 - R^2/\beta_1^2)^{1/2}} \quad , \theta < \theta_c$$

$$3.52 \quad u_1^r(x,y,t) = \frac{-F}{2\pi\mu_1} \operatorname{Im}\left(\frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\right) \frac{H(t-t_c) - H(t - R/\beta_1)}{(R^2/\beta_1^2 - t^2)^{1/2}} +$$

$$+ \frac{F}{2\pi\mu_1} \operatorname{Re}\left(\frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\right) \frac{H(t - R/\beta_1)}{(t^2 - R^2/\beta_1^2)^{1/2}} \quad , \theta > \theta_c$$

where t_c is given by 3.51. Thus if $\beta_1 < \beta_2$ an additional feature is present when $\sin\theta > \beta_1/\beta_2$, This feature corresponds to the critical refraction predicted by ray theory. It is represented by the first term in the second expression in 3.52. From 3.49 we see that the linear relation between x and y on the pulse front means that the pulse front is a plane (in two dimensions; in three dimensions it is a cone); that is - the critical refraction is a non uniform plane wave. Pulse fronts at an arbitrary time after reflection are shown in Fig 3.7

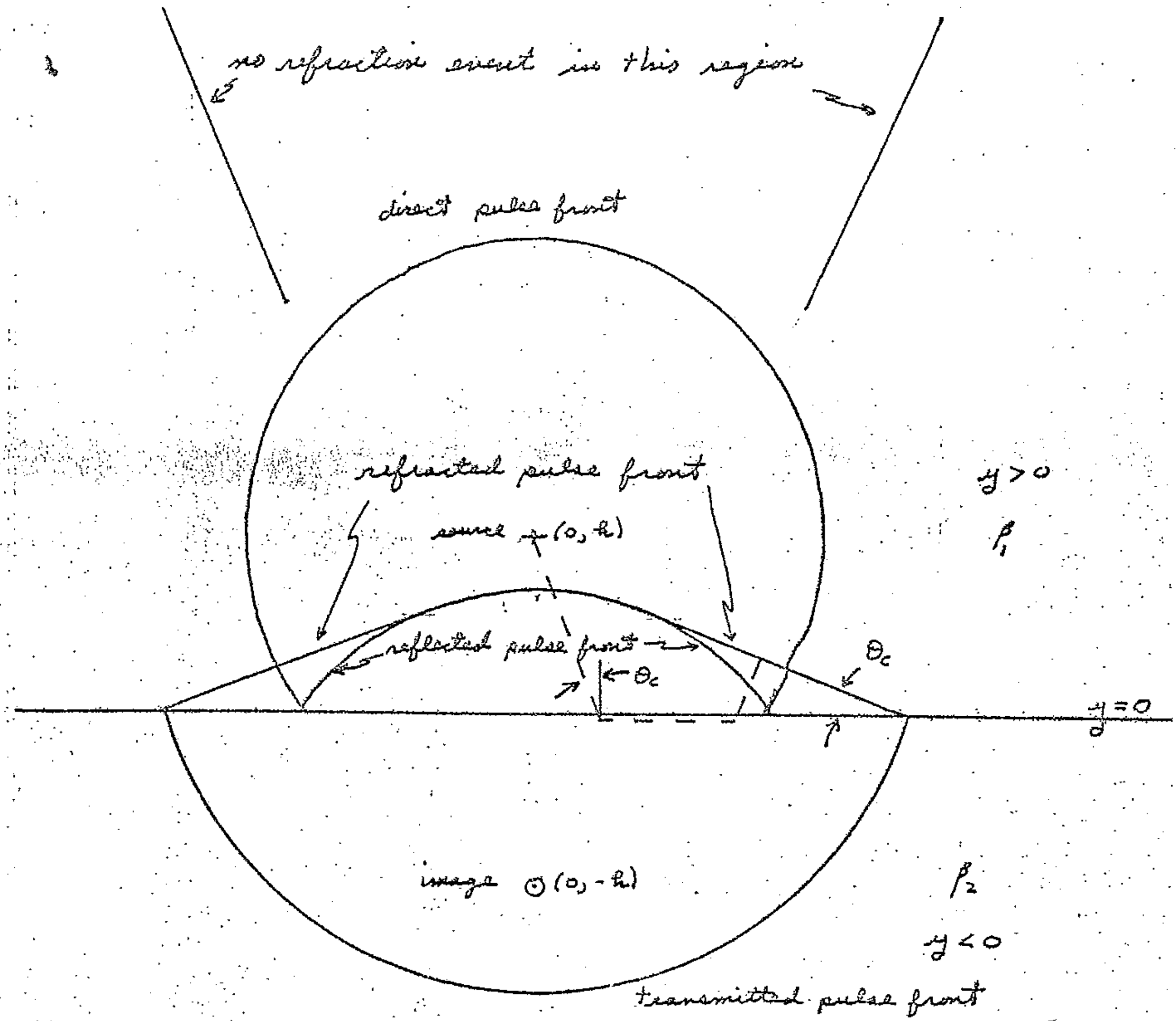

Fig 3.7 Pulse fronts, $\beta_1 < \beta_2$, The dashed line is a possible ray path

We can gain a little more insight into the reflection and refraction phenomena by considering the "first motion" approximation to the exact transient response given by 3.48 and 3.52. The first motion approximations are valid near the pulse fronts. To illustrate the procedure we consider 3.48. Let $t = R/\beta_1 + \tau$, $\tau \ll R/\beta_1$. Then the following approximations are valid

$$p = -\sin\theta/\beta_1$$

$$\eta_1 = \cos\theta/\beta_1$$

3.53 $$\eta_2 = (1/\beta_2^2 - \sin^2\theta/\beta_1^2)^{1/2}$$

$$\frac{dp}{dt} = i\eta/[\tau^{1/2}(2R/\beta_1)^{1/2}]$$

Substituting 3.53 into 3.48 gives

3.54 $$u_1^r(x,y,t) = \frac{F}{2\pi\mu_1}\left[\frac{\mu_1\cos\theta/\beta_1 - \mu_2(1/\beta_2^2 - \sin^2\theta/\beta_1^2)^{1/2}}{\mu_1\cos\theta/\beta_1 + \mu_2(1/\beta_2^2 - \sin^2\theta/\beta_1^2)^{1/2}}\right]\left(\frac{\beta_1}{2R}\right)^{1/2}\frac{1}{(t-R/\beta_1)^{1/2}}$$

and we see that the displacement falls off as $R^{-1/2}$ with a time behavior $(t-R/\beta_1)^{-1/2}$ behind the pulse front. The approximation is also valid for the first expression in 3.52. In 3.54 the bracketed term is the plane wave reflection coefficient. In a similar way these approximations can be applied to the second term in the second expression of 3.52 to find the behavior of the reflection beyond the critical angle. The time behavior is the same as that given in 3.54. The interesting features remaining are the beginning and end of the refraction event.

For $t = t_c + \tau$, $\tau \ll t_c$ we use the approximations

$$p = -1/\beta_2$$

$$\eta_1 = (1/\beta_1^2 - 1/\beta_2^2)^{1/2}$$

3.55 $$\eta_2 = i\,(2\eta_1\tau/\beta_2(R^2/\beta_1^2 - t_c^2)^{1/2})^{1/2}$$

$$\frac{dp}{dt} = -\eta_1/(R^2/\beta_1^2 - t_c^2)^{1/2}$$

Substituting 3.55 into the first term of the second expression in 3.52 yields

3.56 $$u_1^r(x,y,t) = \frac{F}{\pi \mu_1}\,\frac{\mu_2}{\mu_1}\,\frac{(t-t_c)^{1/2}\,2^{1/2}}{(\beta_2^2/\beta_1^2-1)^{1/4}\,(R^2/\beta_1^2-t_c^2)^{3/4}}$$?

The significant feature represented by 3.56 is that the refraction event begins like the time integral of the reflection event. That is, the reflection event begins like $\tau^{-1/2}$ and the refraction event begins like $\tau^{1/2} \propto \int \tau^{-1/2} d\tau$. If the reflection pulse shape is  then the refraction pulse shape is  or .

The first motion approximation for the end of the refraction (perhaps "last motion" is better) shows that it has the same time behavior as the reflection.

Another way to obtain 3.56 that is also useful in more complicated problems is to treat 3.42½ directly. For small times, after the arrival time, we are interested in large s in 3.42½. Consider some $f(t)$ with the Taylor expansion

3.57 $$f(t) = f(0) + \dot{f}(0)\,t + \ddot{f}(0)\,t^2/2 + \cdots$$

The Laplace transform of $f(t)$ is

3.58 $$\bar{f}(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

Using the relation

3.59 $$\int_0^\infty e^{-st}\,t^n\,dt = n!/s^{n+1}$$

and applying 3.58 to 3.57 termwise gives

3.60 $$\bar{f}(s) = \sum_{n=0}^{\infty} f^{(n)}(0)/s^{n+1}$$

Thus

3.61 $$\lim_{s\to\infty} s\,\bar{f}(s) = f(0) = \lim_{t\to 0} f(t)$$

If the expansion 3.57 had been made about some time t_0 then 3.61 would be

3.62 $$\lim_{s\to\infty} s\bar{f}(s)\, e^{st_0} = f(t_0)$$

In our applications t_0 will be some arrival time.

Return now to 3.42½ and shift the path of integration to the negative p axis, when $x > 0$, as shown in Fig. 3.8 and remember $\beta_1 < \beta_2$.

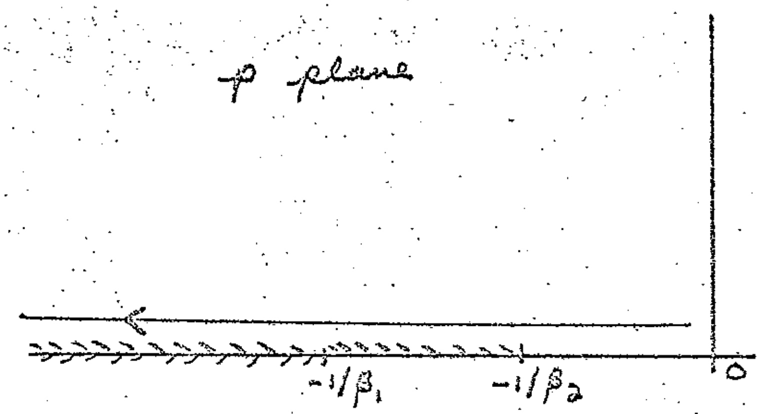

Fig. 3.8

The integrand is real from $p = 0$ to $p = -1/\beta_2$ so 3.42½ is

3.63 $$\bar{u}^n(x,y,s) = \frac{F}{2\pi\mu_1}\, \mathrm{Im}\int_{-1/\beta_2}^{-\infty} dp\, \frac{1}{\eta_1}\left(\frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\right) e^{spx - s\eta_1(y+h)}$$

Let $\eta_2 = i\varpi$ and change the variable of integration from p to ϖ

3.64 $$\bar{u}^n(x,y,s) = \frac{F}{2\pi\mu_1}\int_0^{\infty} d\varpi\, \frac{dp}{d\varpi}\, \frac{1}{\eta_1}\left(\frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\right) e^{spx - s\eta_1(y+h)}$$

For $s \gg 1$ the integrand in 3.64 decays rapidly, when the exponent is negative, as ϖ increases from zero. Most of the contribution to the integral then comes from the vicinity of $\varpi = 0$. For small ϖ

3.65
$$p = -1/\beta_2 - \beta_2 \varpi^2/2$$
$$\eta_1 = (1/\beta_1^2 - 1/\beta_2^2)^{1/2} - \varpi^2/2(1/\beta_1^2 - 1/\beta_2^2)^{1/2}$$

and 3.64 becomes

3.66
$$\overline{u}^n(x, y, s) = \frac{F}{2\pi\mu_1} e^{-s(x/\beta_1 + (y+h)(1/\beta_1^2 - 1/\beta_2^2)^{1/2})}$$
$$\cdot \int_0^\infty d\varpi \, \mathrm{Im}\left(\frac{\mu_1\eta_1 - i\mu_2\varpi}{\mu_1\eta_1 + i\mu_2\varpi} \; \frac{\varpi}{\eta_1 p}\right) e^{-s(\varpi^2 \frac{1}{2}(x\beta_2 - (y+h)(1/\beta_1^2 - 1/\beta_2^2}$$

From 3.66 we see that in order to have the integral converge we must require

3.67 $x\beta_2 - (y+h)(1/\beta_1^2 - 1/\beta_2^2)^{-1/2} > 0$

which leads to

3.68 $\tan\theta - \tan\theta_c > 0$ or $\theta > \theta_c$

Thus 3.66 will converge only beyond the

critical angle. Near $\varpi = 0$ we expand

$$\ln\left(\frac{\mu_1 \eta_1 - i\mu_2 \varpi}{\mu_1 \eta_1 + i\mu_2 \varpi} \; \frac{\varpi}{\eta_1 p}\right)$$ in a Taylor series in

ϖ to get

3.69 $$\ln\left(\frac{\mu_1 \eta_1 - i\mu_2 \varpi}{\mu_1 \eta_1 + i\mu_2 \varpi} \; \frac{\varpi}{\eta_1 p}\right) = 2\,\frac{\mu_2}{\mu_1}\,\frac{\beta_2}{(1/\beta_1^2 - 1/\beta_2^2)}\,\varpi^2 + O(\varpi$$

Using 3.49 and 3.69 in 3.66 we get the approximation

3.70
$$\bar{u}^{7}(x,y,s) = \frac{F}{2\pi\mu_1}\; 2\frac{\mu_2}{\mu_1}\;\frac{\beta_2}{(1/\beta_1^2 - 1/\beta_2^2)}\, e^{-s t_c} \cdot \int_0^\infty d\varpi\, \varpi^2\, e^{-s\varpi^2\left[\frac{1}{2}\left(x\beta_2 - (y+h)(1/\beta_1^2 - 1/\beta_2^2)^{-1/2}\right)\right]}$$

We can use the result

$$\int_0^\infty d\varpi\, \varpi^2\, e^{-a\varpi^2} = 1/4\,(\pi/a^3)^{1/2}$$

to evaluate the integral in 3.70

3.71 $$\bar{u}^n(x,y,s) = \frac{F\mu_2\beta_2 \, e^{-st_c}}{s^{3/2}\mu_1^2(1/\beta_1^2 - 1/\beta_2^2)(2\pi)^{1/2}\left[x\beta_2 - (y+h)(1/\beta_1^2 - 1/\beta_2^2)^{1/2}\right]^{3/2}}$$

Eq 3.59 is valid for non integer values of $n > -1$. In particular

3.72 $$\int_0^\infty dt\, t^{1/2} e^{-st} = (1/2)!\, s^{-3/2} = (\sqrt{\pi}/2)\, s^{-3/2}$$

Thus $s^{-3/2}e^{-st_c}$ is the Laplace transform of $(t-t_c)^{1/2}H(t-t_c)\,2/\sqrt{\pi}$ and the inversion of 3.71 is

3.73 $$u^n(x,y,t) = \frac{2^{1/2}F\mu_2\beta_2(t-t_c)^{1/2}H(t-t_c)}{\pi\mu_1^2(1/\beta_1^2 - \beta_2^2)\left[x\beta_2 - (y+h)(1/\beta_1^2 - 1/\beta_2^2)^{1/2}\right]^{3/2}}$$

which is to be compared to 3.56. We leave it to the reader to show that 3.56 and 3.73 are equal. A more simplified form of 3.56 and 3.73 is

3.74 $$u^n(x,y,t) = \frac{2^{1/2}F\mu_2}{\pi\mu_1^2}\,\frac{(t-t_c)^{1/2}H(t-t_c)(\tan\theta_c)^{1/2}}{\left[(R/\beta_1)\sin(\theta-\theta_c)\right]^{3/2}}$$

4. Reflection and Refraction of an SH pulse in three Dimensions

In rectangular coordinates (x, y, z) or cylindrical coordinates (r, φ, z), the plane $z=0$ separates two different elastic solids. At $r=0$, $z=h$ there is a point source of torque about the z axis. The force is $\underline{f} = \hat{\varphi} f(t)\, \delta(r)\, \delta(z-h)/2\pi r = \hat{\varphi} f(t)\, \delta(x)\delta(y)\delta(z-h)$. For a torque uniform in φ the only component of motion is $s_\varphi(r, z, t)$ which does not depend on φ. Thus $\underline{\nabla}\cdot\underline{s} = 0$ and the linearized momentum equation 1.3 is

$$4.1 \quad \mu \nabla^2 u(r,z,t) - \rho\, \partial^2 u(r,z,t)/\partial t^2 = -f(t)\, \delta(r)\, \delta(z-h)/2\pi r$$

In (x, y, z) coordinates

$$4.2 \quad \mu \nabla^2 u(x,y,z,t) - \rho\, \partial^2 u(x,y,z,t)/\partial t^2 = -f(t)\, \delta(x)\delta(y)\delta(z-h)$$

where $s_\varphi \equiv u$. The total force of the point force is $f(t)$ and we assume $f(t) \equiv 0$ for $t<0$. Also, we assume $u = \partial u/\partial t \equiv 0$ for $t=0$.

To get the particular solution to 4.1 and 4.2 we use the integral transform method in much the same way as in Ch. 3. Our exposition is a slightly modified version of that of de Hoop (1960).

A. T. de Hoop, 1962, Theoretical determination of the surface motion of a uniform elastic half-space produced by a dilatational, impulsive, point source: La propagation des ébranlements dans les milieux hétérogènes; Colloques Internationaux du Centre National de la Recherche Scientifique, No. 111, Marseille, 11-16 Septembre 1961.

The Laplace transform on t is

4.3 $\bar{u}(x,y,z,s) = \int_0^\infty dt\, u(x,y,z,t)\, e^{-st}$; s real, $s>0$

and the transform of 4.2 is

4.4 $\mu(\partial_x^2 + \partial_y^2 + \partial_z^2 - s^2/\beta^2)\,\bar{u}(x,y,z,s) = -\bar{f}(s)\,\delta(x)\,\delta(y)\,\delta(z-h)$

The bilateral transform on x is

4.5 $\bar{u}(\xi, y, z, s) = \int_{-\infty}^{\infty} dx\, \bar{u}(x,y,z,s)\, e^{-\xi x}$

$k = \frac{s}{\beta}$

and the transform of 4.4 is

4.6 $\mu(\xi^2 + \partial_y^2 + \partial_z^2 - k^2)\,\bar{u}(\xi,y,z,s) = -\bar{f}(s)\,\delta(y)\,\delta(z-h)$

We must require $|\mathrm{Re}\,\xi| \leq k = s/\beta$.

The transform on y is

4.7 $\bar{\bar{u}}(\xi, \zeta, z, s) = \int_{-\infty}^{\infty} dy\, \bar{u}(\xi, y, z, s)\, e^{-\zeta y}$

and the transform of 4.6 is

4.8 $\mu(\xi^2 + \zeta^2 + d_z^2 - k^2)\, \bar{\bar{u}}(\xi, \zeta, z, s) = -\bar{f}(s)\, \delta(z-h)$

We must require $|\mathrm{Re}\, \zeta| \leq |\mathrm{Re}\, (k^2 - \xi^2)^{1/2}|$. Let $\nu^2 = k^2 - \xi^2 - \zeta^2$. Then 4.8 is

4.9 $\mu(d_z^2 - \nu^2)\, \bar{\bar{u}}(\xi, \zeta, z, s) = -\bar{f}(s)\, \delta(z-h)$

Eq. 4.9 is the same as 3.6 whose solution is 3.14

4.10 $\bar{\bar{u}}(\xi, \zeta, z, s) = \dfrac{\bar{f}(s)}{2\mu\nu}\, e^{-\nu|z-h|}$, $\mathrm{Re}\, \nu \geq 0$

The inversions of 4.10 with respect to ξ and ζ are

4.11 $\bar{u}(x, y, z, s) = \dfrac{1}{2\pi i} \displaystyle\int_{-i\infty}^{i\infty} d\xi\, \dfrac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\zeta\, \dfrac{\bar{f}(s)}{2\mu\nu}\, e^{\xi x + \zeta y - \nu|z-h|}$

We have that $x = r\cos\varphi$, $y = r\sin\varphi$, $r = x\cos\varphi + y\sin\varphi$. Let us introduce new variables p and q

4.12
$$\xi = p\cos\varphi - q\sin\varphi$$
$$\zeta = p\sin\varphi + q\cos\varphi$$

Then $\xi^2+\zeta^2 = p^2+q^2$ and $d\xi d\zeta = dp\,dq$. In terms of the new variables 4.11 is

4.13 $$\bar{u}(x,y,z,s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp \, \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dq \, \frac{\bar{f}(s)}{2\mu\nu} e^{px - \nu|z-h|}$$

where $\nu^2 = k^2 - p^2 - q^2$, $\operatorname{Re}\nu \geq 0$. Changing the variable q to iq gives

4.14 $$\bar{u}(x,y,z,s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp \, \frac{1}{\pi}\int_{0}^{\infty} dq \, \frac{\bar{f}(s)}{2\mu\nu} e^{px - \nu|z-h|}$$

where $\nu^2 = k^2 - p^2 + q^2$, $\operatorname{Re}\nu \geq 0$. Using the reflection principle (see page 30) we write 4.14 as

4.15 $$\bar{u}(x,y,z,s) = \frac{\bar{f}(s)}{2\mu\pi^2} \operatorname{Im}\int_0^{i\infty} dp \int_0^{\infty} dq \, \frac{e^{px - \nu|z \mp h|}}{\nu}$$

We let $q = sq$, $p = sp$, $\nu = s\eta$ and change the order of integration in 4.15 to get

4.16 $$\bar{u}(x,y,z,s) = \frac{s\bar{f}(s)}{2\mu\pi^2}\int_0^{\infty} dq \operatorname{Im}\int_0^{i\infty} \frac{dp}{\eta} e^{s(px - \eta|z-h|)}$$

where $\eta^2 = 1/\beta^2 - p^2 + q^2$, $\operatorname{Re}\eta \geq 0$. Let $\omega^2(q) = 1/\beta^2 + q^2$. Then $\eta^2 = \omega^2(q) - p^2$ and the p-plane must be cut as shown in Fig 4.1 so that $\operatorname{Re}\eta \geq 0$.

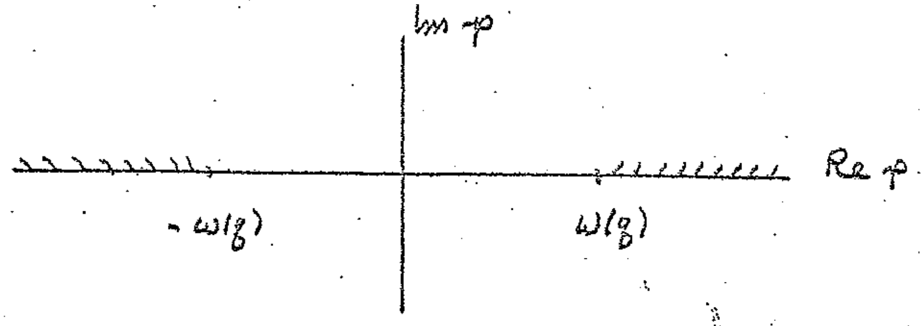

Fig. 4.1

Let $R^2 = r^2 + (z-h)^2$ and $r = R\sin\theta$, $|z-h| = R\cos\theta$. In 4.16 we require

4.17 $pr - \eta|z-h| = -t$

and $s\bar{f}(s) = F$. Then, in the second quadrant,

$$p = -t/R\sin\theta + i(t^2/R^2 - \omega^2(\zeta))^{1/2}\cos\theta$$

4.18 $\eta = t/R\cos\theta + i(t^2/R^2 - \omega^2(\zeta))^{1/2}\sin\theta$

$$dp/dt = i\eta/(t^2 - R^2\omega^2(\zeta))^{1/2}$$

for $t > R\omega(\zeta)$. Changing the integration from p to t in 4.16 gives

4.19 $\bar{u}(x,y,z,s) = \dfrac{F}{2\mu\pi^2}\mathrm{Re}\int_0^\infty d\zeta\int_0^\infty dt\,\dfrac{e^{-st}H(t - R\omega(\zeta))}{(t^2 - R^2\omega^2(\zeta))^{1/2}}$

and interchanging the order of integration gives

4.20 $\bar{u}(x,y,z,s) = \dfrac{F}{2\mu\pi^2}\int_0^\infty dt\, e^{-st}H(t - R/\beta)\int_0^{(t^2/R^2 - 1/\beta^2)^{1/2}}\dfrac{d\zeta}{(t^2 - R^2\omega^2(\zeta))^{1/2}}$

The time solution must then be

4.21 $$u(x,y,z,t) = \frac{F}{2\mu\pi^2} \int_0^{(t^2/R^2 - 1/\beta^2)^{1/2}} dq \, \frac{H(t - R/\beta)}{(t^2 - R^2\omega^2(q))^{1/2}}$$

The integral in 4.21 is $\pi/2R$ as we can show. Let

4.22 $$q = (t^2/R^2 - 1/\beta^2)^{1/2} \sin\psi \quad , 0 \le \psi \le \pi/2$$

Then $(t^2 - R^2\omega^2(q))^{1/2} = R[(t^2/R^2 - 1/\beta^2) - (t^2/R^2 - 1/\beta^2)\sin^2\psi]^{1/2} =$
$= R(t^2/R^2 - 1/\beta^2)^{1/2}\cos\psi$ so

4.23 $$\int_0^{(t^2/R^2 - 1/\beta^2)^{1/2}} \frac{dq}{R(t^2/R^2 - \omega^2(q))^{1/2}} = \frac{1}{R}\int_0^{\pi/2} d\psi = \pi/2R$$

and 4.21 is

4.24 $$\vec{u}(x,y,z,t) = \frac{F H(t - R/\beta)}{4\pi R \mu}$$

In more complex problems the definite integral can seldom be evaluated explicitly. We shall now discuss a few alternative methods for obtaining 4.24, methods that are also useful for other problems.

A second approach begins with 4.13

4.13 $$\bar{u}(x,y,z,s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp \, \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dq \, \frac{\bar{f}(s)}{2\mu} \frac{e^{px - (k^2-p^2-q^2)^{1/2}|z-h|}}{(k^2-p^2-q^2)^{1/2}}$$

From 3.16 and 3.17 we have

4.25 $$K_0[l(x^2+y^2)^{1/2}] = \frac{1}{2i}\int_{-i\infty}^{i\infty} d\xi \, \frac{e^{\xi x - (l^2-\xi^2)^{1/2} y}}{(l^2-\xi^2)^{1/2}} , \quad Re\, l > 0$$

By using 4.25 we can evaluate the p integral in 4.

4.26 $$\bar{u}(x,y,z,s) = \frac{1}{2\pi i} \frac{\bar{f}(s)}{2\mu\pi} \int_{-i\infty}^{i\infty} dq \, K_0[(k^2-q^2)^{1/2}(r^2+(z-h)^2)^{1/2}]$$

Let $R^2 = r^2 + (z-h)^2$ and use the principle of reflection to rewrite 4.26

4.27 $$\bar{u}(x,y,z,s) = \frac{1}{2\mu\pi^2} \bar{f}(s) \, Im \int_0^{i\infty} dq \, K_0[(k^2-q^2)^{1/2} R]$$

In 4.27 let $s\bar{f}(s) = F$; $q = sp$

4.28 $$\bar{u}(x,y,z,s) = \frac{F}{2\mu\pi^2} \, Im \int_0^{i\infty} dp \, K_0[sR(1/\beta^2 - p^2)^{1/2}]$$

$$= \frac{F}{2\mu\pi^2} \, Re \int_0^{\infty} dp \, K_0[sR(1/\beta^2 + p^2)^{1/2}]$$

The inverse transform of K_0 in 4.28 is

$$4.29 \quad K_0[sR(1/\beta^2+p^2)^{1/2}] \Rightarrow \frac{H[t - R(1/\beta^2+p^2)^{1/2}]}{[t^2 - R^2(1/\beta^2+p^2)]^{1/2}}$$

so the inverse of 4.28 is

$$4.30 \quad u(x,y,z,t) = \frac{F}{2\mu\pi^2}\, \mathrm{Re}\int_0^\infty dp\, \frac{H(t - R(1/\beta^2+p^2)^{1/2})}{[t^2 - R^2(1/\beta^2+p^2)]^{1/2}}$$

which is the same as 4.21. An alternative derivation of 4.24 is to let $\tau = R(1/\beta^2+p^2)^{1/2}$, $dp/d\tau = \tau/R(\tau^2-R^2/\beta^2)^{1/2}$ in 4.30

$$4.31 \quad u(x,y,z,t) = \frac{F}{2\mu\pi^2 R}\, \mathrm{Re}\int_0^\infty d\tau\, \tau\, \frac{H(t-\tau)\, H(\tau - R/\beta)}{(\tau^2 - R^2/\beta^2)^{1/2}(t^2-\tau^2)^{1/2}}$$

$$= \frac{F}{2\mu\pi^2 R} H(t - R/\beta)\int_{R/\beta}^{t} d\tau\, \tau\, (\tau^2 - R^2/\beta^2)^{-1/2}(t^2-\tau^2)^{1/2}$$

The integral in 4.31 is $\pi/2$ as can easily be proved by making the substitution

$$4.32 \quad \tau^2 = R^2/\beta^2 + (t^2 - R^2/\beta^2)\sin^2\psi \quad , \quad 0 \le \psi \le \pi/2$$

so we have

$$4.34 \quad \cancel{u(x,y,z)}, \quad u(x,y,z,t) = \frac{F\, H(t - R/\beta)}{4\pi R\mu}$$

As a third approach we return to 4.1 and treat the problem directly in cylindrical coordinates

4.1 $\mu \nabla^2 u(r,z,t) - \rho\, \partial^2 u(r,z,t)/\partial t^2 = -f(t)\,\delta(r)\,\delta(z-h)/2\pi r$

The Laplace transform of 4.1 is

4.33 $\mu\left(\nabla^2 \bar{u}(r,z,s) - s^2/\beta^2\, \bar{u}(r,z,s)\right) = -\bar{f}(s)\,\delta(r)\,\delta(z-h)/2\pi r$

Since $\bar{u}$ is not a function of φ we can write

4.34 $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} = \nabla_r^2 + \partial_z^2$

The bilateral Laplace transform

4.35 $\bar{u}(r,\nu,s) = \int_{-\infty}^{\infty} \bar{u}(r,z,s)\, e^{-\nu z}\, dz$

is applied to 4.33

4.36 $\nabla_r^2 \bar{u}(r,\nu,s) - (k^2-\nu^2)\,\bar{u}(r,\nu,s) = -\bar{f}(s)\, e^{-\nu h}\,\delta(r)/2\pi r \mu$

We must require $|\mathrm{Re}\,\nu| < k = s/\beta$. Let $\xi^2 = (k^2-\nu^2)$, $\mathrm{Re}\,\xi > 0$ and consider the equation

4.37 $\nabla_r^2 v - \xi^2 v = -\delta(r-r_0)/r$

Homogeneous solutions to 4.37 are

4.38 $v_1 = I_0(\xi r)$

$v_2 = K_0(\xi r)$

For large r and fixed ξ (Watson, 1952, p. 202)

4.39 $v_1 \rightarrow e^{\xi r}/(2\pi\xi r)^{1/2}$

$v_2 \rightarrow (\pi/2\xi r)^{1/2} e^{-\xi r}$

Thus for large r we must have

4.40 $v = c_2 K_0(\xi r)$, $|\xi r| >> 1$

and for small r

4.41 $v = c_1 I_0(\xi r)$, $|\xi r| << 1$

Also we require that v be continuous at $r = r_0$

4.42 $v = c\, I_0(\xi r_<) K_0(\xi r_>)$, $r_> = \max(r, r_0)$, $r_< = \min(r, r_0)$

Multiply 4.37 by r and integrate from $r_0 - \epsilon$ to $r_0 + \epsilon$

4.43 $\lim_{\epsilon \to 0} \left[r\, dv/dr \right]_{r_0-\epsilon}^{r_0+\epsilon} = -1$

We substitute 4.42 into 4.43

4.44 $C\xi r_0\left[I_0(\xi r_0)K_0'(\xi r_0) - I_0'(\xi r_0)K_0(\xi r_0)\right] = -1$

The bracketed term in 4.44 is the Wronskian of (I_0, K_0) and is known to be $-1/\xi r_0$. Thus $C = 1$ and the particular solution to 4.37 is

4.45 $v = I_0(\xi r_<)\, K_0(\xi r_>)$

If $r_0 = 0$ as in 4.36

4.46 $v = K_0(\xi r)$

since $I_0(0) = 1$. The particular solution to 4.36 is

4.47 $\bar{u}(r,\nu,s) = \bar{f}(s)\, e^{-\nu h} K_0(\xi r)/2\pi\mu$

The inversion of 4.47 to z is

4.48 $$\bar{u}(r,z,s) = \frac{1}{2\pi i}\,\frac{\bar{f}(s)}{2\pi\mu}\int_{-i\infty}^{i\infty} d\nu\, K_0(\xi r)\, e^{\nu(z-h)}$$

For $z = h$ 4.48 is the same as 4.26 so the time solution is 4.24. For $z \neq h$ in 4.48 we replace $K_0(\xi r)$ by one of its integral representations

4.49 $$K_0(\xi r) = \frac{1}{2i}\int_{-i\infty}^{i\infty} d\zeta\, \frac{e^{\zeta r}}{(\xi^2 - \zeta^2)^{1/2}}$$

and 4.48 becomes

4.50 $$\bar{u}(r,z,s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\nu \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi \, \frac{\bar{f}(s)}{2\mu} \frac{e^{\nu(z-h)+\xi r}}{(k^2-\nu^2-\xi^2)^{1/2}}$$

Let $r = R\sin\theta$, $(z-h) = R\cos\theta$, $0 \leq \theta \leq \pi$. In 4.50 we introduce new variables p, η

4.51 $$p = \nu\cos\theta + \xi\sin\theta \, , \quad \eta = \nu\sin\theta - \xi\cos\theta$$

so that $dp\,d\eta = d\nu\,d\xi$, $\xi^2+\nu^2 = p^2+\eta^2$

4.52 $$\bar{u}(r,z,s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\eta \, \frac{\bar{f}(s)}{2\mu} \frac{e^{pR}}{(k^2-p^2-\eta^2)^{1/2}}$$

which is the same as 4.13 for $z=h$ and therefore leads to 4.24. Alternatively we can use 4.49 to evaluate the p integral in 4.52

4.53 $$\bar{u}(r,z,s) = \frac{\bar{f}(s)}{2\pi\mu} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\eta \, K_0[R(k^2-\eta^2)^{1/2}]$$

$$= \frac{\bar{f}(s)}{2\mu\pi^2} \operatorname{Im}\int_0^{i\infty} d\eta \, K_0[R(k^2-\eta^2)^{1/2}]$$

which is the same as 4.27 and again we obtain 4.24.

A fourth approach begins with 4.48

4.48 $$\bar{u}(r,z,s)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\nu\,\frac{\bar{f}(s)}{2\mu\pi}\,K_0(\xi r)\,e^{\nu(z-h)}$$

where $\xi^2 = k^2-\nu^2$, $Re\,\xi \geq 0$. Using the reflection principle we can write 4.48 as either

4.54 $$\bar{u}(r,z,s)=\frac{\bar{f}(s)}{2\pi^2\mu}\,\mathrm{Im}\int_0^{i\infty} d\nu\,K_0(\xi r)\,e^{\nu(z-h)}$$

or

4.55 $$\bar{u}(r,z,s)=\frac{-\bar{f}(s)}{2\pi^2\mu}\,\mathrm{Im}\int_0^{-i\infty} d\nu\,K_0(\xi r)\,e^{\nu(z-h)}$$

To ensure $Re\,\xi \geq 0$ we cut the ν-plane as shown in Fig 4.2.

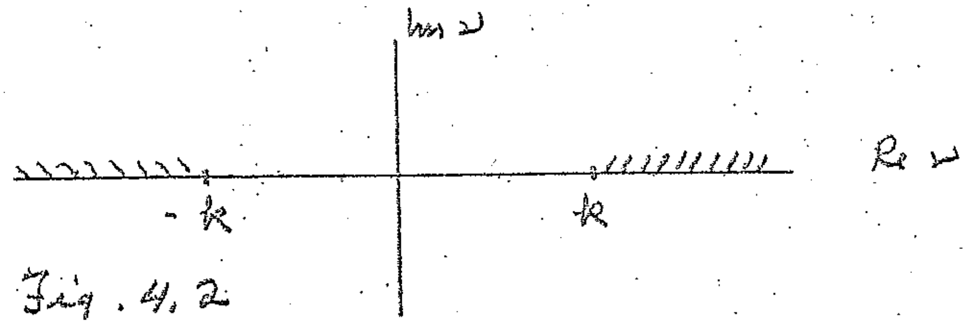

Fig. 4.2

Consider 4.54 for $z > h$. Shift the path of integration to the top of the cut in the left half ν-plane

4.56 $$\bar{u}(r,z,s) = \frac{\bar{f}(s)}{2\pi^3\mu}\,\operatorname{Im}\int_{-k}^{i\infty} d\nu\, K_0(\xi r)\, e^{\nu|z-h|}$$

Changing the variable of integration to ξ gives

4.57 $$\bar{u}(r,z,s) = \frac{-\bar{f}(s)}{2\pi^2\mu}\,\operatorname{Im}\int_{0}^{i\infty} d\xi\, \frac{\xi}{\nu}\, K_0(\xi r)\, e^{\nu|z-h|} \quad ; \ \operatorname{Re}\nu \le 0$$

Let $\nu = -\nu$ in 4.57

4.58 $$\bar{u}(r,z,s) = \frac{\bar{f}(s)}{2\pi^2\mu}\,\operatorname{Im}\int_{0}^{i\infty} d\xi\, \frac{\xi}{\nu}\, K_0(\xi r)\, e^{-\nu|z-h|} \quad ; \ \operatorname{Re}\nu \ge 0$$

If we consider 4.55 for $z < h$ on the bottom of the cut in the right half ν-plane we are led once more to 4.58. Thus 4.58 holds for all z.
In 4.58 let $\xi = sp$, $\nu = s\eta$, $s\bar{f}(s) = F$

4.59 $$\bar{u}(r,z,s) = \frac{F}{2\pi^3\mu}\,\operatorname{Im}\int_{0}^{i\infty} dp\, \frac{p}{\eta}\, K_0(spr)\, e^{-s\eta|z-h|}$$

The operational image of $K_0(spr)$ is

4.60 $$K_0(spr) \Rightarrow H(t-pr)/(t^2-p^2r^2)^{1/2}$$

so the image of $K_0(spr)\,e^{-s\eta|z-h|}$ is

4.61 $$K_0(spr)\,e^{-s\eta|z-h|} \Longrightarrow H(t-pr-\eta|z-h|)/\left[(t-\eta|z-h|)^2-p^2r^2\right]^{1/2}$$

Then the ring solution, the image of 4.59, is

4.62 $$u(r,z,t) = \frac{F}{2\pi^2\mu} \operatorname{Im} \int_0^{i\infty} dp \, \frac{p}{\eta} \frac{H(t - pr - \eta|z-h|)}{[(t - \eta|z-h|)^2 - p^2 r^2]^{1/2}}$$

To evaluate 4.62 we require that the argument of the step function be real. Let

4.63 ~~4.62~~ $$\tau = pr + \eta|z-h| \quad , \quad \tau \text{ real}$$

In the first quadrant of the p-plane

$$p = \tau/R \sin\theta + i(\tau^2/R^2 - 1/\beta^2)^{1/2}\cos\theta = p(\tau)$$

4.64 ~~4.63~~ $$\eta = \tau/R \cos\theta - i(\tau^2/R^2 - 1/\beta^2)^{1/2}\sin\theta$$

$$dp/d\tau = i\eta/(\tau^2 - R^2/\beta^2)^{1/2}$$

In 4.?7 we shift the path to $p(\tau)$ so 4.62 is

4.65 $$u(r,z,t) = \frac{F}{2\pi^2\mu} \operatorname{Im} \int_{p(\tau)} dp \, \frac{p}{\eta} \frac{H(t-\tau)}{[(t-\eta|z-h|)^2 - (\tau - \eta|z-h|)^2]^{1/2}}$$

$$= \frac{F}{2\pi^2\mu} \operatorname{Im} \int_0^{\infty} d\tau \, \frac{dp}{d\tau} \frac{p}{\eta} \frac{H(t-\tau)}{[(t-\eta|z-h|)^2 - (\tau-\eta|z-h|)^2]^{1/2}}$$

For $dp/d\tau$ we use 4.64 in 4.65

4,66 $$u(r,z,t) = \frac{F}{2\pi^2\mu}\, \mathrm{Re}\int_0^t d\tau\, p\,(\tau^2 - R^2/\beta^2)^{1/2}\left[(t-\eta|z-h|)^2 - (\tau-\eta|z-h|)^2\right]^{-1/2}$$

When $t < R/\beta$ we have $\tau \le t < R/\beta$. Then both p and η are real (see 4,64) and $(\tau^2 - R^2/\beta^2)^{1/2}$ is imaginary so $u = 0$ for $t < R/\beta$. We write 4,66 as

4,67. $$u(r,z,t) = \frac{F\,H(t-R/\beta)}{2\pi^2\mu}\, \mathrm{Re}\int_{R/\beta}^{t} d\tau\, p\,(\tau^2 - R^2/\beta^2)^{-1/2}\left[(t-\eta|z-h|)^2 - (\tau-\eta|z-h|)^2\right]$$

For $z = h$ 4.67 is equal to 4,31 which leads to 4,24. The integral in 4,67 is $\pi/2R$ but much manipulation is required to show it. We omit the demonstration.

A fifth approach requires the modified Fourier-Bessel transform pair, sometimes called the Laplace-Bessel pair

4,68
$$F(\xi) = \int_0^\infty dr\, r\, f(r)\, I_n(\xi r)$$

$$f(r) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} d\xi\, \xi\, F(\xi)\, K_n(\xi r)$$

Taking $n = 0$ in 4,68 we apply the first of 4,68 to 4,33

4,69 $$\mu\left[\int_0^\infty dr\, I_0(\xi r)\left\{\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r}\, \bar{u}(r,z,s) + \frac{\partial^2}{\partial z^2}\, \bar{u}(r,z,s) - k^2 \bar{u}(r,z,s)\right\}\right] =$$

$$= -\bar{f}(s)\, \delta(z-h)/2\pi$$

To evaluate the integral in 4.69 we integrate by parts two times. We require $|\mathrm{Re}\,\xi| < k$ so the two terms, obtained by partial integration, will vanish

4.70 $$\int_0^\infty dr\, I_0(\xi r)\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\bar{u}(r,z,s) = \cancelto{0}{I_0(\xi r)\, r\frac{\partial}{\partial r}\bar{u}(r,z,s)\Big|_0^\infty} -$$

$$-\int_0^\infty dr\, \frac{d}{dr} I_0(\xi r)\, r\frac{\partial}{\partial r}\bar{u}(r,z,s) = \cancelto{0}{-r\frac{d}{dr} I_0(\xi r)\,\bar{u}(r,z,s)} +$$

$$+\int_0^\infty dr\, \bar{u}(r,z,s)\frac{d}{dr} r\frac{d}{dr} I_0(\xi r)$$

The function $I_0(\xi r)$ is a solution to

4.71 $$\frac{d}{dr} r\frac{d}{dr} I_0(\xi r) = r\,\xi^2 I_0(\xi r)$$

so 4.70 is

4.72 $$\int_0^\infty dr\, I_0(\xi r)\frac{\partial}{\partial r} r\frac{\partial}{\partial r}\bar{u}(r,z,s) = \int_0^\infty dr\, \bar{u}(r,z,s)\frac{d}{dr} r\frac{d}{dr} I_0(\xi r) =$$

$$= \xi^2\int_0^\infty dr\, r\, I_0(\xi r)\,\bar{u}(r,z,s) = \xi^2\,\bar{\bar{u}}(\xi,z,s)$$

and we can write 4.69 as

4.73 $$\mu\left[\frac{d^2}{dz^2}\bar{u}(\xi,z,s) - (k^2-\xi^2)^{1/2}\,\bar{u}(\xi,z,s)\right] = -\bar{f}(s)\,\delta(z-h)/2\pi$$

Eq. 4.73 is the same as 4.9 and 3.6 so the solution:

4.74 $$\bar{u}(\xi,z,s) = \bar{f}(s)/4\pi\mu(k^2-\xi^2)^{1/2}\, e^{-(k^2-\xi^2)^{1/2}|z-h|}, \quad \mathrm{Re}(k^2-\xi^2)^{1/2} \geq 0$$

With $n=0$ we use the second of 4.68 to get $\bar{u}(r,z,s)$

$$4.75 \quad \bar{u}(r,z,s) = \frac{\bar{f}(s)}{4\pi^2 \mu i} \int_{-i\infty}^{i\infty} d\xi \, \frac{\xi}{(k^2-\xi^2)^{1/2}} K_0(\xi r) \, e^{-(k^2-\xi^2)^{1/2}|z-h|}$$

which is the same as 4.58, and leads to 4.24.

We have discussed five variations on a theme, each of which leads to the transient solution in the form of a real integral over a finite interval. In more complex problems the integral can seldom be evaluated explicitly, but it is in a form that permits easy numerical evaluation.

Let us return to the interface problem stated on page 50. Let the medium $z>0$ be denoted by the subscript (1) and use (2) for $z<0$. The point source is in the first medium

$$\nabla^2 u_1(r,z,t) - \beta_1^{-2}\partial^2 u_1(r,z,t)/\partial t^2 = -F\,H(t)\,\delta(r)\,\delta(z-h)/2\pi r$$

4.76

$$\nabla^2 u_2(r,z,t) - \beta_2^{-2}\partial^2 u_2(r,z,t)/\partial t^2 = 0$$

We transform 4.76 with respect to t and r using 4.3 and 4.68a ($n=0$) to get the pair of equations

$$\partial_z^2 \bar{u}_1(\xi,z,s) - \nu_1^2 \bar{u}_1(\xi,z,s) = -s^{-1}F\,\delta(z-h)/2\pi\mu_1$$

4.77

$$\partial_z^2 \bar{u}_2(\xi,z,s) - \nu_2^2 \bar{u}_2(\xi,z,s) = 0$$

where $|\mathrm{Re}\,\xi| \le \min(k_1,k_2)$, $\nu_1^2 = k_1^2 - \xi^2$, $\nu_2^2 = k_2^2 - \xi^2$. Without loss of generality we require $\mathrm{Re}\,\nu_1 > 0$, $\mathrm{Re}\,\nu_2 > 0$. Eq. 4.77 is the same as 3.34. According to 3.35 solutions to 4.77 are

$$\bar{u}_1 = (s^{-1}F/4\pi\mu_1\nu_1)\left[e^{-\nu_1|z-h|} + A e^{-\nu_1 z}\right]$$

4.78

$$\bar{u}_2 = (s^{-1}F/4\pi\mu_1\nu_1)\left[B e^{\nu_2 z}\right]$$

The only nonzero element of the normal component of stress, $\hat{z}\cdot\underline{\underline{T}}$, is $T_{z\varphi} = T_{\varphi z}$

4.79 $$\bar{T}_{z\varphi}(\xi,z,s) = \mu\,\partial_z \bar{u}(\xi,z,s)$$

The boundary conditions 1.19 and 1.23 require continuity of $\bar{u}$ and $\mu\,\partial_z\bar{u}$ at $z=0$. From 4.78 and 4.79 these conditions lead to 3.37 for A and B. A and B are then given by 3.38 (with z in place of y) so 4.78 becomes

$$\bar{u}_1 = (s^{-1}F/4\pi\mu_1\nu_1)\left[e^{-\nu_1|z-h|} + \frac{\mu_1\nu_1 - \mu_2\nu_2}{\mu_1\nu_1 + \mu_2\nu_2} e^{-\nu_1(z+h)}\right]$$

4.80

$$\bar{u}_2 = (s^{-1}F/2\pi)\,e^{-\nu_1 h + \nu_2 y}\Big/(\mu_1\nu_1 + \mu_2\nu_2)$$

The first term in $\bar{u}_1$ in 4.80 represents the direct pulse $H(t-R/\beta_1)/4\pi R\mu_1$. The inversion of the second term in $\bar{u}_1$ in 4.80 is

$$4.81 \quad \bar{u}_1(r,z,s) = \frac{s^1 F}{4\pi^2\mu_1 i}\int_{-i\infty}^{i\infty} d\xi\, \frac{\xi}{\nu_1} K_0(\xi r)\left[\frac{\mu_1\nu_1-\mu_2\nu_2}{\mu_1\nu_1+\mu_2\nu_2}\right] e^{-\nu_1(z+h)}$$

Problem #1 (due in 2 weeks)

By using the discussion following page 36 as a guide the reader should be able to show that the transient solution corresponding to 4.81 is ($R^2 = r^2+(z+h)^2$)

$$4.82 \quad u_1(r,z,t) = \frac{H(t-R/\beta_1)\, F_\Lambda \mu}{2\pi^2\mu_1}\int_0^t d\tau\, \frac{dp}{d\tau}\, \frac{p}{\eta_1}\left[\frac{\mu_1\eta_1-\mu_2\eta_2}{\mu_1\eta_1+\mu_2\eta_2}\right]\left[(t-\eta_1(z+h))^2-(\tau-\eta_1(z+h$$

where $\tau = pr + \eta_1(z+h)$. The reader should also be able to discuss the critical refraction event when $\beta_1 < \beta_2$ and the first motion approximation using 4.39.

The integral in 4.82 must nearly always be evaluated numerically. One approximation that partially circumvents this problem and that is almost always useful in point source problems is called the large s, or "high frequency" approximation. It is a generalization of the first motion approximation.

We write 4.81 as

4.83 $$\bar{u}(r,z,s) = \frac{s\bar{f}(s)}{2\pi^2\mu_1}\,\mathrm{Im}\int_0^{i\infty} dp\,\frac{p}{\eta_1}\,K_0(spr)\left(\frac{\mu_1\eta_1-\mu_2\eta_2}{\mu_1\eta_1+\mu_2\eta_2}\right)e^{-s\eta_1 z}$$

$Re(spr)\gg 1$ implies $r/\lambda \gg 1$ high freq. or large Dist.

where we have replaced $(z+h)$ by z. For $Re\,p>0$ $r>0$ and $Re\,spr \gg 1$ we replace $K_0(spr)$ by its asymptotic approximation 4.39

4.84 $$K_0(spr) \simeq (\pi/2spr)^{1/2}e^{-spr}$$

and we let $\bar{f}(s) = F s^{-1/2}$ so that $f(t) = FH(t)/(\pi t)^{1/2}$. Then 4.83 becomes

4.85 $$\bar{u}(r,z,s) = \frac{F}{2\pi^3\mu_1}\left(\frac{\pi}{2r}\right)^{1/2}\mathrm{Im}\int_0^{i\infty} dp\,\frac{p^{1/2}}{\eta_1}\left(\frac{\mu_1\eta_1-\mu_2\eta_2}{\mu_1\eta_1+\mu_2\eta_2}\right)e^{-spr-s\eta_1 z}$$

so the time solution is

4.86 $$u(r,z,t) = \frac{F}{2\pi^3\mu_1}\left(\frac{\pi}{2r}\right)^{1/2}\mathrm{Re}\left\{\frac{p^{1/2}}{(t^2-R^2/\beta_1^2)^{1/2}}\left(\frac{\mu_1\eta_1-\mu_2\eta_2}{\mu_1\eta_1+\mu_2\eta_2}\right)\right\}$$

where $pr+\eta_1 z = t$, $R^2 = r^2+z^2$. In this approximat we obtain the solution in closed form.

5. Lamb's Problem in Two Dimensions

The plane $y=0$ is the boundary of an isotropic, homogeneous, perfectly elastic half space $y>0$. We assume that the initial stress state is isotropic and constant. The plane $y=0$ is considered to be a free surface. Then the initial constant isotropic stress state is one of zero pressure. Within the half space the linearized momentum equation is

5.1 $$(\lambda+\mu)\underline{\nabla}\,\underline{\nabla}\cdot\underline{s}+\mu\nabla^2\underline{s}-\rho\,\partial_t^2\,\underline{s}=-\underline{f}$$

We consider a line source parallel to the z axis at $x=0$, $y=h$. First we consider a force that always points normal to the line of the source, that is uniform around the line, and that is not dependent on z. Such a force vector is curl free and can be represented as

5.2 $$\underline{f}=\underline{\nabla}\Psi$$

For a ... source we take

5.3 $$\Psi=-f(t)\,\delta(x)\,\delta(y-h)$$

so the force always points away from the line source when $f(t)>0$. We may regard the source as an idealized line explosion. Due to the uniformity of the source, the displacement will have no z component. In component form we write 5.1 as

$$(\lambda+\mu)[\partial_x^2 u(x,y,t) + \partial_{yx}^2 v(x,y,t)] +$$

$$+\mu[\partial_x^2 u(x,y,t) + \partial_y^2 u(x,y,t)] - \rho\,\partial_t^2 u(x,y,t) =$$

5.4
$$= f(t)\,\delta'(x)\,\delta(y-h)$$

$$(\lambda+\mu)[\partial_{xy}^2 u(x,y,t) + \partial_y^2 v(x,y,t)] +$$

$$+\mu[\partial_x^2 v(x,y,t) + \partial_y^2 v(x,y,t)] - \rho\,\partial_t^2 v(x,y,t) =$$

$$= f(t)\,\delta(x)\,\delta'(y-h)$$

where $u = s_x$, $v = s_y$. We seek the particular solution to 5.4. We use Laplace transforms exactly as in Chapter 3 to get

5.5
$$\begin{bmatrix} (\lambda+2\mu)\xi^2+\mu\zeta^2-\rho s^2 & (\lambda+\mu)\xi\zeta \\ (\lambda+\mu)\xi\zeta & (\lambda+2\mu)\zeta^2+\mu\xi^2-\rho s^2 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} \bar{f}(s)\,\xi\, e^{-\zeta h} \\ \bar{f}(s)\,\xi\, e^{-\zeta h} \end{bmatrix}$$

where $\bar{u} = \bar{u}(\xi,\zeta,s)$, $\bar{v} = \bar{v}(\xi,\zeta,s)$

The solution to 5.5 is

$$\bar{u}(\xi, \zeta, s) = \frac{-f(s)\, e^{-\zeta h}\, \xi}{[\rho s^2 - (\xi^2 + \zeta^2)(\lambda + 2\mu)]}$$

5.6

$$\bar{v}(\xi, \zeta, s) = \frac{-f(s)\, e^{-\zeta h}\, \zeta}{[\rho s^2 - (\xi^2 + \zeta^2)(\lambda + 2\mu)]}$$

Let $\alpha^2 = (\lambda + 2\mu)/\rho$, $k_\alpha^2 = s^2/\alpha^2$, $\nu_\alpha^2 = k_\alpha^2 - \xi^2$, $\sigma = \lambda + 2\mu$

$$\bar{u}(\xi, \zeta, s) = \frac{-f(s)\, e^{-\zeta h}\, \xi}{\sigma(\nu_\alpha^2 - \zeta^2)}$$

5.7

$$\bar{v}(\xi, \zeta, s) = \frac{-f(s)\, e^{-\zeta h}\, \zeta}{\sigma(\nu_\alpha^2 - \zeta^2)}$$

In obtaining 5.5 we require $|\mathrm{Re}\, \xi| \le k_\alpha$, $|\mathrm{Re}\, \zeta| \le |\mathrm{Re}\, \nu_\alpha|$. The inversion of 5.7 is

$$\bar{u}(\xi, y, s) = \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} d\zeta \; \frac{f(s)}{\sigma} \, \xi \, \frac{e^{\zeta(y-h)}}{(\nu_\alpha^2 - \zeta^2)}$$

5.8

$$\bar{v}(\xi, y, s) = \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} d\zeta \; \frac{f(s)}{\sigma} \, \zeta \, \frac{e^{\zeta(y-h)}}{(\nu_\alpha^2 - \zeta^2)}$$

To resolve the ambiguity of sign of ν_α we require $\mathrm{Re}\,\nu_\alpha \geq 0$. Evaluating 5.8 by the residue theorem we have

5.9
$$\bar{u}(\xi, y, s) = -\frac{\bar{f}(s)\,\xi}{2\sigma\nu_\alpha} e^{-\nu_\alpha |y-h|}$$

$$\bar{v}(\xi, y, s) = \frac{\bar{f}(s)}{2\sigma} e^{-\nu_\alpha |y-h|} \operatorname{sgn}(y-h)$$

The ξ inversion of 5.9 is

5.10
$$\bar{u}(x, y, s) = -\frac{\bar{f}(s)}{4\pi\sigma i}\int_{-i\infty}^{i\infty} d\xi \, \frac{\xi}{\nu_\alpha} e^{\xi x - \nu_\alpha |y-h|}$$

$$\bar{v}(x, y, s) = \frac{\bar{f}(s)}{4\pi\sigma i}\int_{-i\infty}^{i\infty} d\xi \, \operatorname{sgn}(y-h)\, e^{\xi x - \nu_\alpha |y-h|}$$

If we let $\bar{f}(s) = F s^{-1}$, $r = x^2 + (y-h)^2$, $x = r \sin\theta$, $(y-h) = r\cos\theta$, then it is clear from our previous remarks that

5.11
$$u(x, y, t) = \frac{F}{2\pi\sigma} H(t - r/\alpha) \frac{(t/r)\sin\theta}{(t^2 - r^2/\alpha^2)^{1/2}}$$

$$v(x, y, t) = \frac{F}{2\pi\sigma} H(t - r/\alpha) \frac{(t/r)\cos\theta}{(t^2 - r^2/\alpha^2)^{1/2}}$$

From 5.11 we see that the motion is purely radial with respect to the line source. Let U be the radial motion

5.12 $$U = u \sin\theta + v \cos\theta = \frac{F}{2\pi\sigma} \frac{H(t - r/\alpha)(t/r)}{(t^2 - r^2/\alpha^2)^{1/2}}$$

Thus the displacement is curl free and can be represented as

$$U = d\varphi/dr$$

where

5.13 $$\varphi(r,t) = \frac{-F}{2\pi\sigma} H(t - r/\alpha) \log\left[\frac{\alpha t + (\alpha^2 t^2 - r^2)^{1/2}}{r}\right]$$

Eq. 5.13 is valid for a step function source. For a delta function $\delta(s) = F$

5.14 $$\varphi(r,t) = \frac{-F}{2\pi\sigma} \frac{H(t - r/\alpha)}{(t^2 - r^2/\alpha^2)^{1/2}}$$

Now that we have obtained the particular solution to 5.5 we seek the homogeneous solution bounded for $y > 0$. For homogeneous solutions the right hand side of 5.5 is zero. Then the determinant of the 2×2 matrix in 5.5 must vanish. The determinant is

5.15 $$\Delta = [\rho s^2 - (\varsigma^2 + \xi^2)(\lambda + 2\mu)][\rho s^2 - (\varsigma^2 + \xi^2)\mu] = 0$$

Let $\beta^2 = \mu/\rho$, $k_\beta^2 = s^2/\beta^2$, $\nu_\beta^2 = k_\beta^2 - \xi^2$.

5.16 $$\Delta = \sigma^{-1}\mu^{-1}(\nu_\alpha^2 - \varsigma^2)(\nu_\beta^2 - \varsigma^2) = 0$$

Thus we must have $\varsigma = \pm\nu_\beta, \pm\nu_\alpha$. Consider the homogeneous part of 5.4. after taking Laplace transforms on t and x we have

5.17 $$\begin{bmatrix} (\lambda+2\mu)\xi^2 + \mu d_y^2 - \rho s^2 & (\lambda+\mu)\xi d_y \\ (\lambda+\mu)\xi d_y & (\lambda+2\mu)d_y^2 + \mu\xi^2 - \rho s^2 \end{bmatrix} \begin{bmatrix} \bar{u}(\xi, y, s) \\ \bar{v}(\xi, y, s) \end{bmatrix} = [0]$$

Eq. 5.17 is a pair of coupled, ordinary, linear differential equations in y with constant coefficients. Then its solutions must be exponentials in y.

5.18 $$\begin{bmatrix} \bar{u}(\xi, y, s) \\ \bar{v}(\xi, y, s) \end{bmatrix} = \begin{bmatrix} \bar{u}(\xi, s)\, e^{\varsigma y} \\ \bar{v}(\xi, s)\, e^{\varsigma y} \end{bmatrix}$$

In order to satisfy 5.17 we must have $\zeta = \pm\nu_\alpha, \pm\nu_\beta$ in 5.18. We take $\operatorname{Re}\nu_\alpha, \nu_\beta \geq 0$ so, for bounded solutions $\zeta = -\nu_\alpha, -\nu_\beta$. The adjoint of 5.5 is

5.19 $$\begin{bmatrix} (\lambda+2\mu)\zeta^2 + \mu\xi^2 - \rho s^2 & -(\lambda+\mu)\xi\zeta \\ -(\lambda+\mu)\xi\zeta & (\lambda+2\mu)\xi^2 + \mu\zeta^2 - \rho s^2 \end{bmatrix}$$

When $\zeta = -\nu_\alpha, -\nu_\beta$ either column of 5.19 is a homogeneous solution to 5.5. For $\zeta = -\nu_\alpha$ 5.18 is

5.20 $$\begin{bmatrix} \bar{u}(\xi,y,s) \\ \bar{v}(\xi,y,s) \end{bmatrix} = A e^{-\nu_\alpha y} \begin{bmatrix} (\lambda+2\mu)\nu_\alpha^2 + \mu\xi^2 - \rho s^2 \\ (\lambda+\mu)\xi\nu_\alpha \end{bmatrix}$$

For $\zeta = -\nu_\beta$ 5.18 is

5.21 $$\begin{bmatrix} \bar{u}(\xi,y,s) \\ \bar{v}(\xi,y,s) \end{bmatrix} = B e^{-\nu_\beta y} \begin{bmatrix} (\lambda+\mu)\xi\nu_\beta \\ (\lambda+2\mu)\xi^2 + \mu\nu_\beta^2 - \rho s^2 \end{bmatrix}$$

Combining 5.20 and 5.21 we have

$$\begin{bmatrix} \bar{u}(\xi, y, s) \\ \bar{v}(\xi, y, s) \end{bmatrix} = A e^{-\nu_\alpha y} \begin{bmatrix} \sigma \nu_\alpha^2 - \mu \nu_\beta^2 \\ (\sigma - \mu)\xi \nu_\alpha \end{bmatrix} + B e^{-\nu_\beta y} \begin{bmatrix} (\sigma - \mu)\xi \nu_\beta \\ -\sigma \nu_\alpha^2 + \mu \nu_\beta^2 \end{bmatrix} \tag{5.22}$$

In order to determine A and B we must apply the boundary conditions. Let $\underline{\underline{\tau}}_0$ be the stress at $y=0$. When pulses generated by the body force impinge the boundary, it will be distorted. A particle initially on the boundary at $x = x_0$, $y = 0$ will move with the boundary to $x = x_0 + s_x(x_0, 0, t)$, $y = s_y(x_0, 0, t)$. The normal to the undeformed boundary is $\hat{j}$. The normal to the deformed boundary is proportional to $\hat{j} + \partial_y \underline{s} + \hat{j} \times (\underline{\nabla} \times \underline{s})$. The stress $\underline{\underline{\tau}}(x_0 + s_x, s_y)$ is expanded in a Taylor series

$$\underline{\underline{\tau}} = \underline{\underline{\tau}}_0 + (\underline{s} \cdot \underline{\nabla}) \underline{\underline{\tau}}_0 + \cdots$$

so the boundary condition is

$$[\hat{j} + \partial_y \underline{s} + \hat{j} \times (\underline{\nabla} \times \underline{s})] \cdot [\underline{\underline{\tau}}_0 + (\underline{s} \cdot \underline{\nabla}) \underline{\underline{\tau}}_0 + \cdots] = 0 \tag{5.23}$$

Since we have already linearized the momentum equation we now linearize 5.23

$$\hat{j} \cdot \underline{\underline{\tau}}_0 = 0 \tag{5.24}$$

In component form 5.24 is

$$\tau_{yx} = \mu\,(\partial_y s_x + \partial_x s_y) = 0$$

5.24 $$\tau_{yy} = \lambda\,(\partial_x s_x + \partial_z s_z) + (\lambda + 2\mu)\,\partial_y s_y = 0$$

$$\tau_{yz} = \mu\,(\partial_z s_y + \partial_y s_z) = 0$$

In our two dimensional problem $s_z = 0$, $\partial_z = 0$ so 5.24 is

$$\tau_{yx} = \mu\,(\partial_y u + \partial_x v) = 0$$

5.25 $$\tau_{yy} = \lambda\,\partial_x u + (\lambda + 2\mu)\,\partial_y v = 0$$

$$\tau_{yz} \equiv 0$$

The Laplace transforms of 5.25 with respect to t and x give

$$\bar{\tau}_{yx}(\xi, 0, s) = \mu\left(d_y\,\bar{u}(\xi, y, s)\Big|_{y=0} + \xi\,\bar{v}(\xi, 0, s)\right) = 0$$

5.26 $$\bar{\tau}_{yy}(\xi, 0, s) = \lambda\,\xi\,\bar{u}(\xi, 0, s) + (\lambda + 2\mu)\,\partial_y\,\bar{v}(\xi, y, s)\Big|_{y=0} = 0$$

In 5.22 we see that $\sigma\nu_\alpha^2 - \mu\nu_\beta^2 = -(\sigma - \mu)\,\xi^2$. Thus $(\mu - \sigma)\,\xi$ is a common factor in the homogeneous solution and we may neglect it to write 5.22 as

5.27 $$\begin{bmatrix} \bar{u}(\xi, y, s) \\ \bar{v}(\xi, y, s) \end{bmatrix} = A e^{-\nu_\alpha y} \begin{bmatrix} \xi \\ -\nu_\alpha \end{bmatrix} + B e^{-\nu_\beta y} \begin{bmatrix} -\nu_\beta \\ -\xi \end{bmatrix}$$

To take account of the source we add the particular solution 5.9 to 5.27

$$\bar{u}(\xi, y, s) = -\frac{\bar{f}(s)\xi}{2\sigma\nu_\alpha} e^{-\nu_\alpha|y-h|} + A\xi e^{-\nu_\alpha y} - B\nu_\beta e^{-\nu_\beta y}$$

5.28 $$\bar{v}(\xi, y, s) = \frac{\bar{f}(s)}{2\sigma} \operatorname{sgn}(y-h) e^{-\nu_\alpha|y-h|} - A\nu_\alpha e^{-\nu_\alpha y} - B\xi e^{-\nu_\beta y}$$

To determine A and B we substitute 5.28 into 5.26 to get

$$\mu\left[-\frac{\bar{f}(s)\xi}{2\sigma} e^{-\nu_\alpha h} - 2A\xi\nu_\alpha + B(\nu_\beta^2 - \xi^2)\right] = 0$$

5.29 $$\lambda\left[-\frac{\bar{f}(s)\xi^2}{2\sigma\nu_\alpha} e^{-\nu_\alpha h} + A\xi^2 - B\nu_\beta\xi\right] + \sigma\left[-\frac{\bar{f}(s)\nu_\alpha}{2\sigma} e^{-\nu_\alpha h} + A\nu_\alpha^2 + B\xi\nu_\beta\right] = 0$$

The relation $\lambda\xi^2 + \sigma\nu_\alpha^2 = \mu(\nu_\beta^2 - \xi^2)$ is used to simplify 5.29

$$\mu\left[-\frac{i\bar{f}(s)\xi}{\sigma}e^{-\nu_\alpha h}-2A\xi\nu_\alpha+B(\nu_\beta^2-\xi^2)\right]=0$$

5.30

$$\mu\left[-\frac{\bar{f}(s)}{2\sigma\nu_\alpha}(\nu_\beta^2-\xi^2)e^{-\nu_\alpha h}+A(\nu_\beta^2-\xi^2)+2B\xi\nu_\beta\right]=0$$

Solving 5.30 for A and B we have

$$A=\frac{\bar{f}(s)}{2\sigma\nu_\alpha}\left[\frac{(\nu_\beta^2-\xi^2)^2-4\xi^2\nu_\alpha\nu_\beta}{(\nu_\beta^2-\xi^2)^2+4\xi^2\nu_\alpha\nu_\beta}\right]e^{-\nu_\alpha h}$$

5.31

$$B=\frac{\bar{f}(s)}{2\sigma\nu_\alpha}\left[\frac{4\xi\nu_\alpha(\nu_\beta^2-\xi^2)}{(\nu_\beta^2-\xi^2)^2+4\xi^2\nu_\alpha\nu_\beta}\right]e^{-\nu_\alpha h}$$

If we substitute 5.31 into 5.28 we get the final expressions for the doubly transformed displacements. For the present we confine our attention to the displacements on the surface of the half-space

$$\bar{u}(\xi,0,s)=-\frac{\bar{f}(s)}{\sigma}\,\frac{2\xi\nu_\beta k_\beta^2}{R(\xi)}\,e^{-\nu_\alpha h}$$

5.32

$$\bar{v}(\xi,0,s)=-\frac{\bar{f}(s)}{\sigma}\,\frac{(\nu_\beta^2-\xi^2)k_\beta^2}{R(\xi)}\,e^{-\nu_\alpha h}$$

where $R(\xi)=(\nu_\beta^2-\xi^2)^2+4\xi^2\nu_\alpha\nu_\beta$.

The inversion of 5.32 is

$$\bar{u}(x,0,s) = -\frac{2k_\beta^2}{\sigma}\bar{f}(s)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi\, \frac{\xi\nu_\beta}{R(\xi)}\, e^{\xi x - \nu_\alpha h}$$

5.33

$$\bar{v}(x,0,s) = -\frac{k_\beta^2}{\sigma}\bar{f}(s)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi\, \frac{(\nu_\beta^2-\xi^2)}{R(\xi)}\, e^{\xi x - \nu_\alpha h}$$

Using the principle of reflection we write 5.33 as

$$\bar{u}(x,0,s) = -\frac{2k_\beta^2\bar{f}(s)}{\sigma\pi}\,\mathrm{Im}\int_{0}^{i\infty} d\xi\, \frac{\xi\nu_\beta}{R(\xi)}\, e^{\xi x - \nu_\alpha h}$$

5.34

$$\bar{v}(x,0,s) = -\frac{k_\beta^2\bar{f}(s)}{\sigma\pi}\,\mathrm{Im}\int_{0}^{i\infty} d\xi\, \frac{(\nu_\beta^2-\xi^2)}{R(\xi)}\, e^{\xi x - \nu_\alpha h}$$

In 5.34 $R(\xi)$ has simple zeros at $\xi = \pm k_\gamma = \pm s/\gamma$ where $\gamma < \beta$. We shall see that these zeros are directly related to the Rayleigh pulse whose speed is γ. To ensure $\mathrm{Re}\,\nu_\alpha, \nu_\beta > 0$ we cut the ξ plane as shown in Fig. 5.1

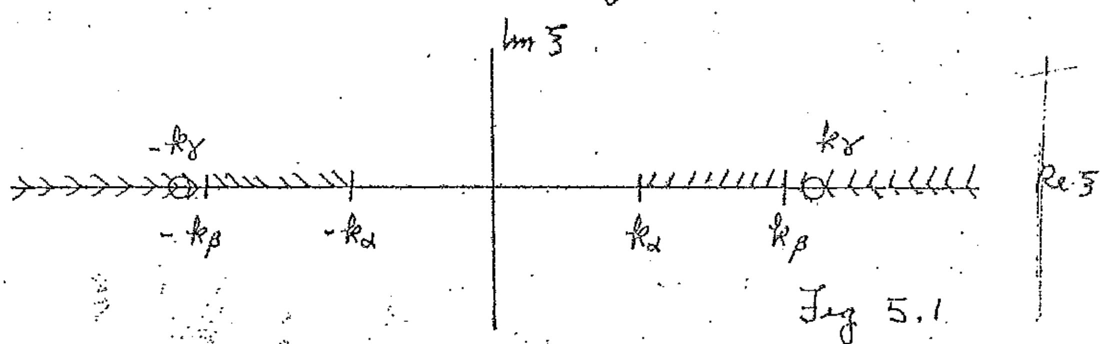

Fig 5.1

Let $\bar{s} = s p$, $\nu_\alpha = s\eta_\alpha$, $\nu_\beta = s\eta_\beta$, $\bar{f}(s) = F/s$
$\eta_\alpha^2 = (1/\alpha^2 - p^2)^{1/2}$, $\eta_\beta^2 = (1/\beta^2 - p^2)^{1/2}$, $Re\ \eta_\alpha, \eta_\beta \geq 0$.

5.35

$$\bar{u}(x,0,s) = \frac{-2F}{\sigma\pi\beta^2} \operatorname{Im} \int_0^{i\infty} dp\, \frac{p\eta_\beta}{R(p)}\, e^{spx - s\eta_\alpha h}$$

$$\bar{v}(x,0,s) = \frac{-F}{\sigma\pi\beta^2} \operatorname{Im} \int_0^{i\infty} dp\, \frac{(\eta_\beta^2 - p^2)}{R(p)}\, e^{spx - s\eta_\alpha h}$$

where $R(p) = (\eta_\beta^2 - p^2)^2 + 4p^2\eta_\alpha\eta_\beta$. $R(p)$ has zeros at $p = \pm\gamma^{-1}$. Let $r = (x^2+h^2)^{1/2}$, $x = r\sin\theta$, $h = r\cos\theta$. We require $px - \eta_\alpha h = -t$. Then

5.36

$$u(x,0,t) = \frac{-2F\,H(t - r/\alpha)}{\sigma\pi\beta^2(t^2 - r^2/\alpha^2)^{1/2}}\ Re\left[\frac{p\,\eta_\alpha\eta_\beta}{R(p)}\right]$$

$$v(x,0,t) = \frac{-F\,H(t - r/\alpha)}{\sigma\pi\beta^2(t^2 - r^2/\alpha^2)^{1/2}}\ Re\left[\frac{\eta_\alpha(\eta_\beta^2 - p^2)}{R(p)}\right]$$

where

5.37 $p = -t/r\sin\theta + i(t^2/r^2 - 1/\alpha^2)^{1/2}\cos\theta$

A more traditional approach to derive 5.36 begins with the representation

5.38 $\underline{s} = \underline{\nabla}\varphi(x, y, t) + \underline{\nabla} \times (\hat{z}\,\omega(x, y, t))$

With $\underline{f}$ given by 5.2 and Ψ by 5.3, 5.1 is written as

5.39 $(\lambda + 2\mu)\underline{\nabla}\,\underline{\nabla}\cdot\underline{s} - \mu\,\underline{\nabla}\times\underline{\nabla}\times\underline{s} - \rho\,\partial_t^2\,\underline{s} = -\underline{\nabla}\Psi$

Using 5.38 we can write the divergence of 5.3 as

5.40 $\nabla^2\left[(\lambda + 2\mu)\nabla^2\varphi - \rho\,\partial_t^2\varphi - \Psi\right] = 0$

and the curl of 5.39 as

5.41 $\nabla^2\left[\mu\nabla^2\omega - \rho\,\partial_t^2\omega\right] = 0$

The bracketed terms in 5.40 and 5.41 are solutions to Laplace's equation which are, therefore, harmonic functions. For $t < 0$ we have $\varphi \equiv 0$, $\omega \equiv 0$. Therefore the bracketed terms must always be zero

5.42

$$(\lambda + 2\mu)\nabla^2\varphi - \rho\,\partial_t^2\varphi = \Psi$$

$$\mu\nabla^2\omega - \rho\,\partial_t^2\omega = 0$$

When transformed solutions to 5.42 are ~~substituted~~ into 5.38 we obtain the particular solution 5.9 and the homogeneous solution 5.27.

Numerical evaluation of 5.36 has been performed for several values of $\sigma = (\alpha^2/\beta^2 - 2)/(2\alpha^2/\beta^2 - 2)$ by Gilbert and Laster (1962)

F. Gilbert and S.J. Laster, 1962) Excitation and Propogation of Pulses on an Interface: Bull. Seismol. Soc. Amer., 52, 299-319

You are referred to that paper for a discussion of the effect of the zeros of R(p) on the transient response.

6. Various approximations in Pulse Propagation Problems

Consider the SH pulse generated by a line source (3.16)

3.16 $$\bar{u}(x, y, s) = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} d\xi \, (f(s)/2\mu\nu) \, e^{\xi x - \nu|y-h|}$$

where $\nu^2 = k^2 - \xi^2$. Let $\xi = sp$, $\nu = s\eta$ and use the principle of reflection to write 3.16 as

6.1 $$\bar{u}(x, y, s) = (f(s)/2\pi\mu) \, \mathrm{Im} \int_0^{i\infty} dp \, \eta^{-1} \, e^{s(px - \eta|y-h|)}$$

To obtain the first motion approximation to $u(x, y, t)$ we seek an approximate evaluation of 6.1 that is good for large s. The classical approach to this problem is to approximate 6.1 by the method of steepest descent. First we find the saddlepoints

6.2 $$\frac{d}{dp}(px - \eta|y-h|) = 0 = x + p\eta^{-1}|y-h|$$

Let $r^2 = x^2 + (y-h)^2$, $x = r \sin\theta$, $|y-h| = r\cos\theta$. Then 6.2 can be written as

6.3 $$\eta \sin\theta + p\cos\theta = 0$$

Those values of p that solve 6.3 are the saddlepoints

Thus when $p = p_0 = -\sin\theta/\beta$ 6.2 and 6.3 are satisfied. Notice that the saddlepoint is the point where the $p(t)$ path in 3.27 leaves the real axis (Fig 3.4, page 32). The path of steepest descent is the path leaving the saddlepoint along which the imaginary part of the exponent has as small magnitude as possible. Along the $p(t)$ path in 3.27 the imaginary part of the exponent is zero so it is clear that the $p(t)$ path is the path of steepest descent. For large s most of the contribution to 6.1 comes from the vicinity of the saddlepoint. Let

6.4 $$g(p) = px - \eta|y-h| \quad ; \quad g(p_0) = -r/\beta$$

Then for $|p - p_0| << 1$ we expand $g(p)$ in a Taylor series

6.5 $$g(p) = g(p_0) + g^{(1)}(p_0)(p-p_0) + \cdots + g^{(n)}(p_0)(p-p_0)^n/n! + \cdots$$

By definition $g^{(1)}(p_0) = 0$. For $g^{(2)}$ and $g^{(3)}$ we have

6.6
$$g^{(2)}(p_0) = r\beta/\cos^2\theta$$
$$g^{(3)}(p_0) = -3r\beta^3\sin\theta/\cos^4\theta$$

If $|p - p_0| << \cos^3\theta / \beta \sin\theta$ we may neglect the $n=3$ term in 6.5. For $p \approx p_0$, $\ddot{g} \approx \cos\theta/\beta$ so we write 6.1 as

6.7 $$\bar{u}(x,y,s) = \frac{\bar{f}(s)}{2\pi\mu} \frac{\beta}{\cos\theta} e^{-sr/\beta} \operatorname{Im} \int_{p_0}^{p_0+i\infty} dp\, e^{s g^{(2)}(p_0)(p-p_0)^2/2}$$

Let $p - p_0 = i\varpi$

$$\bar{u}(x,y,s) = \frac{\bar{f}(s)}{2\pi\mu} \frac{\beta}{\cos\theta} e^{-sr/\beta} \int_0^\infty d\varpi\, e^{-s g^{(2)}(p_0)\varpi^2/2}$$

6.8 $$= \frac{\bar{f}(s)}{2\pi\mu} \frac{\beta}{\cos\theta} e^{-sr/\beta} \frac{1}{2} \left(\frac{2\pi}{s g^{(2)}(p_0)} \right)^{1/2}$$

$$= \frac{\bar{f}(s)}{2\mu (2\pi s r/\beta)^{1/2}} e^{-sr/\beta}$$

Let $\bar{f}(s) = F$, $f(t) = F\delta(t)$. The image of $s^{-1/2}$ is $(\pi t)^{-1/2}$ so the image of 6.8 is

6.9 $$u(x,y,t) \simeq \frac{F\, H(t - r/\beta)}{2\pi\mu (2r/\beta (t - r/\beta))^{1/2}}$$

which is the first motion approximation to 3.31 (page 33).

As a second example we consider the reflected SH pulse (3.42½, page 37)

$$3.42\tfrac{1}{2}.\quad \bar{u}(x,y,s) = \frac{F}{2\mu_1\pi}\, \mathrm{Im} \int_0^{i\infty} \frac{dp}{\eta_1}\, \frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\, e^{spx - s\eta_1(y+h)}$$

Let $r^2 = x^2 + (y+h)^2$, $x = r\sin\theta$, $(y+h) = r\cos\theta$, $g(p) = px - \eta_1(y+h)$. Then

$$p_0 = -\sin\theta/\beta_1$$

$$g(p_0) = -r/\beta_1$$

$$6.10 \qquad g^{(1)}(p_0) = 0$$

$$g^{(2)}(p_0) = r\beta_1/\cos^2\theta$$

$$g^{(3)}(p_0) = -3r\beta_1^2\sin\theta/\cos^4\theta$$

For $\beta_1 > \beta_2$ we have $-1/\beta_2 < -1/\beta_1 \leq p_0$. The saddlepoint, for $x > 0$, lies on the real axis in the left half p-plane to the right of $-1/\beta_1$ as shown in Fig 6.1

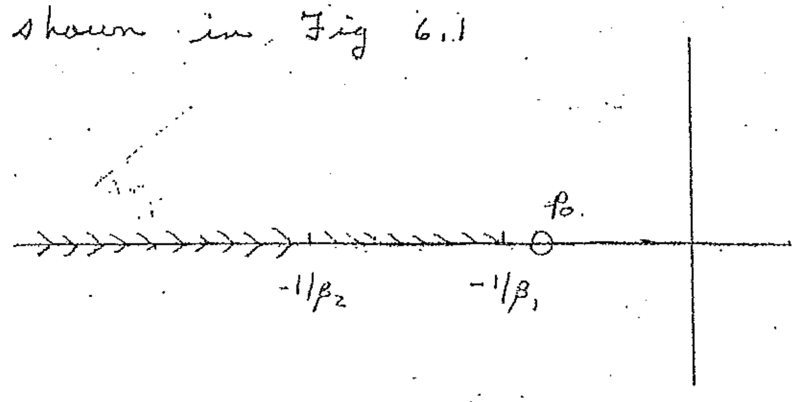

Fig 6.1

In 3.42½ we approximate the reflection coefficient by taking its value at p_0

6,11 $$\mathcal{R}(p_0) = \left.\frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}\right|_{p=p_0} = \frac{\mu_1 \cos\theta/\beta_1 - \mu_2(1/\beta_2^2 - \sin^2\theta/\beta_1^2)^{1/2}}{\mu_1 \cos\theta/\beta_1 + \mu_2(1/\beta_2^2 - \sin^2\theta/\beta_1^2)^{1/2}}$$

For large s the approximation to 3.42½ is

6,12 $$\bar{u}(x,y,s) \simeq \frac{F}{2\mu_1\pi}\,\frac{\beta_1}{\cos\theta}\,\mathcal{R}(p_0)\,e^{-sr/\beta_1}\,\mathrm{Im}\int_{p_0}^{p_0+i\infty} dp\, e^{s g^{(2)}(p_0)(p-p_0)^2/2}$$

Let $p - p_0 = i\varpi$. Then

6,13 $$\bar{u}(x,y,s) \simeq \frac{F}{2\mu_1\pi}\,\frac{\beta_1}{\cos\theta}\,\mathcal{R}(p_0)\,e^{-sr/\beta_1}\int_0^{\infty} d\varpi\, e^{-s g^{(2)}(p_0)\varpi^2/2}$$

which leads directly to 3.54 (page 43).

For $\beta_1 < \beta_2$ we write 3.42½ as an integral from 0 to p_0 to $p_0 + i\infty$. For $\theta < \theta_c$ The part from 0 to p_0 contributes nothing and we have 6,13 again. For $\theta > \theta_c$ η_2 is imaginary, $\eta_2 = i(\sin^2\theta/\beta_1^2 - 1/\beta_2^2)^{1/2}$ and 6.11 becomes

6,14 $$\mathcal{R}(p_0) = \frac{\mu_1 \cos\theta/\beta_1 - i\mu_2(\sin^2\theta/\beta_1^2 - 1/\beta_2^2)^{1/2}}{\mu_1 \cos\theta/\beta_1 + i\mu_2(\sin^2\theta/\beta_1^2 - 1/\beta_2^2)^{1/2}}$$

We write 3.42½ as

6,15 $$\bar{u}(x,y,s) = \frac{F}{2\mu_1\pi} \text{Im}\left[\int_{-1/\beta_2}^{p_0} dp + \int_{p_0}^{p_0+i\infty} dp\right] \frac{1}{\eta_1} R(p)\, e^{spx - s\eta_1(y+h)}$$

where the integration path is shown in Fig 6.2

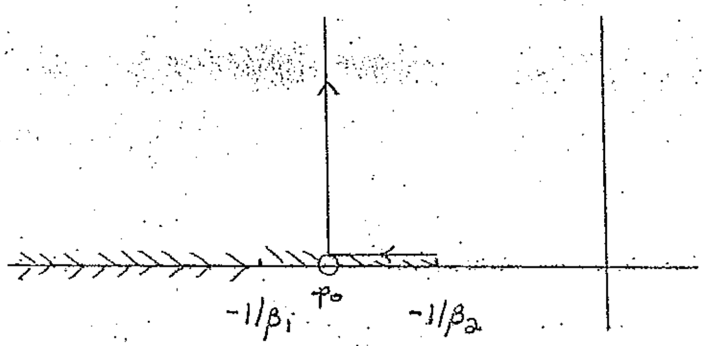

Fig 6.2
$\theta > \theta_c$, $\beta_1 < \beta_2$

For large s we approximate the second integral in 6,15 for $p \simeq p_0$

6,16 $$\bar{u}_2(x,y,s) \simeq \frac{F}{2\mu_1\pi} \frac{\beta_1}{\cos\theta} \text{Re}[R(p_0)]\, e^{-sr/\beta_1} \int_{p_0}^{p_0+i\infty} dp\, e^{s g^{(2)}(p_0)(p-p_0)^2/2}$$

The new feature here is the fact that $R(p_0)$ is complex. But we are still led to 6,13 with $R(p_0)$ replaced by its real part.

The first integral in 6,15 represents a transient that begins at $t = t_c$ (3,49) and that ends at $t = r/\beta_1$. We have already examined the approximation for $t \simeq t_c$ and have seen that it represents the critical refraction (3.56 and pages 45-49). We must now obtain an approximation near $p = p_0$ for the first integral in 6,15 so we can obtain the "last motion" near $t = r/\beta_1$. Consider the simple transient

6,17 $$f(t) = H(t-t_1) - H(t-t_2) \quad , \quad t_2 > t_1 > 0$$

whose Laplace transform is

6,18 $$\bar{f}(s) = s^{-1}\left(e^{-st_1} - e^{-st_2}\right)$$

To find $f(t_1)$, which we know is 1, we use the relation

6,19 $$f(t_1) = \lim_{s\to\infty} e^{st_1} s\bar{f}(s) = \lim_{s\to\infty} \left(1 - e^{s(t_2-t_1)}\right) = 1$$

Now look at $e^{st_2} s\bar{f}(s)$

6,20 $$e^{st_2} s\bar{f}(s) = e^{s(t_2-t_1)} - 1$$

Since $t_2 > t_1$ we see that if we let $s \to \infty$ 6.20 diverges. However if we take the limit as $s \to -\infty$ we recover the last motion which is -1. Thus, for last motions it appears that we should consider s large negative.

Near $p = p_0$ in the first integral of 6,15 we have the approximation

6,21 $$\bar{u}_1(x,y,s) \simeq \frac{F}{2\mu_1\pi} \frac{\beta_1}{\cos\theta} \operatorname{Im}[R(p_0)] e^{-sr/\beta_1} \int_{\infty}^{p_0} dp\, e^{s g^{(2)}(p_0)(p-p_0)^2/2}$$

In the integral $g^{(2)}(p_0) > 0$ and, for the last motion, we let $s = -|s|$, $|s| >> 1$. also we let $p - p_0 = \varpi$

$$\bar{u}_1(x,y,s) \simeq \frac{-F}{2\mu_1\pi} \frac{\beta_1}{\cos\theta} \operatorname{Im}[R(p_0)] e^{-sr/\beta_1} \int_0^{\infty} d\varpi\, e^{-|s| g^{(2)}(p_0)\varpi^2/2}$$

6,22 $$= \frac{-F}{2\mu_1\pi} \frac{\beta_1}{\cos\theta} \operatorname{Im}[R(p_0)] e^{-sr/\beta_1} \frac{1}{2}\left(\frac{2\pi}{|s|\, g^{(2)}(p_0)}\right)^{1/2}$$

To interpret 6,22 consider the integral

6,23 $$f(s) = \int_{-\infty}^{t_0} \frac{e^{-st}}{(t-t_0)^{1/2}}\, dt$$

This integral will converge for $s = -|s|$ and it is

6,24 $$f(s,\quad e^{-st_0}(\pi/|s|)^{1/2}$$

From 6.24 we conclude that 6.22 represents the last motion

6.25 $$u_1(x, y, t) \simeq \frac{-F}{2\mu_1 \pi} H(r/\beta - t) \left(\frac{\beta_1}{2r}\right)^{1/2} \frac{\operatorname{Im}[R(p_0)]}{(r/\beta_1 - t)^{1/2}}$$

From 6.25 and 6.16 we see that the refracted event ends with a relative amplitude $-\operatorname{Im} R(p_0)$ and that the reflected event begins with a relative amplitude $\operatorname{Re} R(p_0)$. Such behavior is usually described by saying that a phase change of $\pi/2$ has occurred.

Next we turn our attention to the transmitted SH pulse of Chapter 3 (page 36, 3.39)

6.26 $$\bar{u}(x, y, s) = \frac{F}{\pi} \operatorname{Im} \int_0^{i\infty} dp \frac{e^{s(px - \eta_1 h + \eta_2 y)}}{\mu_1 \eta_1 + \mu_2 \eta_2} \qquad y < 0$$

We define $g(p) = px - \eta_1 h + \eta_2 y = px - \eta_1 h - \eta_2 |y|$
Then the saddlepoint is that value of p such that

6.27 $$g^{(1)}(p) = x + p\eta_1^{-1} h + p\eta_2^{-1} |y| = 0$$

Let $X = x_1 + x_2$, $r_1^2 = x_1^2 + h^2$, $r_2^2 = x_2^2 + y^2$, $x_1 = r_1 \sin\theta_1$, $x_2 = r_2 \sin\theta_2$, $h = r_1 \cos\theta_1$, $|y| = r_2 \cos\theta_2$. We seek the common solution to

$$x_1 + p\eta_1^{-1} h = 0$$

6.28

$$x_2 + p\eta_2^{-1} |y| = 0$$

The first of 6.28 has the solution $p_1 = -\sin\theta_1/\beta_1$ and the second has $p_2 = -\sin\theta_2/\beta_2$. Thus we must have $p_1 = p_2$ or

6.29 $$\sin\theta_1/\beta_1 = \sin\theta_2/\beta_2$$

which is Snell's law. With $p_0 = -\sin\theta_1/\beta_1 = -\sin\theta_2/\beta_2$ we can draw the ray diagram in Fig 6.3.

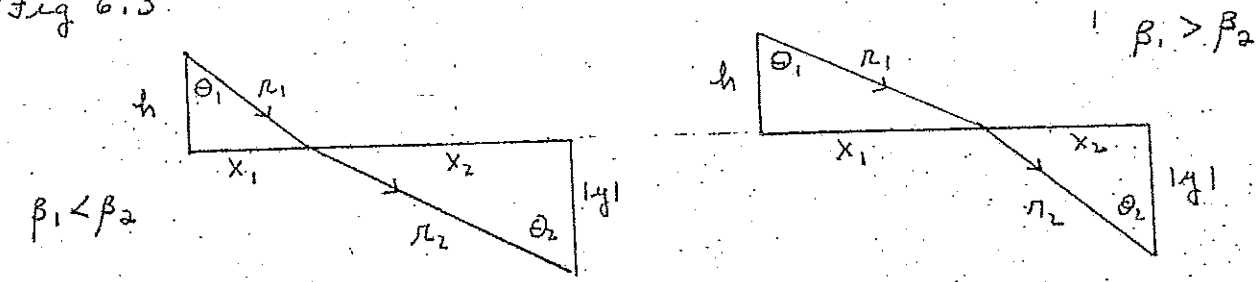

Fig. 6.3 Ray interpretation of the saddlepoint.

For $p = p_0$ the derivatives of $g(p)$ are

$$g^{(0)}(p_0) = -r_1/\beta_1 - r_2/\beta_2$$

$$g^{(1)}(p_0) = 0.$$

6.30 $$g^{(2)}(p_0) = r_1\beta_1/\cos^2\theta_1 + r_2\beta_2/\cos^2\theta_2$$

$$g^{(3)}(p_0) = -3r_1\beta_1^2 \sin\theta_1/\cos^4\theta_1 - 3r_2\beta_2^3 \sin\theta_2/\cos^4\theta_2$$

and the approximation to 6.26 is

6.31 $$\bar{u}(x,y,s) \simeq \frac{F}{\pi}\,\frac{e^{-s(r_1/\beta_1 + r_2/\beta_2)}}{\mu_1 \cos\theta_1/\beta_1 + \mu_2 \cos\theta_2/\beta_2}\int_{p_0}^{p_0+i\infty} dp\; e^{s(p-p_0)^2 g^{(2)}(p_0)/2}$$

$$= \frac{F}{\pi}\,\frac{e^{-s(r_1/\beta_1 + r_2/\beta_2)}}{\mu_1 \cos\theta_1/\beta_1 + \mu_2 \cos\theta_2/\beta_2}\;\frac{1}{2}\left(\frac{2\pi}{s\, g^{(2)}(p_0)}\right)^{1/2}$$

Then

6.32 $$u(x,y,t) \simeq \frac{F H(t - r_1/\beta_1 - r_2/\beta_2)}{\pi(\mu_1 \cos\theta_1/\beta_1 + \mu_2 \cos\theta_2/\beta_2)}\;\frac{(t - r_1/\beta_1 - r_2/\beta_2)^{-1/2}}{(2r_1\beta_1/\cos^2\theta_1 + 2r_2\beta_2/\cos^2\theta_2)^{1/2}}$$

and 6.32 is the first motion approximation for the transmitted pulse. When $\beta_1 = \beta_2$ and $\mu_1 = \mu_2$ 6.32 reduces to 6.9.

As a final example we obtain an approximation for the Rayleigh pulse in 5.36 (page 83). We begin with 5.35

$$\bar{u}(x,0,s) = \frac{-2F}{\sigma\pi\beta^2}\,\mathrm{Im}\int_0^{i\infty} dp\;\frac{p\,\eta_\beta}{R(p)}\,e^{spx - s\eta_\alpha h}$$

5.35 $$\bar{v}(x,0,s) = \frac{-F}{\sigma\pi\beta^2}\,\mathrm{Im}\int_0^{i\infty} dp\;\frac{(\eta_\beta^2 - p^2)}{R(p)}\,e^{spx - s\eta_\alpha h}$$

$R(p)$ has zeros at $p = \pm\gamma^{-1}$, $\gamma < \beta$. The integrands in 5.35 have simple poles at $p = \pm\gamma^{-1}$ and branch points at $p = \pm\beta^{-1}, \pm\alpha^{-1}$ as shown in Fig. 6.4

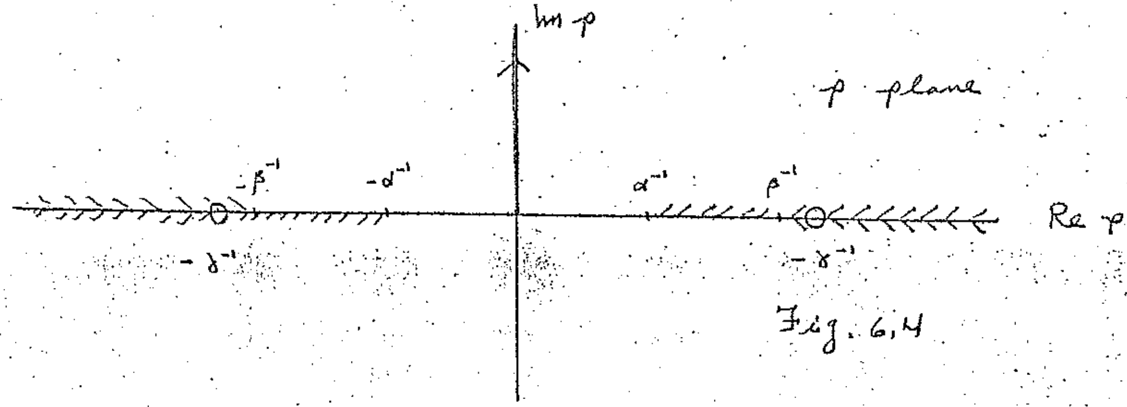

Fig. 6.4

We consider $X > 0$. We wish to shift the path of integration into the second quadrant. To facilitate the discussion we shift the branch cuts in the left half plane as shown in Fig 6.5

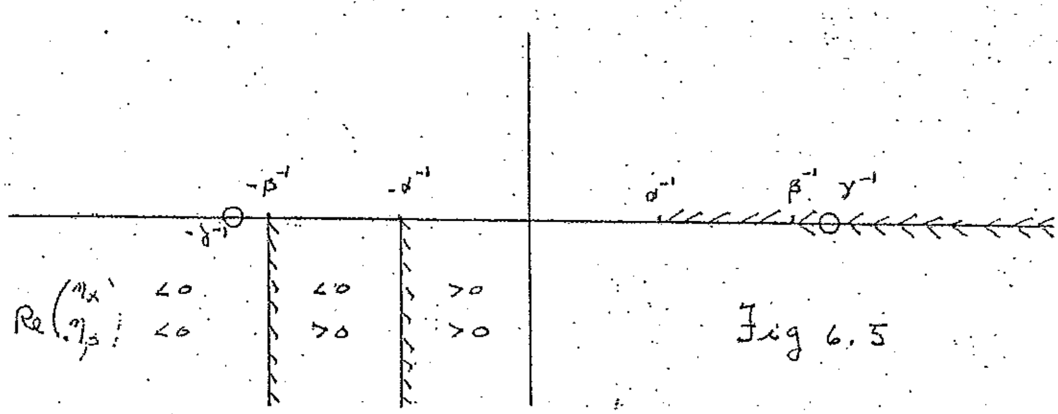

Fig 6.5

Next we shift the integration path as shown in Fig 6.6

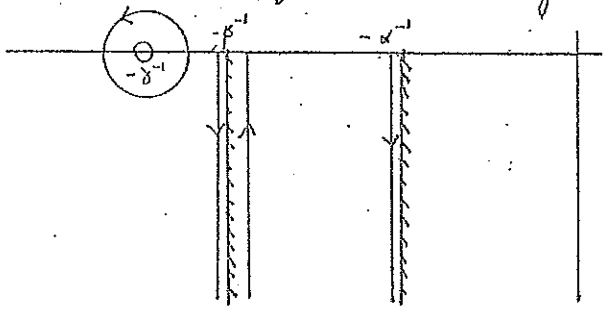

Fig 6.6

To make certain that the path in Fig 6.6 is equivalent to the path in Fig. 6.4 we must require $x > h$. We neglect the branch line integrals to get our approximation. These integrals represent pulse like events that are important near $t = r/\alpha,\ r/\beta$. The remaining part is the residue at $p = -\gamma^{-1}$. For $\bar{v}$ we have

$$\bar{v}(x,0,s) \simeq \frac{-F}{\sigma\pi\beta^2} \operatorname{Im}\left[2\pi i \frac{(\beta^{-2}-2\gamma^{-2})}{R'(-\gamma^{-1})} e^{-s(x\gamma^{-1}+i(\gamma^{-2}-\alpha^{-2})^{1/2}h)}\right]$$

6.33
$$= \frac{2F}{\sigma\beta^2}\frac{(2\gamma^{-2}-\beta^{-2})}{R'(-\gamma^{-1})} e^{-sx\gamma^{-1}} \cos[s(\gamma^{-2}-\alpha^{-2})^{1/2}h]$$

The image of $e^{-sa}\cos sb$ is

6.34 $$e^{-sa}\cos sb \Rightarrow \frac{b/\pi}{(t-a)^2+b^2}$$

Then the image of 6.33 is

6.35 $$v(x,0,t) = \frac{2F}{\sigma\beta^2\pi}\frac{(2\gamma^{-2}-\beta^{-2})}{R'(-\gamma^{-1})}\frac{(\gamma^{-2}-\alpha^{-2})^{1/2}h}{(t-x\gamma^{-1})^2+(\gamma^{-2}-\alpha^{-2})h^2}$$

In a similar way we find

$$6.36 \quad u(x,0,t) \simeq \frac{4F}{\sigma\beta^2\pi}\,\frac{\gamma^{-1}(\gamma^{-2}-\beta^{-2})^{1/2}}{R'(-\gamma^{-1})}\,\frac{t-x\gamma^{-1}}{(t-x\gamma^{-1})^2+(\gamma^{-2}-\alpha^{-2})h^2}$$

The motion is shown schematically in Fig 6.7

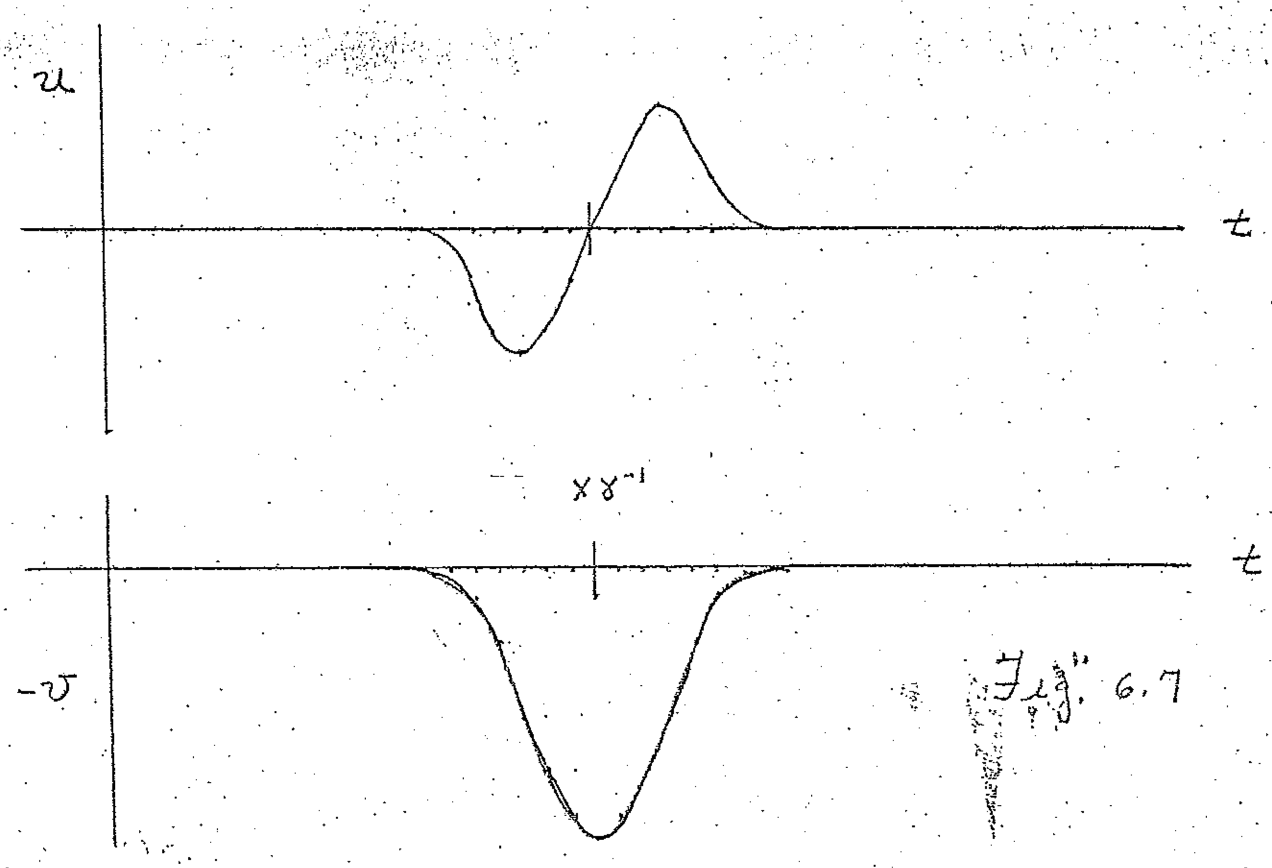

Fig. 6.7

The motion is retrograde elliptical.

7. Propagation of an SH pulse in a Slab

In rectangular coordinates (x, y, z) the planes $y=0$, $y=D$ bound an elastic slab. In the slab there is a line source with the force pointing in the z direction. The motion is also in the z direction, $\underline{s} = \hat{z}\, u(x, y, t)$. The momentum equation is given by 3.2 (page 23)

3.2 $$\mu \nabla^2 u(x, y, t) - \rho\, \partial_t^2 u(x, y, t) = -f(t)\, \delta(x)\, \delta(y-h)$$

The Laplace transforms of 3.2 with respect to t and x are

3.6 $$\mu \left(d_y^2 \bar{u}(\xi, y, s) - \nu^2 \bar{u}(\xi, y, s) \right) = -\bar{f}(s)\, \delta(y-h)$$

To the particular solution 3.14 we add homogeneous solutions and get

7.1 $$\bar{u}(\xi, y, s) = \frac{\bar{f}(s)}{2\mu\nu} \left[e^{-\nu|y-h|} + A e^{-\nu y} + B e^{\nu y} \right], \quad \mathrm{Re}\, \nu \geq 0$$

where A and B are determined from the boundary conditions. For free boundaries we must have $\mu\, d_y \bar{u} = 0$ at $y = 0, D$

7.2
$$-\nu A + \nu B = -\nu e^{-\nu h}$$
$$-\nu A e^{-\nu D} + \nu B e^{\nu D} = \nu e^{-\nu(D-h)}$$

Solving 7.2 gives

7.3 $$A = \frac{\sinh[\nu(D-h)]}{\sinh[\nu D]} \qquad B = \frac{e^{-\nu D}\cosh[\nu h]}{\sinh[\nu D]}$$

Substituting 7.3 into 7.1 gives

7.4 $$\bar{u}(\xi, y, s) = \frac{f(s)}{2\mu}\left[\frac{\cosh[\nu(D-|y-h|)] + \cosh[\nu(D-y-h)]}{\nu \sinh \nu D}\right]$$

ie: The ambiguity of eigenmodes makes no difference to $\bar{u}$

Notice that 7.4 is an even function of ν. Thus $\bar{u}$ has no branch points even though ν does. We shall see that this result is true for any number of layers of finite thickness.

The inversion of 7.4 is

7.5 $$\bar{u}(x, y, s) = \frac{f(s)}{2\mu}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi\, e^{\xi x}\left[\frac{\cosh\nu(D-|y-h|) + \cosh\nu(D-y-h)}{\nu \sinh \nu D}\right]$$

The only singularities of the integrand are the zeros of $\nu \sinh \nu D$. They occur at $\xi = \pm \xi_n$, $n = 0, 1, 2$

7.6 $$\xi_n = (\kappa^2 + n^2\pi^2/D^2)^{1/2}$$

We restrict our attention to $x > 0$. Then we must have $\mathrm{Re}\,\xi < 0$ for the integral to converge

We close the path of integration in the left half ξ plane and evaluate 7.5 by the residue theorem

$$7.7 \quad \bar{u}(x,y,s) = \frac{\bar{f}(s)}{2\mu} \sum_{n=0}^{\infty} \operatorname{Res} \left\{ \left[\frac{\cosh \nu (D-|y-h|) + \cosh \nu (D-y-h)}{\nu \sinh \nu D} \right] e^{\xi x} \right\}_{\xi = -\xi_n}$$

Evaluating 7.7 gives

$$7.8 \quad \bar{u}(x,y,s) = \frac{\bar{f}(s)}{4\mu} \sum_{n=0}^{\infty} (-)^n \epsilon_n \left[\frac{\cos n\pi (1-|y-h|/D) + \cos n\pi (1-(y+h)/D)}{\xi_n D} \right] e^{-\xi_n x}$$

where $\epsilon_0 = 1$, $\epsilon_1 = \epsilon_2 = \cdots = 2$.

Each term in the sum in 7.8 is the Laplace transform of a transient. Let $\bar{f}(s) = F$. The variable s enters the expression

$$7.9 \quad \frac{e^{-\xi_n x}}{\xi_n} = \frac{e^{-(s^2/\beta^2 + n^2\pi^2/D^2)^{1/2} x}}{(s^2/\beta^2 + n^2\pi^2/D^2)^{1/2}}$$

From the relation (Bateman, 1932, 411)

$$7.10 \quad \int_0^{\infty} dx \, \frac{x J_0(\lambda x) e^{ik(x^2+z^2)^{1/2}}}{(x^2+z^2)^{1/2}} = \frac{e^{-|z|(\lambda^2-k^2)^{1/2}}}{(\lambda^2-k^2)^{1/2}}, \quad \lambda^2 > k^2$$

H. Bateman, 1932, Partial Differential Equations of Mathematical Physics; Cambridge University Press (Dover reprint) we derive

$$\beta \int_{|x|/\beta}^{\infty} dt\, J_0\left[(\beta^2 t^2 - x^2)^{1/2} m\pi/D\right] e^{-st} = \frac{e^{-(s^2/\beta^2 + m^2\pi^2/D^2)^{1/2} x}}{(s^2/\beta^2 + m^2\pi^2/D^2)^{1/2}} \qquad 7.11$$

and so we know the image of 7.9. With $\mathcal{J}(s) = F$

$$u(x,y,t) = \frac{F\beta}{4\mu D} \sum_{m=0}^{\infty} (-1)^m \epsilon_m \left[\cos[m\pi(1-|y-h|/D) + \cos[m\pi(1-(y+h)/D]\right] \qquad 7.12$$

$$\cdot\, J_0\left[(\beta^2 t^2 - x^2)^{1/2} m\pi/D\right] H(t - |x|/\beta)$$

which can be written more compactly as

$$u(x,y,t) = \frac{F\beta}{2\mu D} \sum_{m=0}^{\infty} \epsilon_m \cos(m\pi y/D)\cos(m\pi h/D)\, J_0\left[(\beta^2 t^2 - x^2)^{1/2} m\pi/D\right] \cdot \qquad 7.13$$

$$\cdot\, H(t - |x|/\beta)$$

Eq. 7.13 is the exact solution for a delta function source. In this form it is called the transient mode solution. Each term in 7.13 is orthogonal to every other term over the range $0 \leq y \leq D$. If each term is regarded as a generalized vector we say it is normal to all the others. Each term in 7.13 is then called a <u>normal mode</u>.

We know that the least arrival time in this problem is $t_0 = \beta^{-1}(X^2+(y-h)^2)^{1/2}$. For $t < t_0$, $u(x,y,t) \equiv 0$. But each term in 7.13 is not zero for $t < t_0$, in fact for $X/\beta \leq t \leq t_0$ all terms in 7.13 are non zero. But, for $X/\beta \leq t \leq t_0$, the sum in 7.13 must vanish. That is what happens so we must conclude that each normal mode need not satisfy a causality condition.

Another approach to this problem that leads to a ray theoretical interpretation begins with 7.5

7.5 $$\bar{u}(X,y,s) = \frac{f(s)}{2\mu}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi\, e^{\xi X}\left[\frac{\cosh\nu(D-|y-h|)+\cosh\nu(D-y-h)}{\nu\sinh\nu D}\right]$$

We represent the hyperbolic functions as exponentials

7.14 $$\bar{u}(x,y,s) = \frac{f(s)}{2\mu}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{d\xi\, e^{\xi x}}{\nu}\left[\frac{e^{\nu(D-|y-h|)}+e^{-\nu(D-|y-h|)}+e^{\nu(D-y-h)}+e^{-\nu(D-y-h)}}{e^{\nu D}-e^{-\nu D}}\right]$$

Let $\xi = sp$, $\nu = s\eta$, $\mathrm{Re}\,\eta > 0$, $f(s) = F$. We write the denominator of the bracketed term in 7.14 as

7.15 $$e^{\nu D}-e^{-\nu D} = e^{s\eta D}\left(1-e^{-2s\eta D}\right)$$

then
and, [we require s ~~to~~ be large enough so that]
(not necessary, but sufficient

the reciprocal of 7, 15 can be written as

7, 16 $$\left[e^{s\eta D}\left(1-e^{-2s\eta D}\right)\right]^{-1} = e^{-s\eta D}\sum_{m=0}^{\infty} e^{-2ms\eta D}$$

Substituting 7, 16 into 7, 14 gives

7, 17 $$\bar{u}(x,y,s) = \frac{F}{2\mu}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp\,\frac{e^{spx}}{\eta}\sum_{m=0}^{\infty}\sum_{n=1}^{4} e^{-s\eta y_{mn}}$$

where

7, 18
$$y_{m1} = 2mD + |y-h|$$
$$y_{m2} = 2(m+1)D - |y-h|$$
$$y_{m3} = 2mD + y + h$$
$$y_{m4} = 2(m+1)D - y - h$$

A typical term in 7, 17 is

7, 19 $$\bar{u}_{mn}(x,y,s) = \frac{F}{2\mu}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp\,\eta^{-1}\, e^{spx - s\eta y_{mn}}$$

whose image is (Ch. 3)

7, 20 $$u_{mn}(x,y,t) = \frac{F\,H(t - r_{mn}/\beta)}{2\pi\mu\,(t^2 - r_{mn}^2/\beta^2)^{1/2}}$$

where $r_{mn} = (X^2 + y_{mn}^2)^{1/2}$. Thus the image of 7.17 is

$$u(X,y,t) = \sum_{m=0}^{\infty} \sum_{n=1}^{4} u_{mn}(X,y,t)$$

7.21
$$= \frac{F}{2\pi\mu} \sum_{m=0}^{\infty} \sum_{n=1}^{4} \frac{H(t - r_{mn}/\beta)}{(t^2 - r_{mn}^2/\beta^2)^{1/2}}$$

Consider the $m=0$ term in 7.21. The four terms in the series on n can be interpreted as shown in Fig 7.1.

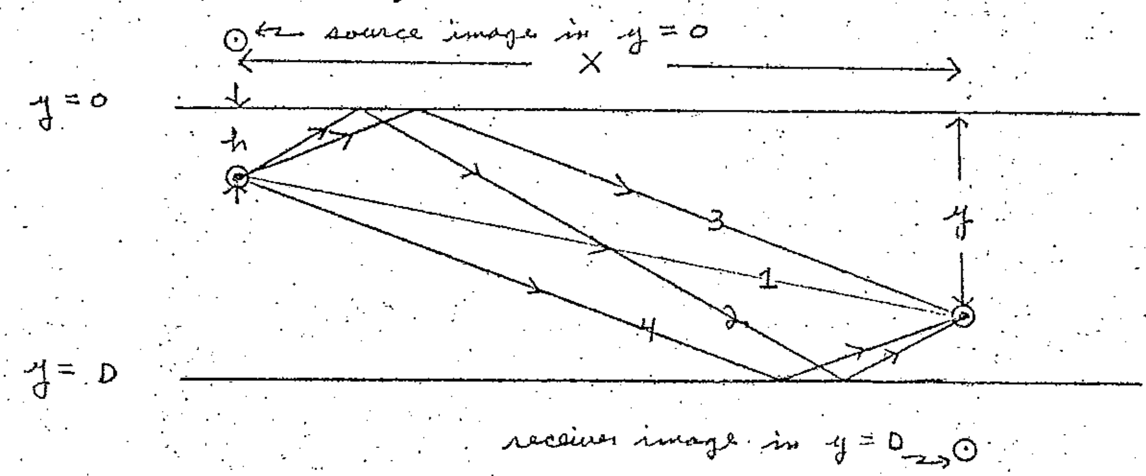

Fig. 7.1 Ray interpretation of $m=0$ terms

The vertical distance travelled by each of the four rays is less than 2D. The length of the n^{th} ray is r_{0n}.

For $m=1$ the vertical distance travelled by each of the four rays is greater than 2D but less than 4D. For the m^{th} term in 7.21 the vertical distance lies between $2mD$ and $2(m+1)D$.

The four terms for $m=0$ appear to come from the four source points shown in Fig. 7.2.

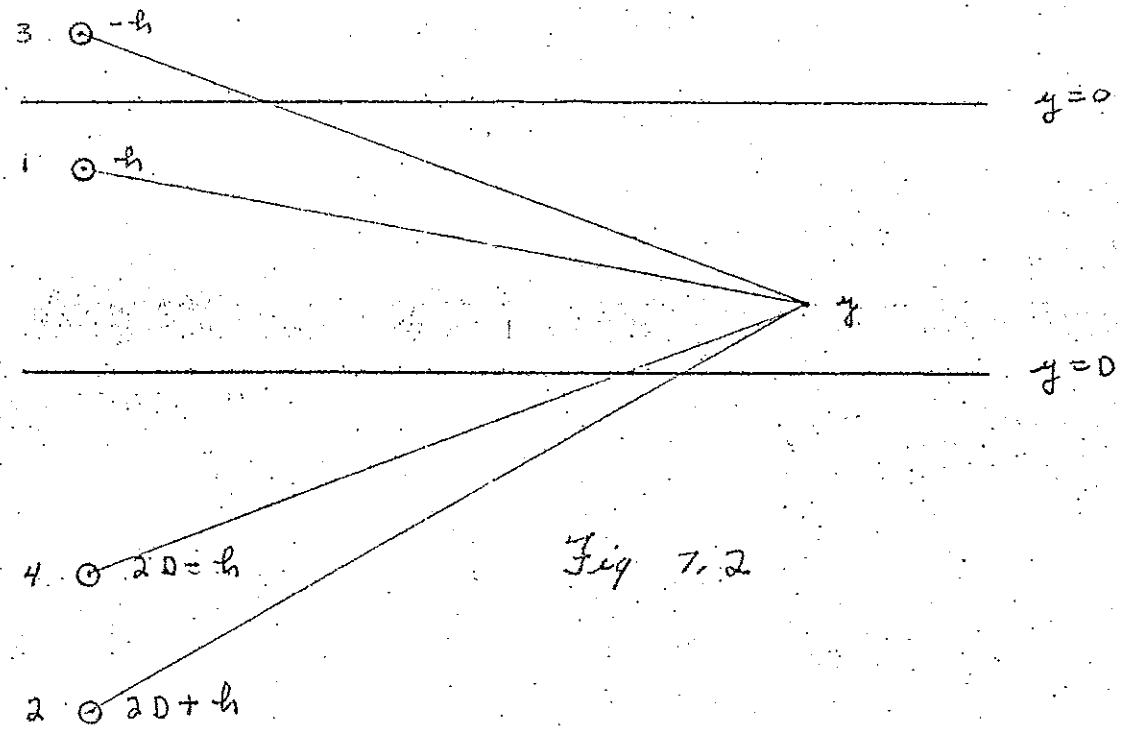

Fig 7.2

The four apparent source points in Fig. 7.2 are the real source and its image in the nearer surface ($y=0$) and the image of that pair in the farther surface ($y=D$). Each m term in 7.21 consists of four n terms that have a similar interpretation.

This problem is particularly simple because the reflection coefficients are unity. In more complex problems the same basic idea of a generalized ray expansion is frequently very useful.

Now that we have examined the line source problem we turn to the point source problem using the methods of Ch. 4.

The reader should have no trouble to show

$$7.22 \quad \bar{u}(r,z,s) = \frac{f(s)}{4\pi^2\mu i}\int_{-i\infty}^{i\infty} d\xi\, \xi K_0(\xi r)\left[\frac{\cosh\nu(D-|z-h|) + \cosh\nu(D-z-h)}{\nu\sinh\nu D}\right]$$

Evaluating the integral in 7.22 by the residue theorem gives

$$7.23 \quad \bar{u}(r,z,s) = \frac{f(s)}{2\pi\mu D}\sum_{n=0}^{\infty} \epsilon_n K_0(\xi_n r)\cos(n\pi z/D)\cos(n\pi h/D)$$

where $\xi_n = (s^2/\beta^2 + n^2\pi^2/D^2)^{1/2}$. Using 4.25 we write $K_0(\xi_n r)$ as

$$7.24 \quad K_0(\xi_n r) = \frac{1}{2i}\int_{-i\infty}^{i\infty} d\varsigma \frac{e^{s\varsigma/\beta - (r^2-\varsigma^2)^{1/2} n\pi/D}}{(r^2-\varsigma^2)^{1/2}} \quad ; \quad Re\,(r^2-\varsigma^2)^{1/2} \geq 0$$

Using the principle of reflection

$$7.25 \quad K_0(\xi_n r) = Im \int_0^{i\infty} d\varsigma (r^2-\varsigma^2)^{-1/2} e^{s\varsigma/\beta - (r^2-\varsigma^2)^{1/2} n\pi/D}$$

Shifting the integration path to the top of the branch cut in the left half s plane

7.26 $$K_0(\xi_n r) = -\int_{r}^{-\infty} ds\,(s^2-r^2)^{-1/2}\, e^{s s/\beta} \cos[n\pi/D\,(s^2-r^2)^{1/2}]$$

Let $s = -\beta t$

7.27 $$K_0(\xi_n r) = \int_{r/\beta}^{\infty} dt\,(t^2-r^2/\beta^2)^{-1/2} \cos[n\pi\beta/D\,(t^2-r^2/\beta^2)^{1/2}]\, e^{-st}$$

Thus $K_0(\xi_n r)$ has the image

7.28 $$K_0(\xi_n r) \Rightarrow \frac{H(t-r/\beta)}{(t^2-r^2/\beta^2)^{1/2}} \cos[n\pi\beta/D\,(t^2-r^2/\beta^2)^{1/2}]$$

If we let $f(s) = F$ the image of 7.23 is

7.29 $$u(r,z,t) = \frac{F}{2\pi\mu D} \frac{H(t-r/\beta)}{(t^2-r^2/\beta^2)^{1/2}} \sum_{n=0}^{\infty} \epsilon_n \cos(n\pi z/D)\cos(n\pi h/D) \cdot \cos[n\pi\beta/D\,(t^2-r^2/\beta^2)^{1/2}]$$

which is the point source normal mode solution.
It should be obvious how to find the point source ray solution.

In waveguide mode problems it is usually more convenient to use Fourier transforms rather than Laplace transforms. Then the concepts of frequency and wave-number can be introduced and the phenomenon of dispersion can be discussed.

We return to 3.2 (pages 23 and 100).

3.2 $$\mu \nabla^2 u(x,y,t) - \rho \partial_t^2 u(x,y,t) = -f(t)\,\delta(x)\,\delta(y-h)$$

For $t<0$ we assume $f(t) \equiv 0$, $u = \partial_t u \equiv 0$. The Fourier transform on t is

7.30 $$\bar{u}(x,y,\omega) = \int_{-\infty}^{\infty} dt\; u(x,y,t)\, e^{i\omega t}$$

and ω is called the frequency in radians per unit time

7.31 $$\mu \nabla^2 \bar{u}(x,y,\omega) + \rho\omega^2 \bar{u}(x,y,\omega) = -\bar{f}(\omega)\,\delta(x)\,\delta(y-h)$$

The Fourier transform on x is

7.32 $$\bar{u}(k,y,\omega) = \int_{-\infty}^{\infty} dx\; \bar{u}(x,y,\omega)\, e^{-ikx}$$

and k is called the wavenumber in radians per unit length. We already know that the solution to 7.31 behaves like $|x|^{-1/2}$ for large $|x|$ so the integration by parts can be done.

7.33 $\mu\left(d_y^2 \bar{u}(k,y,\omega) - k^2 \bar{u}(k,y,\omega)\right) + \rho\omega^2 \bar{u}(k,y,\omega) =$

$$= -\bar{f}(\omega)\,\delta(y-h)$$

We write 7.33 as

7.34 $d_y^2 \bar{u}(k,y,\omega) - \nu^2 \bar{u}(k,y,\omega) = -\bar{f}(\omega)\,\delta(y-h)/\mu$

where $\nu^2 = k^2 - \omega^2/\beta^2$. The particular solution to 7.34 is (Eg 3.14)

7.35 $\bar{u}(k,y,\omega) = \dfrac{\bar{f}(\omega)}{2\mu\nu} e^{-\nu|y-h|}$; $\mathrm{Re}\,\nu \geq 0$

To 7.35 we add the homogeneous solutions 7.1 and apply the boundary conditions to get 7.2 The result is 7.4 $\frac{1}{2\pi}$

7.4 $\bar{u}(\ ,y,\omega) = \dfrac{\bar{f}(\omega)}{2\mu}\left[\dfrac{ch[\nu(D-|y-h|)] + ch[\nu(D-y-h)]}{\nu\, sh\, \nu D}\right]$

here $ch \equiv \cosh$, $sh \equiv \sinh$. The inversion of 7.4 in k and ω is

7.36 $u(x,y,t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} dk\, e^{ikx} \dfrac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \bar{u}(k,y,\omega)$

We consider the ω integral first. In the ω plane there are branch points at $\omega = \pm \beta k$. For $\mathrm{Re}\,\nu \geq 0$ there must be branch cuts as shown in Fig. 7.3.

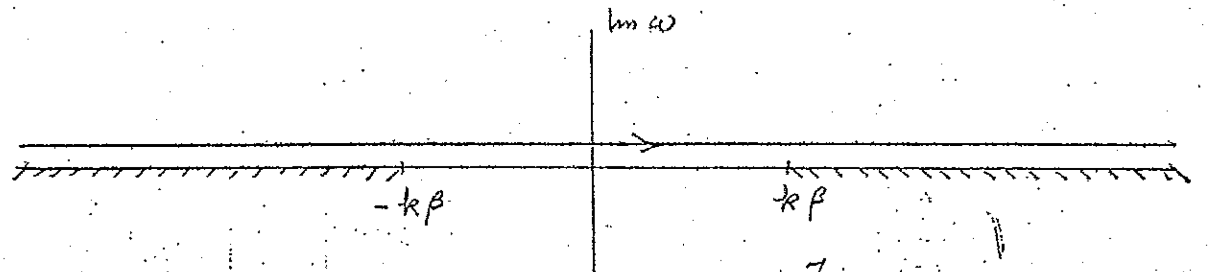

Fig. 7.3

The poles of the integrand in 7.36 occur where $\mathrm{sh}\,\nu D = 0$, or $\nu D = \pm i n \pi$

7.37 $$\omega^2 = \beta^2 k^2 + n^2 \pi^2 \beta^2 / D^2 \quad , \quad n = 0, 1, 2, \ldots$$

The equation $\mathrm{sh}\,\nu D = 0$ is called the secular equation or dispersion equation. The solutions 7.37 are conveniently represented by the (ω, k) diagram, Fig 7.4

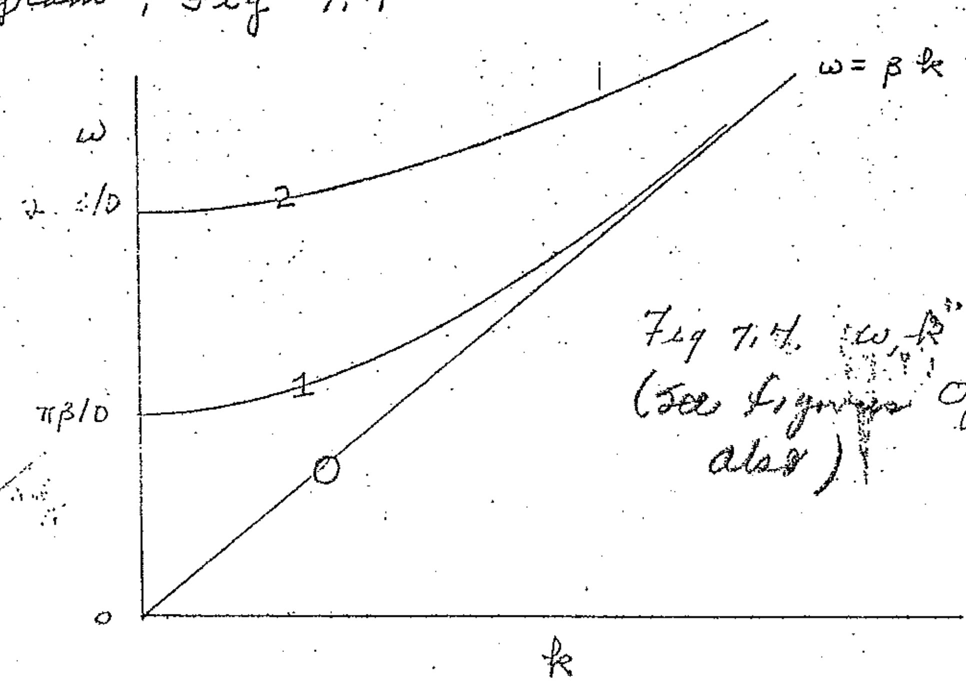

Fig 7.4 (ω, k) diagram (See figures opposite page also)

The values of ω at $k = 0$ are called cutoff frequencies

7.38 $\omega_n^0 = n\pi\beta/D$

We recognize 7.38 as the frequencies of vibration of an air column in an open organ pipe. For that reason the cutoff frequencies are often called organ pipe frequencies and the normal modes are called organ pipe modes.

All the poles of 7.36 are on the real ω axis, $-\beta k \le \omega \le \beta k$. The integrand in 7.36 has no branch point so we can evaluate the ω integral in 7.36 by the residue theorem. For $t < 0$ we deform the ω contour into the upper half ω plane. There are no poles and the integrand vanishes exponentially. Thus $u(x, y, t) \equiv 0$ for $t < 0$. For $t > 0$ we deform the contour into the lower half ω plane. We encounter the poles on the real axis. The rest of the integral vanishes exponentially. The poles are located at $\omega = \pm\omega_n$, $n = 0, 1, 2, \ldots$ where

7.39 $\omega_n = \beta(k^2 + n^2\pi^2/D^2)^{1/2}$

Thus the evaluation of the ω integral in 7.36 with $\bar{u}(k, y, \omega)$ given by 7.4 (page 111) is

7.40 $$u(x,y,t) = \frac{\beta^2}{4\mu D}\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\, e^{ikx} \sum_{n=0}^{\infty} \epsilon_n \cos(n\pi y/D)\cos(n\pi h/D) \, *$$

$$* \left[\frac{\bar{f}(\omega_n)}{\omega_n} e^{-i\omega_n t} - \frac{\bar{f}(-\omega_n)}{\omega_n} e^{i\omega_n t} \right]$$

where ω_n is given by 7.39. The bracketed term in 7.40 is an even function of k. Also $\bar{f}(-\omega) = \bar{f}(\omega)$ because $f(t)$ is real. Thus we can write 7.40 as

7.41 $$u(x,y,t) = \frac{\beta^2}{4\mu D}\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\, e^{ikx} \sum_{n=0}^{\infty} \epsilon_n \cos(n\pi y/D)\cos(n\pi h/D) \, *$$

$$* \frac{\bar{f}(\omega_n)}{\omega_n}\left[e^{-i\omega_n t} - e^{i\omega_n t} \right]$$

We know that 7.41 leads to 7.13 when $\bar{f}(\omega_n) = F$.

In more complex problems integrals like 7.41 can seldom be evaluated explicitly. We must resort either to some numerical integration method or to some analytic approximation. The most common approximation is the method of stationary phase applied to integrals of the form

7.42 $$u_n(x,t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\, e^{ikx} F_n(k)\left(e^{-i\omega_n t} - e^{i\omega_n t}\right)$$

where F_n and ω_n are even functions of k. Then we can write 7.42 as

7.43 $$u_n(x,t) = \frac{1}{2\pi i}\int_{0}^{\infty} dk\, F_n(k)\left(e^{ikx} + e^{-ikx}\right)\left(e^{-i\omega_n t} - e^{i\omega_n t}\right)$$

We restrict our attention to $x, t > 0$. Also we consider only $\omega_n(k) > 0$. Let

$$\varphi_1(x,t,k) = kx - \omega_n(k)\,t$$
$$\varphi_2 = -\varphi_1$$
7.44 $$\varphi_3 = kx + \omega_n(k)\,t$$
$$\varphi_4 = -\varphi_3$$

Then

$$u_n(x,t) = \frac{1}{2\pi i}\int_{0}^{\infty} dk\, F_n(k)\left(e^{i\varphi_1(k)} - e^{i\varphi_2(k)} - e^{i\varphi_3(k)} + e^{i\varphi_4(k)}\right)$$

7.45 $$= I_1 + I_2 + I_3 + I_4$$

$$= I_1 + I_1^* + I_3 + I_3^* = 2\,\mathrm{Re}\,(I_1 + I_3)$$

where

$$I_1 = \frac{1}{2\pi i}\int_0^\infty dk\, F_n(k)\, e^{i(kx - \omega_n(k)t)}$$

7.46

$$I_3 = \frac{-1}{2\pi i}\int_0^\infty dk\, F_n(k)\, e^{i(kx + \omega_n(k)t)}$$

The heuristic argument for the method of stationary phase proceeds as follows. For large x and t the exponential in 7.46 will generally be a rapidly fluctuating function. Only where the phase, $kx \mp \omega_n(k)t$, varies slowly will there be much contribution to 7.46. Values of k for which the phase does not vary are called points of stationary phase. They are solution to

7.47 $$\frac{d}{dk}(kx \mp \omega_n(k)t) = x \mp \omega_n^{(1)}(k)\,t = 0$$

$$\frac{d\omega}{dk} = \omega_n^{(1)}(k) = \pm x/t$$

Thus, for I_1 to have points of stationary phase it is necessary that $\omega_n^{(1)}(k) \geq 0$. For I_3 to have points of stationary phase $\omega_n^{(1)}(k) \leq 0$. If $\omega_n^{(1)}(k) \geq 0$ for $0 \leq k \leq \infty$ then we neglect

ii. I_3 has no stationary points and is ~~therefore~~ small.

I_3 with respect to I_1. If $\omega_n^{(1)}(k) \doteq 0$ for $0 \leq k \leq \infty$ we neglect I_1. Let k_s be one of the points of stationary phase. Let

7.48 $$I = \frac{1}{2\pi i}\int_0^\infty dk\, F_n(k)\, e^{i\varphi(k)} \qquad \varphi(k) = \psi(k)t$$

where $\varphi = \varphi_1$ or φ_3. To approximate 7.4.8 we expand $\psi(k) = kx/t \mp \omega_n(k)$ about $k = k_s$

7.49 $$\psi(k) = \psi^{(0)}(k_s) + \psi^{(1)}(k_s)(k-k_s) + \psi^{(2)}(k_s)(k-k_s)^2/2 + \cdots$$

By definition $\psi^{(1)}(k_s) = 0$. Let $k = k_s + \zeta$. The contribution to 7.48 from the vicinity of k_s is approximately

7.50 $$I \simeq \frac{F_n(k_s)}{2\pi i} e^{i\psi^{(0)}(k_s)t} \int_{-\infty}^{\infty} d\zeta\, e^{it(\psi^{(2)}\zeta^2/2 + \psi^{(3)}\zeta^3/6 + \cdots)}$$

We assume that most of the contribution to 7.50 comes from $\zeta \simeq 0$ so we use the approximation

7.51 $$e^{it(\psi^{(3)}\zeta^3/6 + \psi^{(4)}\zeta^4/24 + \cdots)} \simeq 1 + it(\psi^{(3)}\zeta^3/6 + \psi^{(4)}\zeta^4/24 + \psi^{(5)}\zeta^5/120 + \psi^{(6)}\zeta^6/720) - (\psi^{(3)}t)^2\zeta^6/36 + \cdots$$

and we disregard the odd terms since they vanish on integration.

Using the result

7.52 $$\int_{-\infty}^{\infty} d\xi\, \xi^{2m}\, e^{-a\xi^2} = (-)^m (d/da)^m (\pi/a)^{1/2}$$

we obtain the approximation to 7.50

7.53 $$I \simeq \frac{F(k_s)}{2\pi i}\, e^{i\psi^{(0)}t} \left[\frac{2\pi}{t\,|\psi^{(2)}|}\right]^{1/2} e^{\pm i\pi/4} \left\{ 1 + \frac{i}{t}\left[\frac{\psi^{(4)}}{8(\psi^{(2)})^2} - \frac{5}{24}\frac{(\psi^{(3)})^2}{(\psi^{(2)})^3}\right] + O(t^{-2}) \right\}$$

where the $\pm i\pi/4$ term is taken according as $\psi^{(2)} \lessgtr 0$. For the first term in 7.53 to be a good approximation the bracketed term within braces must be small or t must be very large and $\psi^{(2)} \neq 0$. When $\psi^{(2)} = 0$ we must use a different approach.

Using 7.53 we can evaluate 7.46 and 7.45 as

7.54 $$u_n(X,t) \simeq F(k_s)\left[\frac{2}{\pi t\,|\psi_1^{(2)}|}\right]^{1/2} \sin[\psi_1^{(0)} t \pm i\pi/4]$$

when $\omega_n^{(1)}(k_s) = X/t$ and

7.55 $$u_n(X,t) \simeq -F(k_s)\left[\frac{2}{\pi t\,|\psi_3^{(2)}|}\right]^{1/2} \sin[\psi_3^{(0)} t \pm i\pi/4]$$

when $\omega_n^{(1)}(k_s) = -X/t$ where

7.56
$$\psi_1^{(0)}(k_s)\,t = k_s X - \omega_n(k_s)\,t = \varphi_1^{(0)}$$

$$\psi_3^{(0)}(k_s)\,t = k_s X + \omega_n(k_s)\,t = \varphi_3^{(0)}$$

From 7.45 we see that $k_s > 0$ and we have already restricted $\omega_n(k_s) > 0$. In 7.54 when $\psi_1^{(0)} t$ is constant the motion has constant phase

7.57
$$\frac{\partial}{\partial t}\psi_1^{(0)} t = 0 = k_s \frac{dX}{dt} - \omega_n(k_s)$$

The velocity, $dX/dt = \omega_n/k_s$, is called the phase velocity. Thus 7.54 represents phases propagating in the direction of increasing X while 7.55 represent phases propagating in the direction of decreasing X. The symbol $C_{ph} = \omega/k$ is often used to denote the phase velocity. A surface on which $\psi^{(0)}$ remains constant is often called a wave surface. We have ... that a wave surface propagates with the phase velocity.

From the first of 7.56 we derive

7.58
$$k_s = \frac{\partial \varphi_1^{(0)}}{\partial X}, \quad \omega_n(k_s) = -\frac{\partial \varphi_1^{(0)}}{\partial t}$$

Therefore $k_s c_{ph}$

7.59 $$\frac{\partial k_s}{\partial t} + \frac{\partial \omega_n(k_s)}{\partial x} = 0$$

Eq. 7.59 has the form of a conservation equation (1.14, page 4 and Table 1.1, page 5) where k_s is the conserved quantity, $\hat{x}\,\omega_n(k_s)$ is the k_s flux vector, and $\omega_n(k_s)/k_s = c_{ph}$ is the instantaneous velocity of k_s transport in the x direction. The rate of production of k_s is zero. Another form of 7.59 is

7.60 $$\frac{\partial k_s}{\partial t} + \frac{\partial \omega_n(k_s)}{\partial k_s}\frac{\partial k_s}{\partial x} = 0 \quad ; \quad \frac{\partial \omega_n(k_s)}{\partial k_s} = \omega_n^{(1)}(k_s)$$

According to 7.60, which is a simple first order "wave" equation, k_s remains constant on curves in x, t space which are defined by

7.61 $$\frac{\partial x}{\partial t} = \omega_n^{(1)}(k_s) = x/t$$

Since $\omega_n^{(1)}$ is a function of k_s these curves are straight lines as can be seen by integrating 7.61. Thus a wave number k_s moves with the velocity $\omega_n^{(1)}(k_s)$ which is called the <u>group velocity</u> $c_{gr}(k_s) = x/t$. We can also write 7.59 as

$$\frac{\partial k}{\partial \omega}\frac{\partial \omega}{\partial t}+\frac{\partial \omega}{\partial x}=0 \quad (7.62)$$

and we see that a frequency $\omega_m(k_s)$ also moves with the group velocity. A more detailed discussion is given by Whitham (1961)

G. B. Whitham, 1961, Group velocity and energy propagation for three dimensional waves: Commun. Pure & Appl. Math., 14, 675-691.

and by Backus (1962a, 1962b)

G. E. Backus, 1962a, The effect of the earth's rotation on the propagation of ocean waves over long distances: Deep-Sea Research, 9, 185-197.

G. E. Backus, 1962b, The propagation of short elastic waves on a slowly rotating earth: Bull. Seism. Soc. Amer., 52, 823-846.

Further references are cited by Backus and Whitham.

In 7.45 the function $F_n(k)$ for the n^{th} mode depends only on the initial conditions, the source behavior and the boundary conditions. It does not depend on x, t. Since each wave number k propagates with its group velocity and since $F_n(k)$ depends only on k, then $F_n(k)$ is carried along with each k at the group velocity. The elastic energy density is proportional to $F_n(k) F_n^*(k)$ and so is the kinetic energy density. Therefore the energy density in each mode propagates with the group velocity.

When $\partial\omega_n/\partial k > 0$ we use 7.54. The phase velocity dx/dt is positive. When $\partial\omega_n/\partial k < 0$ we use 7.55. The phase velocity is negative, and the sign is changed in 7.60.

7.62½ $$\frac{\partial k_s}{\partial t} - \frac{\partial \omega_n}{\partial k_s}\frac{\partial k_s}{\partial x} = 0.$$

But $\partial\omega_n/\partial k < 0$ so the group velocity is still positive.

In the slab problem $\omega_n^{(1)}/k \geq 0$ so the phase velocity is always positive. In some problems $\omega_n^{(1)}(k) \leq 0$. Then the group velocity is still positive but the phase velocity is negative because we must use 7.55.

The (ω, k) diagram Fig 7.4 conveniently presents the dispersive properties of the problem. The magnitude of $d\omega/dk$ is the group velocity. The phase velocity is $\pm \omega/k$ according as $d\omega/dk \gtrless 0$.

In the slab problem $(\omega/k)(d\omega/dk) = \beta^2$.

~~pict the dispersive properties of the problem. For a given mode pick a point on its ω, k curve. Draw the chord from the origin to that point. Then the slope of the chord is the phase velocity and the slope of the (ω, k) curve is the group velocity.~~

~~In the slab problem $c_{ph}\, c_{gr} = \beta^2$.~~

The approximation 7.53 breaks down near $\psi_n^{(2)} = 0$. Let $k = k_a$ be the place where $\psi_n^{(1)}(k_a) = \psi_n^{(2)}(k_a) = 0$. Assume $\psi_m^{(3)}(k_a) \neq 0$. We never encounter this difficulty in the slab problem (except for the degenerate mode $n = 0$) except for $k \to \infty$. But for other problems it frequently happens that $\psi_n^{(2)}(k_a) = 0$.

If we make a Taylor expansion of ψ_n about $k = k_a$ where both $\psi_n^{(1)} = \psi_n^{(2)} = 0$ then we shall get an approximation valid only for $t = t_a$ at some x; $t_a = x/\omega_n^{(1)}(k_a)$. Such an approximation, valid only for a single value of t, is not very useful. Instead we let t differ from t_a and make the expansion about $k = k_a$ where $\psi_n^{(2)}(k_a) = 0$ for all t, but $\psi_n^{(1)}(k_a) = 0$ only for $t = t_a$. Therefore the derivatives in the expansion are independent of both t and x, except for $\psi_n^{(1)}$.

For $k = k_a + \xi$ we approximate $\psi(k)$ in 7.48 (page 117) by

7.63 $\psi(k) \simeq \psi^{(0)}(k_a) + \psi^{(1)}(k_a)\,\varsigma + \psi^{(3)}(k_a)\,\varsigma^3/6 + \cdots$

Then in place of 7.50 we have

7.64 $$I \simeq \frac{F_n(k_a)}{2\pi i}\, e^{i\psi^{(0)}(k_a)t} \int_{-\infty}^{\infty} d\varsigma\, e^{it\left(\psi^{(1)}\varsigma + \psi^{(3)}\varsigma^3/6 + \psi^{(4)}\varsigma^4/24\right)}$$

In 7.64 we approximate the integral

7.65 $$\int_{-\infty}^{\infty} d\varsigma\, e^{it\left(\psi^{(1)}\varsigma + \psi^{(3)}\varsigma^3/6 + \psi^{(4)}\varsigma^4/24\right)} \simeq$$

$$\simeq \int_{-\infty}^{\infty} d\varsigma\, e^{it\left(\psi^{(1)}\varsigma + \psi^{(3)}\varsigma^3/6\right)} \left(1 + it\,\psi^{(4)}\varsigma^4/24\right)$$

Let $\alpha = t\,\psi^{(1)}$, $\beta = t\,\psi^{(3)}/6$, $\gamma = t\,\psi^{(4)}/24$

$$I \simeq \frac{F_m(k_a)}{2\pi i}\, e^{i\psi^{(0)}(k_a)t} \int_{-\infty}^{\infty} d\varsigma\,(1 + i\gamma\varsigma^4)\, e^{i(\alpha\varsigma + \beta\varsigma^3)}$$

7.66 $$= \frac{F_n(k_a)}{2\pi i}\, e^{i\psi^{(0)}(k_a)t} \int_{-\infty}^{\infty} d\varsigma\,(1 + i\gamma\varsigma^4)\cos(\alpha\varsigma + \beta\varsigma^3)$$

We can evaluate 7.66 by using the definition of the Airy function (Jeffreys and Jeffreys, 1956).

7.67 $$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\left(zx + \tfrac{1}{3}x^3\right) dx$$

H. and B. S. Jeffreys, 1956, Methods of Mathematical Physics: 3rd ed. page 508 eq. 5, Cambridge Univ. Press

Let $\frac{1}{3}X^3 = \beta\zeta^3 = \psi^{(3)}(k_a)t/6$ Then

7.68 $$\int_{-\infty}^{\infty} \cos(\alpha\zeta + \beta\zeta^3) = (3\beta)^{-1/3}\int_{-\infty}^{\infty} \cos\left[(3\beta)^{-1/3}\alpha X + \tfrac{1}{3}X^3\right] dX =$$

$$= 2\pi(3\beta)^{-1/3}\, Ai\left[\alpha(3\beta)^{-1/3}\right]$$

Also

7.69 $$\int_{-\infty}^{\infty} d\zeta\, \zeta^4 \cos(\alpha\zeta + \beta\zeta^3) = -\frac{\partial^2}{\partial\alpha\,\partial\beta}\int_{-\infty}^{\infty} d\zeta \cos(\alpha\zeta + \beta\zeta^3)$$

Therefore

7.70 $$\int_{-\infty}^{\infty} d\zeta(1 + i\gamma\zeta^4)\cos(\alpha\zeta + \beta\zeta^3) = \left(1 - i\gamma\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)\left(2\pi(3\beta)^{-1/3} Ai\left[\alpha(3\beta)^{-1/3}\right]\right)$$

For the first term in the approximation 7.70 to be valid we must have $\psi^{(3)} \neq 0$ and the operator $\gamma\,\partial^2/\partial\alpha\,\partial\beta$ must be small. For 7.66 we have

7.71 $$I \simeq \frac{F_n(k_a)}{i} e^{i\psi^{(0)}(k_a)t}\left(\frac{2}{\psi^{(3)}t}\right)^{1/3} Ai\left[t^{2/3}\psi^{(1)}(2/\psi^{(3)})^{1/3}\right]$$

We have obtained 7.71 with the tacit assumption

$\psi_1^{(3)} > 0$ ($\beta > 0$). When $\psi^{(3)} < 0$ we obtain

7.72 $$I \simeq \frac{F_n(k_a)}{i} e^{i\psi^{(0)}(k_a)t} \left(\frac{2}{|\psi^{(3)}|t}\right)^{1/3} Ai\left[-t^{2/3}\psi^{(1)}(2/|\psi^{(3)}|)^{1/3}\right]$$

Combining 7.71 and 7.72 we have

7.73 $$I \simeq \frac{F_n(k_a)}{i} e^{i\psi^{(0)}(k_a)t} \left(\frac{2}{|\psi^{(3)}|t}\right)^{1/3} Ai\left[\pm t^{2/3}\psi^{(1)}(2/|\psi^{(3)}|)^{1/3}\right]$$

according as $\psi^{(3)} \gtrless 0$. Using 7.73 we can evaluate 7.46 and 7.45

7.74 $$u_n(x,t) = 2F_n(k_a)\left(\frac{2}{|\psi_1^{(3)}|t}\right)^{1/3} \sin(\psi_1^{(0)}(k_a)t)\, Ai\left(\pm t^{2/3}\psi_1^{(1)}(2/|\psi_1^{(3)}|)^{1/3}\right)$$

when $\omega_n^{(1)}(k_a) > 0$, $t \simeq t_a$ and

7.75 $$u_n(x,t) = -2F_n(k_a)\left(\frac{2}{|\psi_3^{(3)}|t}\right)^{1/3} \sin(\psi_3^{(0)}(k_a)t)\, Ai\left(\pm t^{2/3}\psi_3^{(1)}(2/|\psi_3^{(3)}|)^{1/3}\right)$$

when $\omega_n^{(1)}(k_a) < 0$, $t \simeq t_a$. Because of the appearance of the Airy function in 7.74 and 7.75 the phase $\psi^{(0)}(k_a)t_a$ is called the Airy phase. This appelation is usually generalized to describe the motion for $t \simeq t_a$.
Of course when $\psi^{(1)}(k_a) = \psi^{(2)}(k_a) = \psi^{(3)}(k_a) = 0$

and $\psi^{(4)}(k_a) \neq 0$ we must deal with integrals of the form

7.76 $$I = \int_{-\infty}^{\infty} d\zeta \cos(\alpha \zeta + \beta \zeta^4)$$

or in general

7.77 $$I = \int_{-\infty}^{\infty} d\zeta \cos(\alpha \zeta + \beta \zeta^n)$$

These integrals are sometimes said to represent generalized Airy functions and lead to time behavior like $O(t^{-1/n})$.

In using normal mode theory and the traditional approximations to it we must remember that each normal mode propagates phases (wave surfaces) with the phase velocity $\pm \omega/k$, and propagates frequencies and wavenumbers, and therefore energies, (group surfaces) with the group velocity $|d\omega/dk|$. ~~[illegible] energy propagates away from the source when the group velocity is [illegible] propagates toward the source [illegible] [illegible])~~ When the group velocity is zero or very small the energy does not propagate and we speak of standing waves. Organ pipe modes are one example of standing waves.

We have seen that a single normal mode may violate the causality condition so it is not always physically meaningful, but the sum of normal modes does satisfy the causality condition and does have physical meaning. We often speak of the sum of normal modes as a wave interference pattern.

A classic paper on waveguide mode theory is the one by Pekeris (1948)

C. L. Pekeris, 1948, Theory of propagation of explosive sound in shallow water: G. S. A. Memoir 27.

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8. SH waveguide modes in a stratified medium

A line source of SH waves at $(0, h)$ in the x, y plane is parallel to the two planes $y = 0$, $y = D > h$. The two planes bound an isotropic elastic slab. For $y > D$ there is vacuum. For $y < 0$ there is an isotropic elastic half space. The boundary $y = D$ is stress free. The boundary $y = 0$ is "welded." Boundary conditions require continuity of s_z and τ_{zy} at $y = 0$, and the vanishing of τ_{zy} at $y = D$. In Fig 8.1 we depict the problem

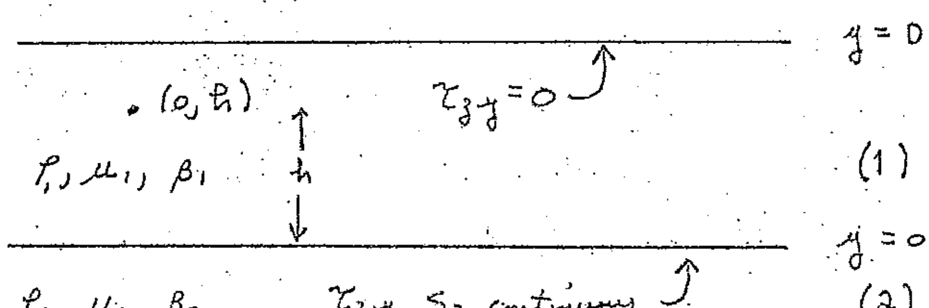

Fig 8.1 The simple SH waveguide

Expressions for the doubly Laplace transformed displacements are

$$\bar{u}_1(\xi, y, s) = \frac{\bar{f}(s)}{2\mu_1\nu_1}\left[e^{-\nu_1|y-h|} + A e^{-\nu_1 y} + B e^{\nu_1 y}\right]$$

8.1

$$\bar{u}_2(\xi, y, s) = \frac{\bar{f}(s)}{2\mu_1\nu_1} C e^{\nu_2 y}$$

The parameters A, B, C are chosen to satisfy the three boundary conditions. Solving the three boundary equations for A, B, C lets us write $\bar{u}_1(\xi, y, s)$ in the form

$$8.2 \quad \bar{u}_1(\xi,y,s) = \frac{f(s)}{2\mu_1\nu_1}\left[\frac{\left[ch\,\nu_1(D-|y-h|)+ch\,\nu_1(D-y-h)\right] + \frac{\mu_2\nu_2}{\mu_1\nu_1}\left[sh\,\nu_1(D-|y-h|) - sh\,\nu_1(D-y-h)\right]}{sh\,\nu_1 D + \frac{\mu_2\nu_2}{\mu_1\nu_1}\, ch\,\nu_1 D}\right]$$

When $\mu_2 = 0$ 8.2 becomes 7.4. The inversion of 8.2 on ξ is

$$8.3 \quad \bar{u}_1(x,y,s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi\, \bar{u}_1(\xi,y,s)\, e^{\xi x}$$

To get the ray theoretical interpretation of 8.3 we follow Ch 7. (pages 104-107) and represent the hyperbolic functions as exponentials. Let $f(s) = F$, $\xi = sp$; $\nu_1 = s\eta_1$, $\nu_2 = s\eta_2$, $R = 1+\mu_2\eta_2/\mu_1\eta_1$, $S = 1-\mu_2\eta_2/\mu_1\eta_1$, $Q = S/R$. Then we can write 8.2 as

$$8.4 \quad \bar{u}_1(sp,y,s) = \frac{Fs^{-1}}{2\mu_1\eta_1}\left[\frac{R\,e^{s\eta_1 D}\left(e^{-s\eta_1|y-h|} + e^{-s\eta_1(2D-y-h)} + Q\,e^{-s\eta_1(y+h)} + Q\,e^{-s\eta_1(2D-|y-h|)}\right)}{R\,e^{s\eta_1 D}\left(1-Q\,e^{-2s\eta_1 D}\right)}\right]$$

For s large enough we can expand the denominator in 8.4 in a geometrical series

$$8.5 \quad \bar{u}_1(sp,y,s) = \frac{Fs^{-1}}{2\mu_1\eta_1}\left[e^{-s\eta_1|y-h|} + e^{-s\eta_1(2D-y-h)} + Q\,e^{-s\eta_1(y+h)} + Q\,e^{-s\eta_1(2D-|y-h|)}\right]\sum_{m=0}^{\infty} Q^m e^{-s2m\eta}$$

which we represent as

8.6 $$\bar{u}_1(sp, y, s) = \frac{F}{2\mu_1\eta_1} \sum_{m=0}^{\infty} \sum_{n=1}^{4} U_{mn}(p)\, e^{-s\eta_1 y_{mn}}$$

where the y_{mn} are given by 7.18 (page 105) and

$U_{m1} = Q^m$

$y_{m1} = 2mD + |y-h|$

$y_{m2} = 2(m+1)D - |y-h|$

etc.

$U_{m2} = Q^{m+1}$

8.7 $U_{m3} = Q^{m+1}$

$U_{m4} = Q^m$

Notice that $Q = \frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2}$ is the generalized reflection coefficient for a plane interface (Ch. 3, pages 34-44). A general term of 8.3 is:

8.8 $$\bar{u}_1^{mn}(x, y, s) = \frac{F}{2\mu_1\pi} \operatorname{Im} \int_0^{i\infty} dp\, U_{mn}(p)\, e^{spx - s\eta_1 y_{mn}}$$

Q has branch points at $p = \pm\beta_1^{-1}, \pm\beta_2^{-1}$. For $\beta_1 > \beta_2$ the image of 8.8 is

8.9 $$u_1^{mn}(x, y, t) = \frac{F\, H(t - r_{mn}/\beta_1)}{2\mu_1\pi\,(t^2 - r_{mn}^2/\beta_1^2)^{1/2}} \operatorname{Re} U_{mn}(p(t))$$

where $r_{mn}^2 = X^2 + y_{mn}^2$, $X = r_{mn} \sin\theta_{mn}$, $y_{mn} = r_{mn} \cos\theta_{mn}$ and

8,10 $$p = \frac{-t}{r_{mn}} \sin\theta_{mn} + i\left(t^2/r_{mn}^2 - 1/\beta_1^2\right)^{1/2} \cos\theta_{mn}$$

For $\beta_2 > \beta_1$ ~~$\beta_1 > \beta_2$~~ we define the critical angle $\theta_c = \sin^{-1} \beta_1/\beta_2$. For each mn there is a critical time, when $\theta_{mn} > \theta_c$,

8,11 $$t_{mnc} = X/\beta_2 + (1/\beta_1^2 - 1/\beta_2^2)^{1/2} y_{mn}$$

and when $\theta_{mn} > \theta_c$ there will be a critical refraction.

When $m=0$ $u_{01} = u_{04} = 1$. The two rays appear to come from the source and its image in $y = 0$ (Fig. 7.2 page 107). Since $y = 0$ is a free surface the reflection coefficient is 1 and for these two rays there is no critical refraction. Excluding these two rays the refractions are

8,12 $$u_1^{mn}(x, y, t) = \frac{-F\left[H(t - t_{mnc}) - H(t - r_{mn}/\beta_1)\right]}{2\mu_1 \pi \left(r_{mn}^2/\beta_1^2 - t^2\right)^{1/2}} \operatorname{Im} U_{mn}(p(t))$$

For any finite time t there are only a finite number of terms in the sums

8,8. As t increases so does the number of terms in the sum.

As in the case of the slab the motion in our SH waveguide is exactly represented by the sum of generalized rays. When $\beta_1 < \beta_2$ there is refraction at the interface $y=0$. The sum of reflections and refractions is said to represent Love waves (Love 1911)

A. E. H. Love, 1911, Some problems of Geodynamics: Cambridge Univ. Press.

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although Love treated the problem from the point of view of mode theory. The ray theoretical interpretation was made by Jeffreys! Some numerical calculations have been published by Knopoff (1958)

L. Knopoff, 1958, Love waves from a line SH source: J. G. R. 63, 619-630.

The mode theoretical treatment of the problem begins with the momentum equation (3.2) Following the discussion on page 110 we take Fourier transforms on t (ω) and X (k) to get 7.34 for $\bar{u}_1$ and its homogeneous counterpart for $\bar{u}_2$

$$d_y^2\,\bar{u}_1(k,y,\omega) - \nu_1^2\,\bar{u}_1(k,y,\omega) = -f(\omega)\,\delta(y-h)/\mu_1$$

8,13

$$d_y^2\,\bar{u}_2(k,y,\omega) - \nu_2^2\,\bar{u}_2(k,y,\omega) = 0$$

where $\nu_1^2 = k^2 - \omega^2/\beta_1^2$, $\nu_2^2 = k^2 - \omega^2/\beta_2^2$. The solution to 8.13 is 8.1. Applying the boundary conditions we recover 8, 2 whose inversion is

8,14 $$u_1(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\,\bar{u}_1(k,y,\omega)$$

Eq. 8.2 is an even function of ν_1 so $\bar{u}_1(k,y;\omega)$ has no branch points at $\nu_1 = 0$. However there are branch points at $\nu_2 = 0$. The ω plane has branch cuts as shown in Fig. 8.2

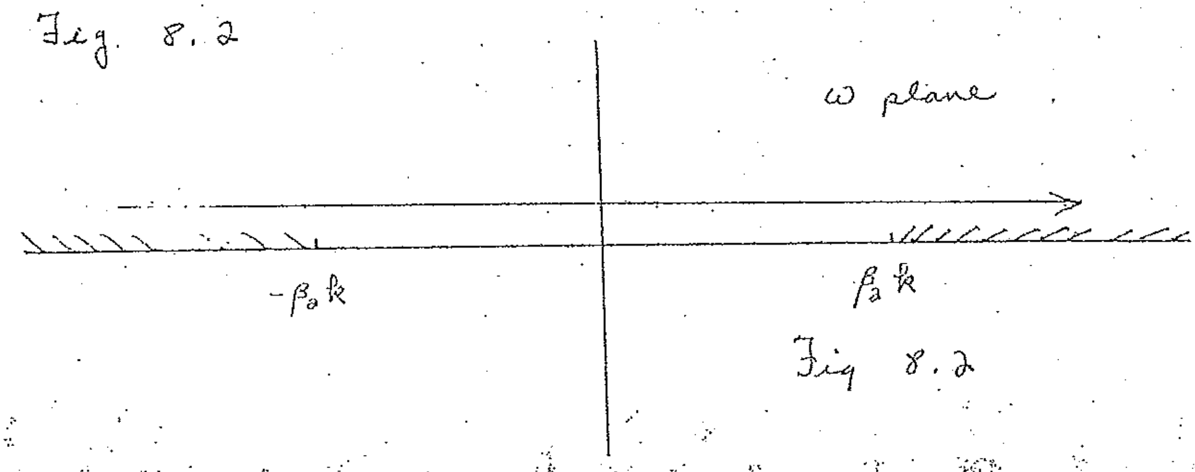


Fig 8.2

To evaluate the ω integral in 8,14 we must find the poles of $\overline{u}_1(k,y,\omega)$. We write $\overline{u}_1$ as

8,15 $$\overline{u}_1(k,y,\omega) = \frac{\overline{f}(\omega)}{2\mu_1\nu_1}\;\frac{N(k,y,\omega)}{\mathcal{D}(k,\omega)}$$

where

8,16 $$N = \mu_1\nu_1\left[\operatorname{ch}\nu_1(D-|y-h|) + \operatorname{ch}\nu_1(D-y-h)\right] + \mu_2\nu_2\left[\operatorname{sh}\nu_1(D-|y-h|) - \operatorname{sh}\nu_1(D-y-h)\right]$$

$$\mathcal{D} = \mu_1\nu_1\operatorname{sh}\nu_1 D + \mu_2\nu_2\operatorname{ch}\nu_1 D$$

and we seek the ω zeros of $\nu_1\mathcal{D}$. In general $\omega = \pm\beta_1 k$ is not a ~~zero of $\nu_1\mathcal{D}$~~ singularity of $\overline{u}_1$ (ie. $N/2\mu_1\nu_1\mathcal{D}$) so we restrict our attention to the ω zeros of $\mathcal{D}$. Since $\mathcal{D}$ is an even function of k we regard $k \geq 0$. Since $u_1(x,y,t) \equiv 0$ for $t<0$ there can be no roots of $\mathcal{D}$ in the upper half ω plane. Since ω enters $\mathcal{D}$ as ω^2 there can be no roots in the lower half ω plane. Thus the only roots of $\mathcal{D}$ on the ω plane where $\operatorname{Re}\nu_2 \geq 0$ (the $(+)$ sheet) must lie on the real axis. Let $|\omega| < \beta_1 k, \beta_2 k$. Then ν_1 and ν_2 are both real so $\mathcal{D}$ is positive and has no roots. Let $|\omega| > \beta_1 k, \beta_2 k$. Then $\nu_1 = \pm i\overline{\nu}_1$ $\nu_2 = \pm i\overline{\nu}_2$ and

8,17 $$\mathcal{D} = -\mu_1\overline{\nu}_1\sin\overline{\nu}_1 D \pm i\mu_2\overline{\nu}_2\cos\overline{\nu}_1 D$$

The real part of $\mathcal{D}$ vanishes when $\bar{\nu}_1 D = n\pi$ and the imaginary part of $\mathcal{D}$ vanishes when $\bar{\nu}_1 D = (n+1/2)\pi$. Therefore they cannot vanish together so $\mathcal{D}$ has no roots. Let $\beta_1 > \beta_2$. When $|\omega| < \beta_2 k$ or $|\omega| > \beta_1 k$ there are no roots. Only when $\beta_2 k < |\omega| < \beta_1 k$ may there be roots. Then

8.18 $$\mathcal{D} = \mu_1 \nu_1 \, sh \, \nu_1 D \pm i \mu_2 \bar{\nu}_2 \, ch \, \nu_1 D$$

and again there are no roots. Thus when $\beta_1 > \beta_2$ $\mathcal{D}$ has no roots on the + sheet for all k. Now let $\beta_1 < \beta_2$. Only when $\beta_1 k < |\omega| < \beta_2 k$ may there be roots. Then

8.19 $$\mathcal{D} = -\mu_1 \bar{\nu}_1 \sin \bar{\nu}_1 D + \mu_2 \nu_2 \cos \bar{\nu}_1 D$$

When $\beta_1 < \beta_2$ there may be roots on the (+) sheet for $\beta_1 k \le |\omega| \le \beta_2 k$ as shown in Fig 8.3

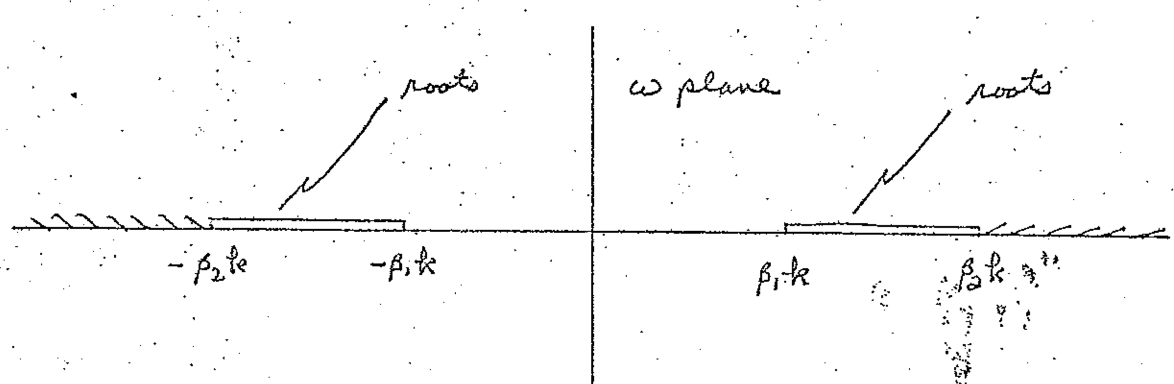


Fig 8.3

Let us examine the two end points of the region $\beta_1 k \leq \omega \leq \beta_2 k$. First let $\omega = \beta_1 k$

8.20 $$\theta = \mu_2 k (1 - \beta_1^2/\beta_2^2)^{1/2}$$

so there is a root at $k=0$ and, since $\omega = \beta_1 k$, there is a root at $\omega = 0$. Now let $\omega = \beta_2 k$

8.21 $$\theta = -\mu_1 k (\beta_2^2/\beta_1^2 - 1)^{1/2} \sin[kD(\beta_2^2/\beta_1^2 - 1)^{1/2}]$$

Thus there is a root at $k=0$ ($\omega=0$) which makes the $\omega=0$ root double when $k=0$. Also there are roots when

8.22 $$kD(\beta_2^2/\beta_1^2 - 1)^{1/2} = n\pi \quad , \quad n = 1, 2, \ldots$$

so

8.23 $$\omega = \frac{n\pi\beta_2}{(\beta_2^2/\beta_1^2 - 1)^{1/2} D} = \frac{n\pi}{(\beta_1^{-2} - \beta_2^{-2})^{1/2} D} = \omega_n^c = \beta_2 k_n$$

Thus when $\beta_1 k \leq \omega \leq \beta_2 k$ the number of roots on the + sheet increases as k increases. Where do they arise? We see that they appear at the ν_2 branch point so they must come from the (−) sheet.

When $k=0$ there is a double root at $\omega=0$

on the (+) sheet, Let us examine D on the $(-)$ sheet for $k = 0$.

8.24
$$D = -\mu_1 \omega \beta_1^{-1} \sin \omega \beta_1^{-1} D \pm i \mu_2 \omega \beta_2^{-1} \cos \omega \beta_1^{-1} D \qquad \text{according as } \operatorname{Im} \omega \gtrless 0$$
$$= \omega(-\rho_1 \beta_1 \sin \omega \beta_1^{-1} D \pm i \rho_2 \beta_2 \cos \omega \beta_1^{-1} D)$$

Clearly there are no roots for real ω except $\omega = 0$. The roots must be complex. Let $z_1 = \rho_1 \beta_1$; $z_2 = \rho_2 \beta_2$; $\omega \beta_1^{-1} D = p + iq$. We seek the solution to the equation

8.25 $$z_1 (\sin p \operatorname{ch} q + i \cos p \operatorname{sh} q) - i z_2 (\cos p \operatorname{ch} q - i \sin p \operatorname{sh} q) = 0$$

for $q > 0$. We require p and q to be real so 8.25 is a pair of equations

8.30
$$(z_1 \operatorname{ch} q - z_2 \operatorname{sh} q) \sin p = 0$$
$$(z_1 \operatorname{sh} q - z_2 \operatorname{ch} q) \cos p = 0$$

First we try $p = n\pi$ so

8.31 $$(z_1 \operatorname{sh} q - z_2 \operatorname{ch} q) = 1/2 (z_1 - z_2) e^{q} - 1/2 (z_1 + z_2) e^{-q} = 0$$

Thus

8.32 $$q = 1/2 \log \left(\frac{z_1 + z_2}{z_1 - z_2} \right)$$

Eq. 8.32 is real only when $z_1 > z_2$

8.33 $$\frac{\omega D}{\beta_1} = \pm\left[n\pi \mp i\,\tfrac{1}{2}\log\left(\frac{\rho_1\beta_1 - \rho_2\beta_2}{\rho_1\beta_1 + \rho_2\beta_2}\right)\right] \quad ; \ \rho_1\beta_1 > \rho_2\beta_2$$

If $z_1 < z_2$

8.34 $$\frac{\omega D}{\beta_1} = \pm\left[(n+1/2)\pi \mp i\,\tfrac{1}{2}\log\left(\frac{\rho_2\beta_2 - \rho_1\beta_1}{\rho_1\beta_1 + \rho_2\beta_2}\right)\right] \quad ; \ \rho_2\beta_2 > \rho_1\beta_1$$

The product $\rho\beta$ has the dimensions [stress]/[velocity] and is often called the wave impedance. A plane SH pulse $S_z(y-\beta t)$ has particle velocity $v_z = \partial_t S_z = -\beta S_z'$ and stress $\tau_{zy} = \mu\,\partial_y S_z = \mu S_z'$. The ratio τ_{zy}/v_z has magnitude $\mu\beta^{-1} = \rho\beta$.

The argument of the logarithm in 8.33 and 8.34 is the value of the reflection coefficient Q (page 131) at $p=0$ (normal incidence). When $Q(p=0) = +1$ (stress free interface) 8.33 reduces to the organ pipe formula 7.38 (page 113). When $Q(p=0) = -1$ (rigid interface) 8.34 becomes the organ pipe formula for a "pipe" closed at one end. When the interface is neither free nor rigid so that motion can "leak" through it there can be no real standing wave pattern. Part of the energy propagates into the half space

Wayne Thatcher To here

and never returns. Consequently the wave pattern in the waveguide decays with time according to the imaginary part of 8.33 or 8.34. It is customary to refer to 8.33 and 8.34 as the leaking organ pipe mode formulas. The dispersion curves that start at $k = 0$ like 8.33 or 8.34 are called leaking mode dispersion curves as long as the root loci $\omega_n(k)$ remain on the (−) sheet.

From now on we consider $\beta_1 < \beta_2$, $P_1\beta_1 < P_2\beta_2$. When $k=0$ there is a double root of $\nu_1 D$ at $\omega = 0$ on the (+) sheet, a double root at $\omega = 0$ on the (−) sheet, and simple roots given by 8.34 on the (−) sheet. Let $k_n^c = n\pi D^{-1}(\beta_2^2/\beta_1^2 - 1)^{-1/2}$. These values of k are the ones for which a root of D appears on the (+) sheet at the branch point $\omega = \beta_2 k$. These values of k are called cut off wave numbers and the corresponding values of ω (8.33) are called cut off frequencies.

In Chapter 7 the the ω, k relation is given by 7.37.

7.37 $$\omega_n = \beta_1 (k^2 + n^2\pi^2/D^2)^{1/2}$$

The ω, k diagram is shown in Fig. 7.4

When $\omega = \beta_2 k$ 7.37 leads to 8.23. Thus where the dispersion curves 7.37 cross the line $\omega = \beta_2 k$ in the ω, k diagram the roots of ν, D are at cutoff. The situation is sketched in Fig. 8.4

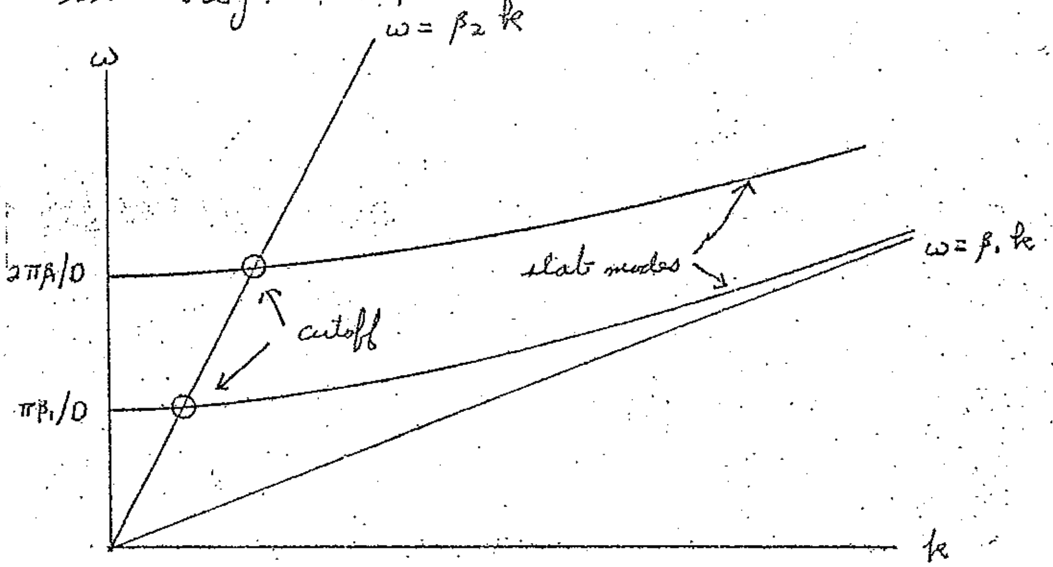


Fig 8.4

If the present wave guide is called a Love waveguide and the one of Chapter 7 is called a slab waveguide, then we say that Love waveguide modes are at cutoff where the equivalent slab waveguide modes cross the Love waveguide cutoff line $\omega = \beta_2 k$

Let $\omega = \beta k$, $\beta_1 \le \beta \le \beta_2$. When $\beta = \beta_2$ the modes are at cutoff. We can write D as

$$D = -\mu_1 k(\beta^2/\beta_1^2 - 1)^{1/2} \sin[kD(\beta^2/\beta_1^2-1)^{1/2}] +$$

8.35

$$+ \mu_2 k (1-\beta^2/\beta_2^2)^{1/2} \cos[kD(\beta^2/\beta_1^2-1)^{1/2}]$$

When β is nearly equal to β_1 the first term

in 8.35 is small compared to the second. As $\beta_1 \to \beta_1$
We seek the zeros of

8.36 $$\mathcal{D} = \mu_2 k (1-\beta^2/\beta_2^2)^{1/2} \cos[kD(\beta^2/\beta_1^2-1)^{1/2}] \quad , \beta \to \beta_1$$

We have already seen that $k=0$ is a root. The other roots are the cosine zeros

$$kD(\beta^2/\beta_1^2-1)^{1/2} = (n+1/2)\pi$$

8.37 $$k = \frac{(n+1/2)\pi}{D(\beta^2/\beta_1^2-1)^{1/2}}$$

As $\beta \to \beta_1$ $k \to \infty$ so the roots asymptotically approach the line $\omega = \beta_1 k$ just as the slab modes do.

Along each root locus $\mathcal{D} = 0$ so

8.38 $$\frac{d\omega}{dk} = -\frac{\partial \mathcal{D}}{\partial k} \Big/ \frac{\partial \mathcal{D}}{\partial \omega}$$

For each of the cutoff frequencies (8.23) we find

8.39 $$\frac{d\omega}{dk} = \beta_2 \quad , \omega = \omega_n^c$$

When $\omega > \omega_i$ we find $d\omega/dk < \beta_2$. When $k \to \infty$, $\omega = \beta k$, $\beta \to \beta_1$ we find $d\omega/dk \to \beta_1$. Thus on the (+) sheet the (ω, k) diagram has the appearance shown in Fig. 8.5.

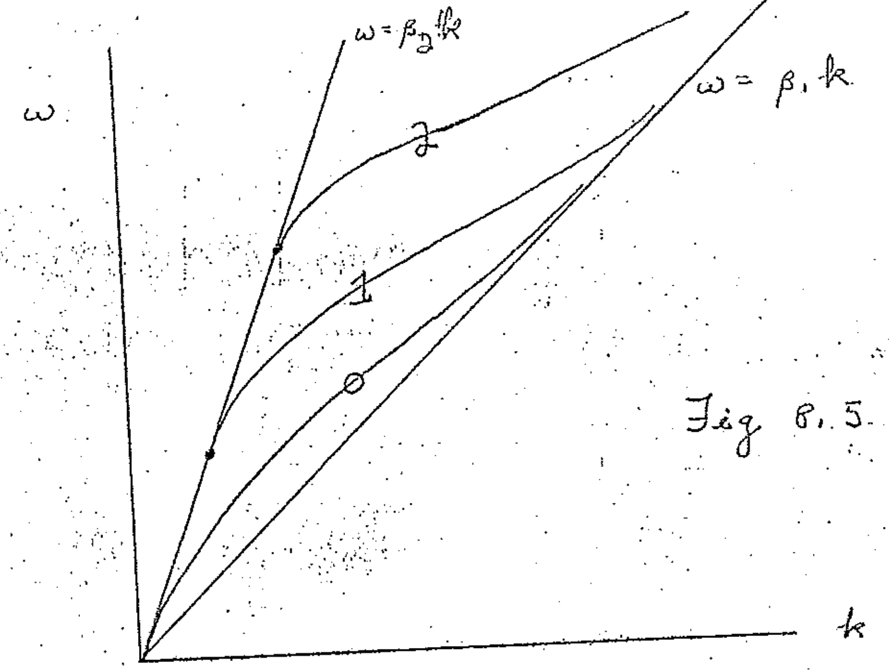


Fig 8.5.

For each mode the ω, k curve has at least one inflection and the curves $d\omega/dk$ have the appearance shown in Fig 8.6

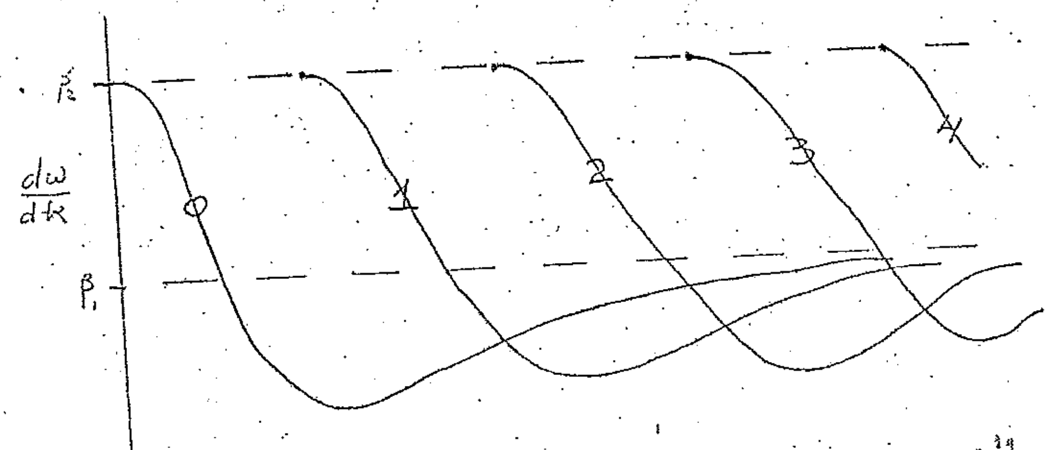
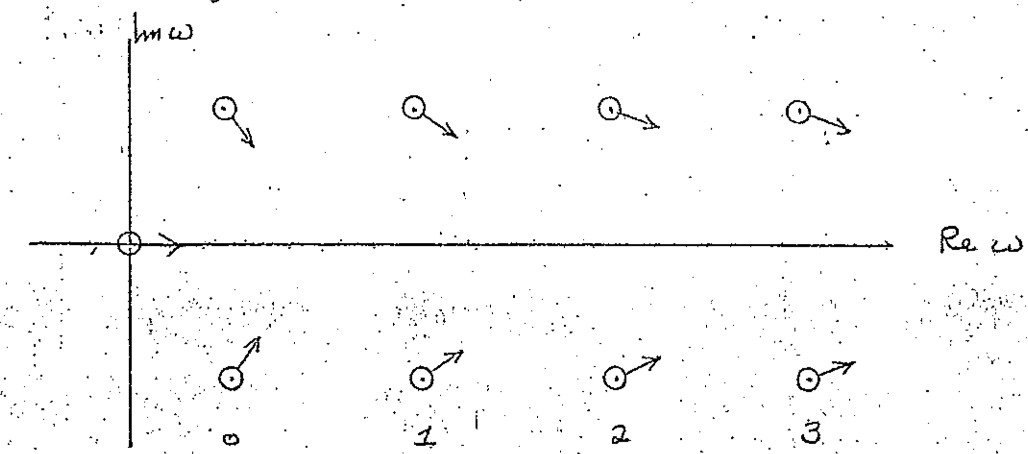


Fig. 8.6.

On the (−) sheet the initial motion of the roots $\omega(k)$ as k increases from zero is shown in Fig. 8.7

Fig 8.7



As k increases the roots move to the right in Fig 8.7 and approach the real axis until the first pair touch the real axis and form a double root as shown in Fig. 8.8.

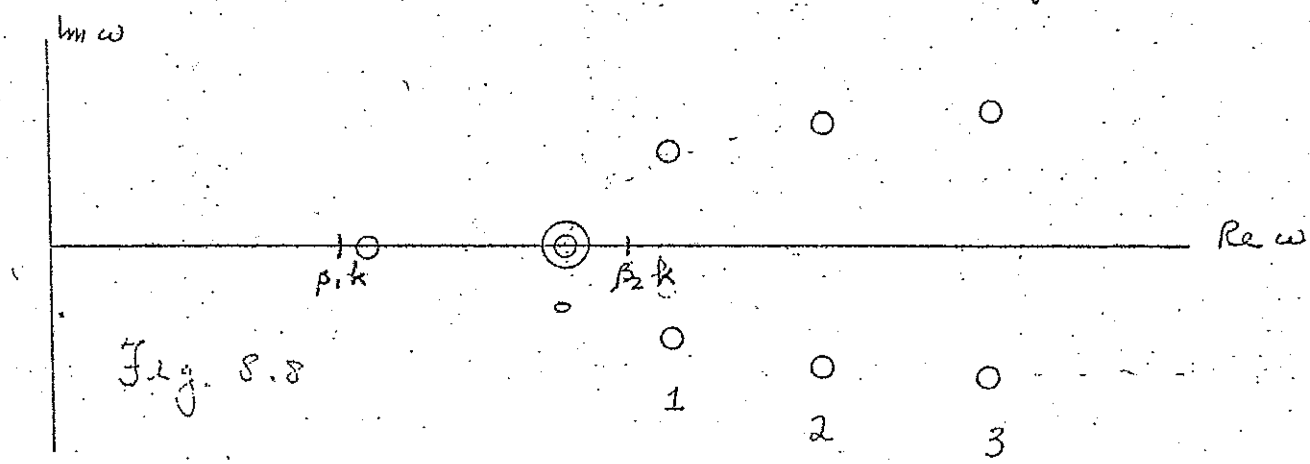


Fig. 8.8

As k increases further the double root separates into two real roots. The one nearer $\beta_2 k$ moves at a rate $d\omega/dk > \beta_2$ and the other moves at a rate $d\omega/dk < \beta_2$. The nearer root "decelerates" until it moves at the rate $d\omega/dk = \beta_2$ when $\omega = \beta_2 k$. At this point the root is on the ν_2 branch point. As k increases it continues to decelerate and moves through the ν_2 branch point into the real axis of the (+) sheet ($\omega < \beta_2 k$) to become the $n=1$ locked mode. We call it a locked mode because its frequency is real.

The (ω, k) diagram can be represented as in Fig. 8.9. The arrows show where a

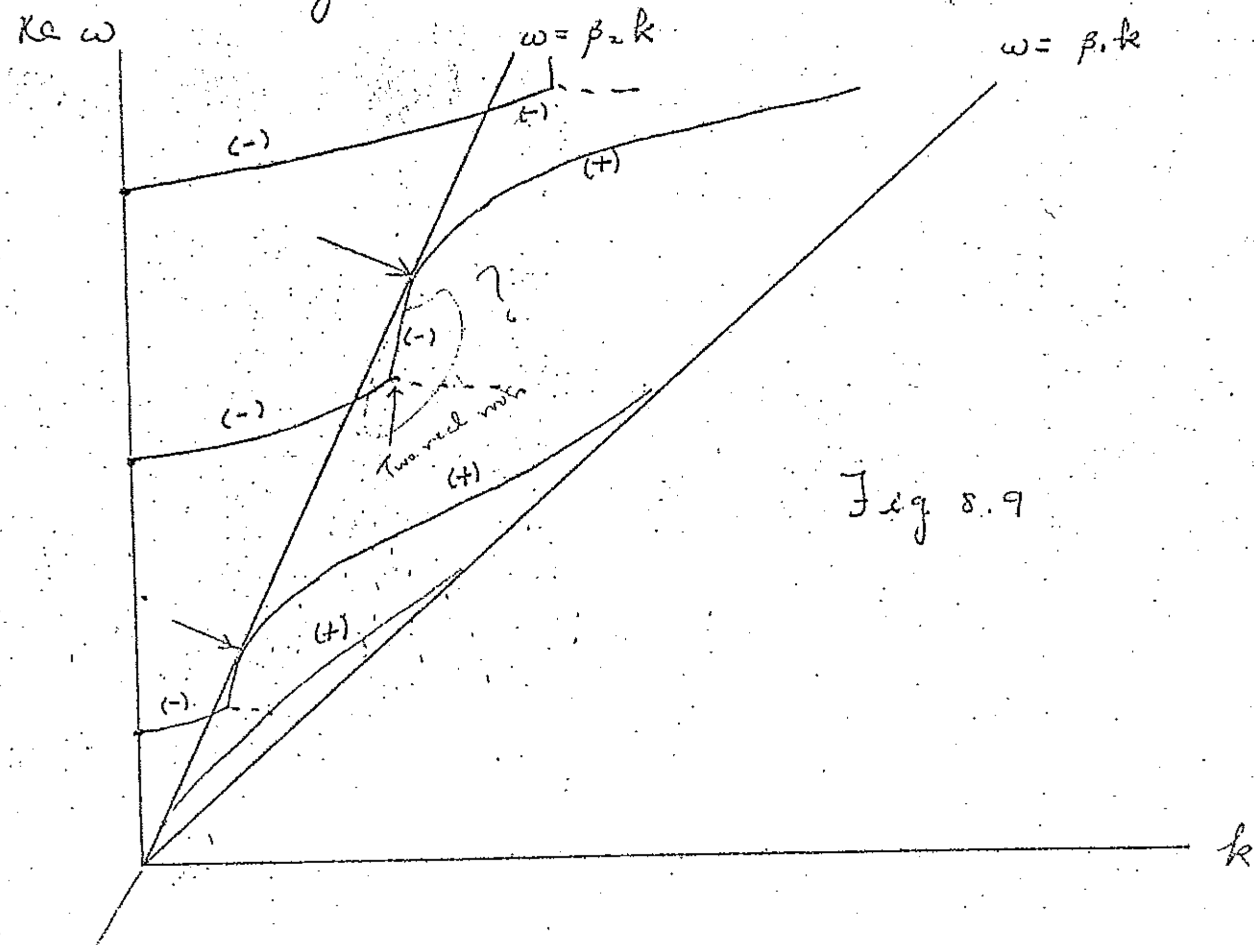


Fig 8.9

root passes from the (-) sheet through the ν_2 branch point to the (+) sheet.

Let us now consider the evaluation of the ω integral in 8,14

$$8.14 \quad u_1(x,y,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk\, e^{ikx} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \overline{u}_1(k,y,\omega)$$

where $\overline{u}_1(k,y,\omega)$ is given by 8.2 and 8.15. We see that $\overline{u}_1(k,y,\omega)$ is an even function of k so we can write 8.14 as

$$8.40 \quad u_1(x,y,t) = \frac{1}{2\pi^2}\int_{0}^{\infty} dk \cos kx \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \overline{u}_1(k,y,\omega)$$

Also $\overline{u}_1(k,y,\omega)$ is an even function of ω so we can write 8.40 as

$$8.41 \quad u_1(x,y,t) = \frac{Re}{\pi^2}\int_{0}^{\infty} dk \cos kx \int_{0}^{\infty} d\omega\, e^{-i\omega t}\, \overline{u}_1(k,y,\omega)$$

The integration path in the ω plane is shown in Fig. 8,10

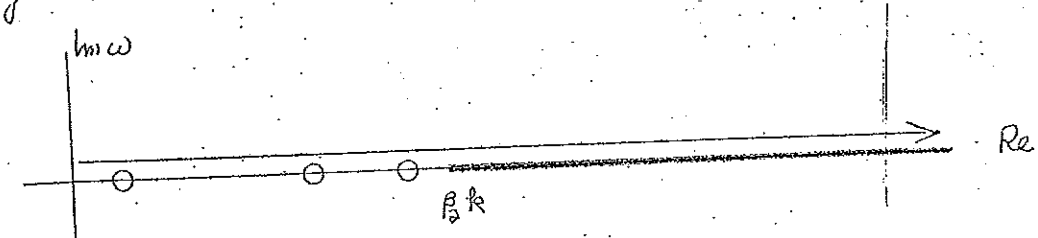


Fig. 8.10

On the top of the branch cut in the ω plane we have

8.42 $$\nu_2 = -i\,(\omega^2/\beta_2^2 - k^2)^{1/2} = -i\,\bar{\nu}_2 \qquad \text{top of cut}$$

and on the bottom of the cut

8.43 $$\nu_2 = i\,\bar{\nu}_2 \qquad \text{bottom of cut}$$

The path in Fig. 8.10 is equivalent to the path in Fig 8.11 because there are

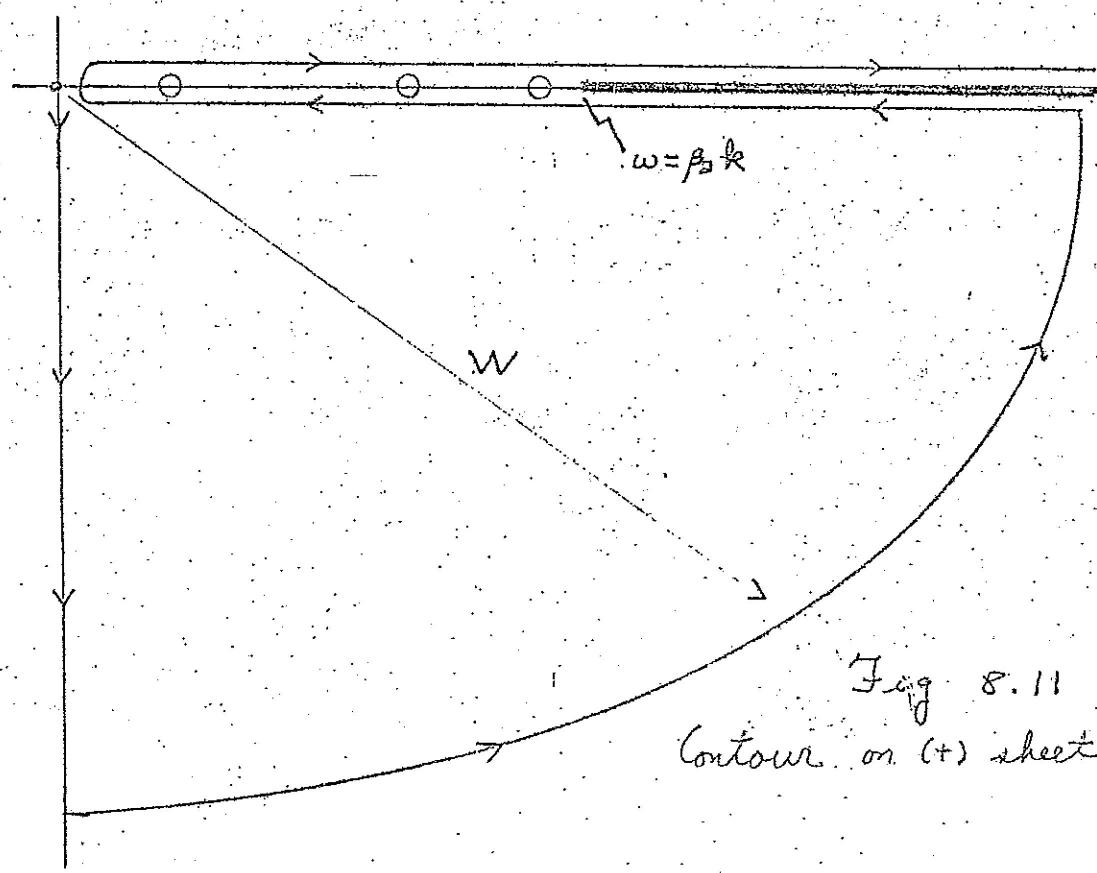


Fig 8.11
Contour on (+) sheet

no singularities in the fourth quadrant of the (+) sheet.

The path from $\omega = 0$ to $\omega = -i\infty$ makes no contribution because $\exp(-i\omega t)$ is real, $\bar{u}_1(k, y, \omega)$ is real and $d\omega$ is imaginary. and $u_1(x,y,t) = Re \int d\omega$?

The arc of radius W also makes no contribution, if t is large enough, as $W \to \infty$ because the integrand vanishes exponentially as $W \to \infty$. Between $\omega = 0$ and $\omega = \beta_2 k$ we can use the residue theorem. Thus we can write 8.41 as

8.44 $$u_1(x,y,t) = \frac{Re}{\pi^2} \sum_{n=0}^{\infty} \int_{k_n^0}^{\infty} dk \cos kx \, e^{-i\omega_n t} \left[-2\pi i \, \underline{\mathrm{Res}} \, \bar{u}_1(k,y,\omega_n) \right]$$

$$+ \frac{Re}{\pi^2} \int_0^{\infty} dk \cos kx \int_{\beta_2 k}^{\infty} d\omega \, e^{-i\omega t} \left[\bar{u}_1^{+}(k,y,\omega) - \bar{u}_1^{-}(k,y,\omega) \right]$$

In 8.44 k_n^0 is that value of k for which the n^{th} root appears at the branch point on the (+) sheet. Also $\bar{u}_1^{+}$ is calculated using 8.42 and $\bar{u}_1^{-}$ is calculated using 8.43. Therefore $\bar{u}_1^{+}$ and $\bar{u}_1^{-}$ are a complex conjugate pair and

8.45 $$\bar{u}_1^{+} - \bar{u}_1^{-} = 2i \, \mathrm{Im} \, \bar{u}_1^{+}$$

so we can use 8.45 to write 8.44 in the form

$$8.46 \quad u_1(x,y,t) = -\frac{2}{\pi} \sum_{n=0}^{\infty} \int_{k_n^0}^{\infty} dk \cos kx \sin \omega_n t \ \underline{\text{Res}}\ \bar{u}_1(k,y,\omega_n)$$

$$+ \frac{\text{Re}}{\pi^2} \int_0^{\infty} dk \cos kx \int_{\beta_2 k}^{\infty} d\omega\, e^{-i\omega t} \left[2i \operatorname{Im} \bar{u}_1^+(k,y,\omega) \right]$$

In 8.46 the sum represents the contribution from the poles on the (+) sheet. Each term in the sum is called a <u>locked</u> <u>normal</u> <u>mode</u>. From Fig. 8.9 we see that each (+) mode has a phase velocity $c_{ph} = \omega/k$ and a group velocity $c_{gr} = d\omega/dk$ both of which are never greater than β_2. Thus ν_2 is always positive <u>real</u> or zero so there is no radiation into the halfspace. Of course part of the motion in the half space is represented by the locked normal modes, but the integrand of the k integral, for motion in the halfspace, decays exponentially into the halfspace. The exponential decay is what leads one to speak of <u>locked</u> modes.

Let us now consider the branch line integral in the second term in 8.46. The function $\bar{u}_1^+(k,y,\omega)$ is continuous when ν_2 is continuous. That is, they are both continuous through the cut from the top of the cut on the (+) sheet to the bottom of the cut on the (−) sheet. Let us write 8.46 as

8.47 $$u_1(x,y,t) = \overset{\text{locked}}{u_1}(x,y,t) + \overset{\text{leaking}}{u_1}(x,y,t)$$

where $\overset{\text{locked}}{u_1}(x,y,t)$ is the sum in 8.46 and

8.48 $$\overset{\text{leaking}}{\bar{u}_1}(x,y,t) = \frac{Re}{\pi^2}\int_0^\infty dk \cos kx \int_{\beta_2 k}^\infty d\omega\, e^{-i\omega t}\left[2i \,\mathrm{Im}\, \bar{u}_1^+(k,y,\omega)\right]$$

We repeat that the ω integral in 8.48 can be regarded as taken along the top of the cut in the (+) sheet or along the bottom of the cut in the (−) sheet. We select the
* latter alternative. In 8.45 the function u_1^+ is evaluated on the (−) sheet. Also u_1^- is evaluated on the (−) sheet. From the way we have defined ν_2, u_1^- on the (−) sheet is equal to u_1^+ on the (+) sheet. That is, on the (−) sheet the singularities of u_1^- are the same as those of u_1^+ on the (+) sheet. In 8.48 the ω integral from $\omega = \beta_2 k$ to $\omega = \infty$ along the bottom of the cut in the (−) sheet is equivalent to the ω integral taken over the path shown in Fig. 8.12.

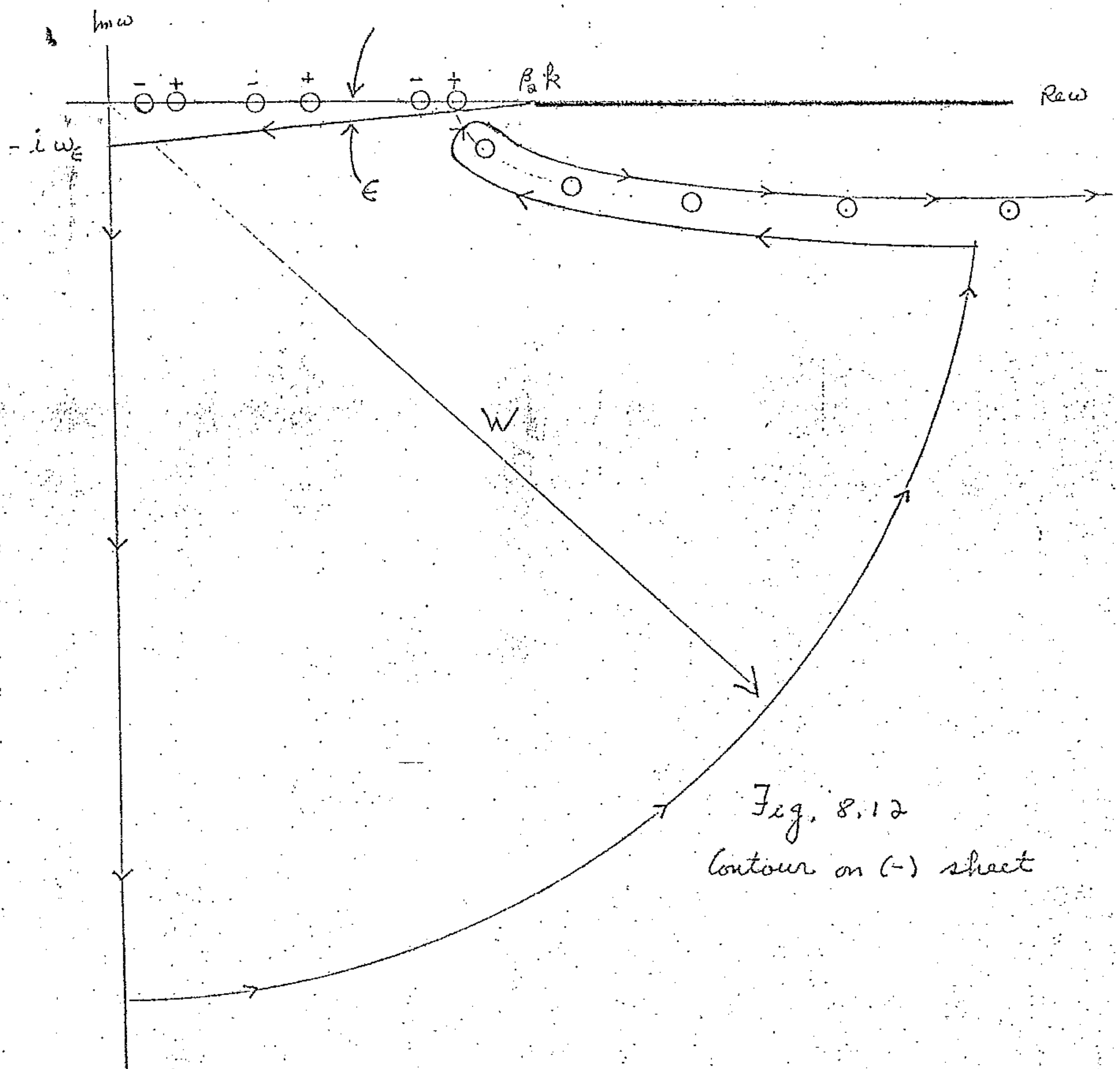


Fig. 8.12
Contour on (-) sheet

The poles labelled (-) on the real axis belong to $\bar{u}_i^+$ and the poles labelled (+) belong to $\bar{u}_i^-$. The poles in the fourth quadrant belong to $\bar{u}_i^+$ because $\bar{u}_i^-$ on the (-) sheet has the same poles as $\bar{u}_i^+$ on the (+) sheet and these always lie on the real axis $\omega \le \beta_2 k$. The (+) and (-) poles on the real axis always appear in pairs. When $0 \le k \le k_1^0$ there is one pair that lies on the origin when $k = 0$.

When $k_1^\circ < k \le k_2^\circ$ there are two pairs; the first coming from $\omega = 0$ and the second derived from the $n=0$ pair of 8.34. When $k_2^\circ < k \le k_3^\circ$ there are three pairs. In general, when $k_{n-1}^\circ < k \le k_n^\circ$ there are n pairs.

Along the path from $\omega = -i\omega_E$ to $\omega = -i\infty$ both $\bar{u}_i^+$ and $\bar{u}_i^-$ are real. Also $e^{-i\omega t}$ is real and $d\omega$ is imaginary so there is no contribution to 8.48. Jordan's lemma (Whittaker and Watson, 1952, p. 115) can be

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E. T. Whittaker and G. N. Watson, 1952, Modern Analysis, Cambridge University Press.

There is a contribution from the real poles on the (-) sheet. This paragraph must be corrected.

used to show that the arc of radius W contributes nothing as $W \to \infty$. On the path from $\omega = \beta_2 k$ to $\omega = -i\omega_E$ in Fig 8.12 the phase of ν_2 in $\bar{u}_i^+$ is $-\pi + \epsilon/2$ and the phase of ν_2 in $\bar{u}_i^-$ is $\pi - \epsilon/2$ so we can still use 8.45. As $\epsilon \to 0$, $\operatorname{Im} \bar{u}_i^+ \to 0$ so there is no contribution. Thus, even though there are poles on the real axis, they make no contribution. In other words, the residue sum of real axis poles is zero on the (-) sheet.

We are left with the contour enclosing the fourth quadrant poles. We can use the residue theorem to evaluate the integral in 8.48. The poles belong to $\bar{u}_i^+$, which is just $\bar{u}_1$ on the (-) sheet, so we have

8.49 $$\bar{u}_1^{\text{leaking}}(x,y,t) = \frac{2}{\pi}\sum_{n=0}^{\infty}\int_0^{K_n^\circ} dk \cos kx \,\mathrm{Im}\left\{ e^{-i\omega_n t} \,\underline{\mathrm{RES}}\, \bar{u}_1(k,y,\omega_n)\right\}$$

+ real poles on (−) sheet

where K_n° is that value of k for which the n^{th} root (given by 8.34 for $k=0$) touches the real ω axis, $\omega < \beta_2 k$, on the (−) sheet.

In 8.44 $\underline{\mathrm{RES}}\ \bar{u}_1$ is real but in 8.49 $\underline{\mathrm{RES}}\ \bar{u}_1$ is complex. Since the $n=0$ root in 8.49 leads to the $n=1$ root in 8.44 we rewrite 8.34 as

8.50 $$\omega = \frac{\beta_1}{D}\left[(n-1/2)\pi + i\,\frac{1}{2}\log\left(\frac{\rho_2\beta_2 - \rho_1\beta_1}{\rho_2\beta_2 + \rho_1\beta_1}\right)\right] \quad ; \quad n = 1, 2, \ldots$$

We now combine 8.44 and 8.49

$$\bar{u}_1(x,y,t) = -\frac{2}{\pi}\int_0^{\infty} dk \cos kx \sin \omega_0^+ t \ \underline{\overset{+}{\mathrm{RES}}_0}\ \underline{\bar{u}_1}(k,y,\omega_0^+) -$$

8.51 $$-\frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{2}\int_{k_n^\circ}^{\infty} dk \cos kx \sin \omega_n^+ t\ \underline{\overset{+}{\mathrm{R}}\mathrm{ES}_n}\ \bar{u}_1(k,y,\omega_n^+) - \right.$$

$$\left. - \int_0^{K_n^\circ} dk \cos kx \,\mathrm{Im}\left[e^{-i\omega_n^- t}\ \overset{-}{\mathrm{R}}\mathrm{ES}_n\ \bar{u}_1(k,y,\omega_n^-)\right]\right\}$$

includes real poles on (−) sheet

In 8.51 the k integrals are to be evaluated either numerically or by some analytic approximation method such as the plus real poles on (−) sheet

method of stationary phase. The residues on the (−) sheet come from poles in the fourth quadrant so the corresponding modes decay exponentially with time. In the stationary phase approximation the phase velocity becomes $c_{ph} = \pm \mathrm{Re}\,\omega / k$ and the group velocity becomes $c_{gr} = |\mathrm{Re}\, d\omega/dk|$. These complex modes have $c_{ph} > \beta_2$. The integrand of the k integral for each mode, for motion in the halfspace, increases exponentially into the half space and also oscillates. Thus it appears that there is radiation into the halfspace and the modes are called *leaking normal modes*. Each mode diverges as we go deeper into the half space. But the motion is the sum of all modes and we have seen that the motion does not diverge, so even though each term in the sum diverges, the sum itself must converge.

The expression in 8.51 is said to represent Love waves. Love waves can be regarded as either the sum of generalized ray pulses or as the sum of locked and leaking waveguide modes.

9. Propagator Matrices for Waveguide Modes

Now that we have completed our examination of the simple Love waveguide, we consider more general waveguides. To do so we use the Love waveguide problem as a starting place and we introduce the concept of propagator matrices or propagators (Gilbert and Backus, 1966).

F. Gilbert and G. E. Backus, 1966, Propagator matrices in elastic wave and vibration problems: Geophysics, 31, 326-332.

For the line source SH problems that we have been considering the displacement has only a z component. When μ and ρ are functions of y the momentum equation becomes

9.1 $$\partial_x \tau_{zx} + \partial_y \tau_{zy} + \partial_z \tau_{zz} - \rho \partial_t^2 s_z = -f(t)\delta(x)\delta(y-h)$$

where

$$\tau_{zx} = \mu(y)\partial_x s_z$$

9.2 $$\tau_{zy} = \mu(y)\partial_y s_z$$

$$\tau_{zz} = 0$$

We take Fourier transforms of 9.1 and 9.2 with respect to t ($e^{i\omega t}$) and x (e^{-ikx}). The doubly transformed equations are

$$ik\bar{\tau}_{zx} + \partial_y \bar{\tau}_{zy} + \rho\omega^2 \bar{s}_z = -\bar{f}(\omega)\,\delta(y-h)$$

9, 3 $$\bar{\tau}_{zy} = \mu(y)\,\partial_y \bar{s}_z$$

$$\bar{\tau}_{zx} = ik\,\mu(y)\,\bar{s}_z$$

We can write 9, 3 in matrix form

9.4 $$\partial_y \begin{bmatrix} \bar{s}_z \\ \bar{\tau}_{zy} \end{bmatrix} = \begin{bmatrix} 0 & \mu^{-1} \\ \mu k^2 - \rho\omega^2 & 0 \end{bmatrix} \times \begin{bmatrix} \bar{s}_z \\ \bar{\tau}_{zy} \end{bmatrix} - \begin{bmatrix} 0 \\ \bar{f}(\omega)\,\delta(y-h) \end{bmatrix}$$

which is a special case of

9, 5 $$\partial_y \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \times \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} + \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

For the moment we consider the homogeneous equation, $g_1 = \cdots = g_n \equiv 0$.

9, 6 $$\partial_y \underline{f} = \underline{\underline{A}} \cdot \underline{f}$$

Eq. 9, 6 is a system of n linear, homogeneous, first order, coupled differential equations for the functions $f_i(y)$, $i = 1, \ldots, n$. An $n \times n$ matrix $\underline{\underline{F}}(y)$ is called an <u>integral</u> <u>matrix</u> of 9.6 if

9.7 $\partial_y \underline{\underline{F}}(y) = \underline{\underline{A}}(y) \cdot \underline{\underline{F}}(y)$

The integral matrices of 9.6 are those $n \times n$ matrices each of whose columns is a solution to 9.6.

An integral matrix of 9.6 is called a <u>fundamental</u> <u>matrix</u> of 9.6 if it is nonsingular for every y in its domain of definition (Coddington and Levinson, 1955, p. 68).

E. A. Coddington and N. Levinson, 1955, Theory of ordinary differential equations: Mc Graw Hill, New York.

An integral matrix, $\underline{\underline{F}}(y)$, of 9.6 is called a <u>propagator</u> <u>matrix</u> from y_0 if $\underline{\underline{F}}(y_0)$ is the $n \times n$ identity matrix, $\underline{\underline{F}}(y_0) = \underline{\underline{I}}$. Gantmacher (1959, $\underline{2}$, p. 113) calls the propagator matrix the matricant. Frazer, Duncan, and Collar (1960, p. 53)

F. R. Gantmacher, 1959, The theory of matrices: 2 vols., Chelsea, New York, translated from the Russian by K. A. Hirsch

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use the word matrizant.

R. A. Frazer, W. J. Duncan, and A. R. Collar, 1960, Elementary Matrices: Cambridge University Press.

When $A_{ij}(y)$ are continuous functions of y it is well known (Coddington and Levinson, 1955, p. 20) that for any $n \times 1$ column matrix $\underline{b}$ and any y_0 there is exactly one solution, $\underline{f}(y)$, of 9.6 such that $\underline{f}(y_0) = \underline{b}$. It follows that for any $n \times n$ matrix $\underline{\underline{B}}$ and any y_0 there is exactly one solution, $\underline{\underline{F}}(y)$, of 9.7 such that $\underline{\underline{F}}(y_0) = \underline{\underline{B}}$. In particular, for any y_0, 9.6 has a unique propagator from y_0 which we will denote by $\underline{\underline{P}}(y, y_0)$; $\underline{\underline{P}}(y_0, y_0) = \underline{\underline{I}}$. The matrices $\underline{\underline{F}} = \underline{\underline{P}}(y, y_0)$ and $\underline{\underline{F}} = \underline{\underline{P}}(y, y_1) \cdot \underline{\underline{P}}(y_1, y_0)$ are both solutions of 9.7 and are equal when $y = y_1$. The uniqueness theorem assures their equality for all y. For any y_1, y_2, y_3

9.8 $$\underline{\underline{P}}(y_3, y_1) = \underline{\underline{P}}(y_3, y_2) \cdot \underline{\underline{P}}(y_2, y_1) = \prod_{l=2}^{3} \underline{\underline{P}}(y_l, y_{l-1})$$

Since $\underline{\underline{P}}(y_1, y_1)$ is the identity matrix it follows from 9.8 that the inverse of $\underline{\underline{P}}(y_2, y_1)$ is $\underline{\underline{P}}(y_1, y_2)$. In particular, for any y, $\underline{\underline{P}}(y, y_0)$ has an inverse. Therefore $\underline{\underline{P}}(y, y_0)$ is nonsingular. Hence any propagator is a fundamental matrix.

If $\underline{M}(y)$ is any fundamental matrix for 9.6 then not only $\underline{M}(y)$ but also $\underline{M}(y)\underline{M}^{-1}(y_0)$ satisfies 9.7. Since $\underline{M}(y_0)\underline{M}^{-1}(y_0)$ is the identity matrix, $\underline{M}(y)\underline{M}^{-1}(y_0)$ must be the propagator from y_0. That is, if $\underline{M}(y)$ is any fundamental matrix for 9.6,

9.9 $$\underline{P}(y, y_0) = \underline{M}(y)\underline{M}^{-1}(y_0) \quad *$$

Therefore, the propagator from any point y_0 can be calculated immediately by matrix inversion, once n linearly independent $n \times 1$ column matrix solutions of 9.6 are known.

The name propagator derives from the fact that if $\underline{f}(y)$ is an $n \times 1$ column matrix solution of 9.6, then

9.10 $$\underline{f}(y) = \underline{P}(y, y_0)\, \underline{f}(y_0)$$

This is a consequence of the uniqueness theorem.

The propagator can be used to solve the inhomogeneous system 9.5

9.5 $$\partial_y \underline{f}(y) = \underline{A}(y)\, \underline{f}(y) + \underline{g}(y)$$

where $\underline{g}(y)$ is a given $n \times 1$ column matrix function of y. The solution, verified by direct substitution, is

9.11 $$\underline{f}(y) = \underline{\underline{F}}(y)\left[\int_{y_0}^{y} \underline{\underline{F}}^{-1}(s)\,\underline{g}(s)\,ds + \underline{\underline{F}}^{-1}(y_0)\,\underline{f}(y_0)\right]$$

where $\underline{\underline{F}}(y)$ is any fundamental matrix of 9.6. Using 9.9 we may write 9.11 as

9.12 $$\underline{f}(y) = \int_{y_0}^{y} \underline{\underline{P}}(y,s)\,\underline{g}(s)\,ds + \underline{\underline{P}}(y,y_0)\,\underline{f}(y_0)$$

Thus the solution to the inhomogeneous counterpart of 9.7

9.13 $$\partial_y \underline{\underline{F}}(y) = \underline{\underline{A}}(y)\,\underline{\underline{F}}(y) + \underline{\underline{G}}(y)$$

is

9.14 $$\underline{\underline{F}}(y) = \int_{y_0}^{y} \underline{\underline{P}}(y,s)\,\underline{\underline{G}}(s)\,ds + \underline{\underline{P}}(y,y_0)\,\underline{\underline{F}}(y_0)$$

If $\underline{F}(y)$ is any solution of 9.7, let $\underline{\underline{F}}^a(y)$ denote the classical adjoint of $\underline{\underline{F}}$, that is, F^a_{ij} is the cofactor of F_{ji}. Then $\underline{\underline{F}}^a\underline{\underline{F}} = \underline{\underline{I}}\,|\underline{\underline{F}}|$ where $\underline{\underline{I}}$ is the $n\times n$ identity matrix and $|\underline{\underline{F}}|$ denotes the determinant of $\underline{\underline{F}}$. Also

$$d_y\,|\underline{\underline{F}}(y)| = (d_y F_{ij}(y))\,F^a_{ji}(y)$$

9.15 $$= A_{ik}(y)\,F_{kj}(y)\,F^a_{ji}(y)$$

$$= A_{ik}(y)\,\delta_{ki}\,|\underline{\underline{F}}(y)| = tr(\underline{\underline{A}}(y))\,|\underline{\underline{F}}(y)|$$

Eq 9,15 is an ordinary linear first order differential equation for $|\underline{\underline{F}}(y)|$ whose solution is

9,16 $$|\underline{\underline{F}}(y)| = |F(y_0)| \cdot e^{\int_{y_0}^{y} tr(\underline{\underline{A}}(\xi))\, d\xi}$$

In particular

9,17 $$|\underline{\underline{P}}(y, y_0)| = e^{\int_{y_0}^{y} tr(\underline{\underline{A}}(\xi))\, d\xi}$$

For Love waves, 9,4, the trace of $\underline{\underline{A}}(y)$ is zero and $|\underline{\underline{P}}(y, y_0)| \equiv 1$.

In case $\underline{\underline{A}}(y)$ and $\int_{y_0}^{y} \underline{\underline{A}}(\xi)\, d\xi$ commute for every y

9,18 $$\underline{\underline{P}}(y, y_0) = e^{\int_{y_0}^{y} \underline{\underline{A}}(\xi)\, d\xi}$$

Eq. 9,18 is applicable in particular if $\underline{\underline{A}}(y)$ is independent of y.

Let us return to 9.4. We consider μ and ρ independent of y. We have

9,19 $$\underline{\underline{A}} = \begin{bmatrix} 0 & \mu^{-1} \\ \mu k^2 - \rho\omega^2 & 0 \end{bmatrix} ; \quad \int_0^D \underline{\underline{A}}\, d\xi = \underline{\underline{A}}\, D$$

Then

9,20 $$\underline{\underline{P}}(D, 0) = e^{\underline{\underline{A}}\, D}$$

Using the power series expansion of the exponential we have ($\nu^2 = k^2 - \omega^2/\beta^2$)

9,21

$$\underline{\underline{P}}(D,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \mu^{-1} D \\ \mu \nu^2 D & 0 \end{bmatrix} + \begin{bmatrix} \nu^2 D^2 & 0 \\ 0 & \nu^2 D^2 \end{bmatrix} \frac{1}{2} + $$

$$+ \begin{bmatrix} 0 & \mu^{-1} \nu^2 D^3 \\ \mu \nu^4 D^3 & 0 \end{bmatrix} \frac{1}{6} + \begin{bmatrix} \nu^4 D^4 & 0 \\ 0 & \nu^4 D^4 \end{bmatrix} \frac{1}{24} + \cdots$$

Grouping the diagonal and anti diagonal matrices we have

9,22

$$\underline{\underline{P}}(D,0) = \begin{bmatrix} 1 + \nu^2 D^2/2 + \nu^4 D^4/24 + \cdots & 0 \\ 0 & 1 + \nu^2 D^2/2 + \nu^4 D^4/24 + \cdots \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & \mu^{-1} \nu^{-1} (\nu D + \nu^3 D^3/6 + \cdots) \\ \mu \nu (\nu D + \nu^3 D^3/6 + \cdots) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cosh \nu D & 0 \\ 0 & \cosh \nu D \end{bmatrix} + \begin{bmatrix} 0 & \mu^{-1} \nu^{-1} \sinh \nu D \\ \mu \nu \sinh \nu D & 0 \end{bmatrix}$$

Thus the propagator for Love waves in a homogeneous medium from $y=0$ to $y=D$ is

9.23 $$\underline{\underline{P}}(D,0) = \begin{bmatrix} \cosh \nu D & \mu^{-1}\nu^{-1}\sinh \nu D \\ \mu\nu \sinh \nu D & \cosh \nu D \end{bmatrix}$$

Alternatively we can use the Cayley-Hamilton theorem and the Lagrange-Sylvester interpolation formula to compute $\exp(\underline{\underline{A}}D)$ (Gantmacher, 1959; Frazer, Duncan and Collar, 1960) The characteristic equation for $\underline{\underline{A}}D$ is

9.24 $$\gamma^2 - D^2 A_{21} A_{12} = \gamma^2 - \nu^2 D^2 = 0$$

Since $\underline{\underline{A}}$ is 2×2 we know that $\exp(\underline{\underline{A}}D)$ is a first degree polynomial in $\underline{\underline{A}}D$. Since the characteristic roots of $\underline{\underline{A}}D$ are $\pm\nu D$, the characteristic roots of $\exp(\underline{\underline{A}}D)$ are $\exp(\pm\nu D)$ and

9.25 $$\underline{\underline{P}}(D,0) = \exp(\underline{\underline{A}}D) = a_1 \underline{\underline{A}}D + a_0 \underline{\underline{I}}$$

where

9.26 $$a_1 = \sinh \nu D / \nu D, \quad a_0 = \cosh \nu D$$

Thus 9.25 and 9.23 are the same.

Consider the simple Love waveguide. In the halfspace we have

9.27 $\bar{S}_z = C e^{\nu_2 y}$; $\bar{\tau}_{zy} = \mu_2 \nu_2 C e^{\nu_2 y} = \mu_2 \nu_2 \bar{S}_z$

at $y = 0$ we have, for no sources,

9.28 $$\begin{bmatrix} \bar{S}_z \\ \bar{\tau}_{zy} \end{bmatrix} = \underline{P}(0,0) \begin{bmatrix} C \\ \mu_2 \nu_2 C \end{bmatrix}$$

The boundary conditions at $y=0$ have automatically been satisfied. at $y = D$, $\bar{\tau}_{zy}$ must vanish. Thus

9.28 $$0 = C\left(P_{21}(D,0) + \mu_2 \nu_2 P_{22}(D,0)\right)$$

and using 9.23 we get

9.29 $$0 = C\left(\mu_1 \nu_1 \sinh \nu_1 D + \mu_2 \nu_2 \cosh \nu_1 D\right)$$

which is the correct secular equation for the simple Love waveguide discussed in Ch. 8.

We can also use the propagator matrix to solve 9.4 and obtain 8.2.

In most Love waveguide problems the region $0 \leq y \leq D$ is a stratified medium and the region $y \leq 0$ is a homogeneous halfspace.

The traditional approach to such problems is to represent the stratified medium $0 \leq y \leq D$ as a stack of homogeneous layers. A solution, 9.27, is taken in the half space. The propagator for each layer, 9.23, is used to propagate the solution to the free surface $y = D$. Then the secular function, 9.28, is evaluated. For given values of k its ω zeros are found numerically. The root loci, $\omega(k)$, are then the dispersion curves.

Of course it is completely unnecessary to approximate the stratified medium with a stack of homogeneous layers. Once 9.27 is taken as the starting solution, then the homogeneous part of 9.4 can be integrated numerically to the free surface where the boundary condition is applied and the secular function is obtained. Once the $\omega(k)$ diagram is known the source term in 9.4 is computed from 9.11. Then the ω and k inversion integrals are evaluated, the first by the residue theorem and the second by stationary phase or some numerical quadrature method. In this way any piecewise continuous μ and ρ stratification can be treated as accurately as desired.

Experience has shown that the most useful method for integrating first order equations like 9.4 is one of the Runge-Kutta-Gill methods.

Eq. 9.9 has been used for homogeneous layers by Thomson (1950) and Haskell (1953)

W. T. Thomson, 1950, Transmission of elastic waves through a stratified solid medium: J. Appl. Phys., 21, 89-93.

N. Haskell, 1953, The dispersion of surface waves on multilayered media: Bull. Seis. Soc. Amer., 43, 17-34

10. P-SV Generalized Rays and Waveguide Modes

A line source of P waves (see ch. 5) at $x=0$, $y=h$ is parallel to the z axis. The two planes $y=0$, $y = D > h > 0$ bound an isotropic, homogeneous, elastic slab. The boundary, $y = D$, is stress free. The boundary, $y=0$, is "welded" to an isotropic, homogeneous, elastic half-space $y<0$. Boundary conditions require the continuity of s_x, s_y, τ_{yx}, τ_{yy} at $y=0$, and the vanishing of τ_{yx}, τ_{yy} at $y=D$. The boundary conditions have been linearized. In Fig. 10.1 we depict the problem.

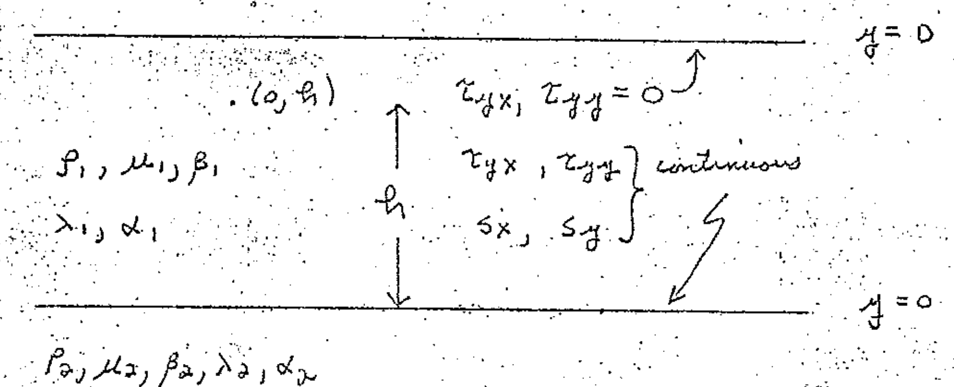

Fig 10.1 The simple P-SV waveguide

In component form we write the momentum equation

$$\partial_x \tau_{xx} + \partial_y \tau_{xy} - \rho \partial_t^2 s_x = - f_x$$

10.1

$$\partial_x \tau_{yx} + \partial_y \tau_{yy} - \rho \partial_t^2 s_y = - f_y$$

where

$$f_x = -f(t)\,\delta'(x)\,\delta(y-h)$$

10.2

$$f_y = -f(t)\,\delta(x)\,\delta'(y-h)$$

The Laplace transforms of 10.1 and 10.2 on $x\,(e^{-\xi x})$ and $t\,(e^{-st})$ give

$$\xi\,\bar{\tau}_{xx} + d_y\,\bar{\tau}_{xy} - \rho s^2\,\bar{s}_x = \bar{f}(s)\,\xi\,\delta(y-h)$$

10.3

$$\xi\,\bar{\tau}_{yx} + d_y\,\bar{\tau}_{yy} - \rho s^2\,\bar{s}_y = \bar{f}(s)\,\delta'(y-h)$$

For the stress components we have

$$\bar{\tau}_{xx} = \lambda\,(\xi\,\bar{s}_x + d_y\,\bar{s}_y) + 2\mu\,\xi\,\bar{s}_x = \sigma\,\xi\,\bar{s}_x + \lambda\,d_y\,\bar{s}_y$$

10.4 $$\bar{\tau}_{xy} = \bar{\tau}_{yx} = \mu\,(d_y\,\bar{s}_x + \xi\,\bar{s}_y)$$

$$\bar{\tau}_{yy} = \lambda(\xi\,\bar{s}_x + d_y\,\bar{s}_y) + 2\mu\,d_y\,\bar{s}_y = \lambda\,\xi\,\bar{s}_x + \sigma\,d_y\,\bar{s}_y$$

where $\sigma = \lambda + 2\mu$ (not Poisson's ratio). We rewrite 10.4 in the form

$$d_y\,\bar{s}_y = \sigma^{-1}(\bar{\tau}_{yy} - \lambda\,\xi\,\bar{s}_x)$$

10.5 $$d_y\,\bar{s}_x = \mu^{-1}\,\bar{\tau}_{yx} - \xi\,\bar{s}_y$$

$$\bar{\tau}_{xx} = \lambda\,\sigma^{-1}\,\bar{\tau}_{yy} + 4\mu(\lambda+\mu)\sigma^{-1}\,\bar{s}_x$$

We combine 10.3 and 10.5 and write the combination in matrix form

$$
10.6 \quad d_y \begin{bmatrix} \bar{S}_y \\ \bar{S}_x \\ \bar{\tau}_{yy} \\ \bar{\tau}_{yx} \end{bmatrix} = \begin{bmatrix} 0 & -\lambda\sigma^{-1}\xi & \sigma^{-1} & 0 \\ -\xi & 0 & 0 & \mu^{-1} \\ \rho s^2 & 0 & 0 & -\xi \\ 0 & \rho s^2 - 4\mu(\lambda+\mu)\sigma^{-1}\xi^2 & -\lambda\sigma^{-1}\xi & 0 \end{bmatrix} \times \begin{bmatrix} \bar{S}_y \\ \bar{S}_x \\ \bar{\tau}_{yy} \\ \bar{\tau}_{yx} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \bar{f}(s)\,\xi\,\delta(y-h) \\ \bar{f}(s)\,\delta'(y-h) \end{bmatrix}
$$

which is a special case of 9.5. Since the trace of the coefficient matrix in 10.6 is zero, we see from 9.17 that the propagator has unit determinant. For the simple P-SV waveguide the coefficient matrix in 10.6 is independent of y, $0 \le y \le D$, so we can calculate $\underline{P}(y, y_0)$ from 9.18 or 9.9. Then we can use 9.12 to get the solution to 10.6.

We shall use 9.9 to calculate $\underline{P}$. Eq. 5.27 (page 80) represents two of the four independent solutions. In 5.16 (page 76) we found that $\pm\nu_\alpha$, $\pm\nu_\beta$ were the four possible exponents. Thus to 5.27 we add the other two possibilities

$$
\bar{S}_y = -A\nu_\alpha e^{-\nu_\alpha y} - \xi B e^{-\nu_\beta y} + \nu_\alpha C e^{\nu_\alpha y} - \xi D e^{\nu_\beta y}
$$

10.7

$$
\bar{S}_x = A\xi e^{-\nu_\alpha y} - B\nu_\beta e^{-\nu_\beta y} + C\xi e^{\nu_\alpha y} + D\nu_\beta e^{\nu_\beta y}
$$

We use 5.26 ($y \neq 0$) to determine the stresses

$$\bar{\tau}_{yy} = \left[(\nu_\beta^2-\xi^2)A e^{-\nu_\alpha y} + 2\xi\nu_\beta B e^{-\nu_\beta y} + (\nu_\beta^2-\xi^2) C e^{\nu_\alpha y} - 2\xi\nu_\beta D e^{\nu_\beta y}\right]\mu$$

10.8

$$\bar{\tau}_{yx} = \left[-2\xi\nu_\alpha A e^{-\nu_\alpha y} + (\nu_\beta^2-\xi^2) B e^{-\nu_\beta y} + 2\xi\nu_\alpha C e^{\nu_\alpha y} + (\nu_\beta^2-\xi^2) D e^{\nu_\beta y}\right]\mu$$

If we choose A, B, C, D so that each one of 10.7 and 10.8 is unity when $y=0$ then we shall have the propagator from $y=0$.

10.9

$$\begin{bmatrix} -\nu_\alpha & -\xi & \nu_\alpha & -\xi \\ \xi & -\nu_\beta & \xi & \nu_\beta \\ (\nu_\beta^2-\xi^2) & 2\xi\nu_\beta & (\nu_\beta^2-\xi^2) & -2\xi\nu_\beta \\ -2\xi\nu_\alpha & (\nu_\beta^2-\xi^2) & 2\xi\nu_\alpha & (\nu_\beta^2-\xi^2) \end{bmatrix} \times \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \mu^{-1} \\ \mu^{-1} \end{bmatrix}$$

Let $A = \mathcal{A} + \mathcal{C}$, $C = \mathcal{A} - \mathcal{C}$, $B = \mathcal{B} + \mathcal{D}$, $D = \mathcal{B} - \mathcal{D}$ in 10.9

10.10

$$\begin{bmatrix} 0 & -2\xi & -2\nu_\alpha & 0 \\ 2\xi & 0 & 0 & -2\nu_\beta \\ 2(\nu_\beta^2-\xi^2) & 0 & 0 & 4\xi\nu_\beta \\ 0 & 2(\nu_\beta^2-\xi^2) & -4\xi\nu_\alpha & 0 \end{bmatrix} \times \begin{bmatrix} \mathcal{A} \\ \mathcal{B} \\ \mathcal{C} \\ \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \mu^{-1} \\ \mu^{-1} \end{bmatrix} = \begin{bmatrix} \bar{S}_y \\ \bar{S}_x \\ \bar{\tau}_{yy}/\mu \\ \bar{\tau}_{yx}/\mu \end{bmatrix}_{y=0}$$

We recognize the 4×4 system 10.10 as a decoupled pair of 2×2 systems. The solution of 10.10 is

10.11
$$\begin{bmatrix} \mathcal{A} \\ \mathcal{B} \\ \mathcal{C} \\ \mathcal{D} \end{bmatrix} = \begin{bmatrix} 0 & \xi/k_\beta^2 & 1/2\mu k_\beta^2 & 0 \\ -\xi/k_\beta^2 & 0 & 0 & 1/2\mu k_\beta^2 \\ \dfrac{-(\nu_\beta^2-\xi^2)}{2\nu_\alpha k_\beta^2} & 0 & 0 & -\xi/2\mu\nu_\alpha k_\beta^2 \\ 0 & \dfrac{-(\nu_\beta^2-\xi^2)}{2\nu_\beta k_\beta^2} & \xi/2\mu\nu_\beta k_\beta^2 & 0 \end{bmatrix} \times \begin{bmatrix} \bar{S}_y \\ \bar{S}_x \\ \bar{\tau}_{yy} \\ \bar{\tau}_{yx} \end{bmatrix}_{y=0}$$

We combine 10.7 and 10.8 in matrix form and write the combination in terms of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$,

10.12
$$\begin{bmatrix} \bar{S}_y \\ \bar{S}_x \\ \bar{\tau}_{yy} \\ \bar{\tau}_{yx} \end{bmatrix} = \begin{bmatrix} 2\nu_\alpha \operatorname{sh}\nu_\alpha y & -2\xi \operatorname{ch}\nu_\beta y & -2\nu_\alpha \operatorname{ch}\nu_\alpha y & 2\xi \operatorname{sh}\nu_\beta y \\ 2\xi \operatorname{ch}\nu_\alpha y & 2\nu_\beta \operatorname{sh}\nu_\beta y & -2\xi \operatorname{sh}\nu_\alpha y & -2\nu_\beta \operatorname{ch}\nu_\beta y \\ 2\mu(\nu_\beta^2-\xi^2)\operatorname{ch}\nu_\alpha y & -4\mu\xi\nu_\beta \operatorname{sh}\nu_\beta y & -2\mu(\nu_\beta^2-\xi^2)\operatorname{sh}\nu_\alpha y & 4\mu\xi\nu_\beta \operatorname{ch}\nu_\beta y \\ 4\mu\xi\nu_\alpha \operatorname{sh}\nu_\alpha y & 2\mu(\nu_\beta^2-\xi^2)\operatorname{ch}\nu_\beta y & -4\mu\xi\nu_\alpha \operatorname{ch}\nu_\alpha y & -2\mu(\nu_\beta^2-\xi^2)\operatorname{sh}\nu_\beta y \end{bmatrix} \times \begin{bmatrix} \mathcal{A} \\ \mathcal{B} \\ \mathcal{C} \\ \mathcal{D} \end{bmatrix}$$

Substituting 10.11 into 10.12 gives us the propagator matrix, $\underline{P}(y,0)$, whose elements are

$$P_{11} = 2\xi^2 k_\beta^{-2}\, ch\,\nu_\beta y + (\nu_\beta^2 - \xi^2) k_\beta^{-2}\, ch\,\nu_\alpha y$$ $- 2 \xi^{-4}\, ch\, k y$

$$P_{21} = -2\xi \nu_\beta k_\beta^{-2}\, sh\,\nu_\beta y + (\nu_\beta^2 - \xi^2)\xi \nu_\alpha^{-1} k_\beta^{-2}\, sh\,\nu_\alpha y$$

$$P_{31} = 4\mu \xi^2 \nu_\beta k_\beta^{-2}\, sh\,\nu_\beta y + \mu(\nu_\beta^2 - \xi^2)^2 \nu_\alpha^{-1} k_\beta^{-2}\, sh\,\nu_\alpha y$$

$$P_{41} = -2\mu(\nu_\beta^2 - \xi^2)\xi k_\beta^{-2}\, ch\,\nu_\beta y + 2\mu(\nu_\beta^2 - \xi^2)\xi k_\beta^{-2}\, ch\,\nu_\alpha y$$

$$P_{12} = 2\xi \nu_\alpha k_\beta^{-2}\, sh\,\nu_\alpha y - (\nu_\beta^2 - \xi^2)\nu_\beta^{-1} k_\beta^{-2}\, sh\,\nu_\beta y$$

$$P_{22} = 2\xi^2 k_\beta^{-2}\, ch\,\nu_\alpha y + (\nu_\beta^2 - \xi^2) k_\beta^{-2}\, ch\,\nu_\beta y$$

10.13

$$P_{32} = 2\mu(\nu_\beta^2 - \xi^2)\xi k_\beta^{-2}\, ch\,\nu_\alpha y - 2\mu(\nu_\beta^2 - \xi^2)\xi k_\beta^{-2}\, ch\,\nu_\beta y$$

$$P_{42} = 4\mu \xi^2 \nu_\alpha k_\beta^{-2}\, sh\,\nu_\alpha y + \mu(\nu_\beta^2 - \xi^2)^2 \nu_\beta^{-1} k_\beta^{-2}\, sh\,\nu_\beta y$$

$$P_{13} = \mu^{-1} \nu_\alpha k_\beta^{-2}\, sh\,\nu_\alpha y + \mu^{-1} \xi^2 \nu_\beta^{-1} k_\beta^{-2}\, sh\,\nu_\beta y$$

$$P_{23} = \mu^{-1} \xi k_\beta^{-2}\, ch\,\nu_\alpha y - \mu^{-1} \xi k_\beta^{-2}\, ch\,\nu_\beta y$$

$$P_{33} = (\nu_\beta^2 - \xi^2) k_\beta^{-2}\, ch\,\nu_\alpha y + 2\xi^2 k_\beta^{-2}\, ch\,\nu_\beta y$$

$$P_{43} = 2\xi \nu_\alpha k_\beta^{-2}\, sh\,\nu_\alpha y - (\nu_\beta^2 - \xi^2)\xi \nu_\beta^{-1} k_\beta^{-2}\, sh\,\nu_\beta y$$

$$P_{14} = -\mu^{-1}\xi k_\beta^{-2}\,ch\,\nu_\beta y + \mu^{-1}\xi k_\beta^{-2}\,ch\,\nu_\alpha y.$$

10.13
$$P_{24} = \mu^{-1}\nu_\beta k_\beta^{-2}\,sh\,\nu_\beta y + \mu^{-1}\xi^2\nu_\alpha^{-1}k_\beta^{-2}\,sh\,\nu_\alpha y$$

$$P_{34} = -2\xi\nu_\beta k_\beta^{-2}\,sh\,\nu_\beta y + (\nu_\beta^2-\xi^2)\xi\nu_\alpha^{-1}k_\beta^{-2}\,sh\,\nu_\alpha y$$

$$P_{44} = (\nu_\beta^2-\xi^2)k_\beta^{-2}\,ch\,\nu_\beta y + 2\xi^2 k_\beta^{-2}\,ch\,\nu_\alpha y$$

In the half space, $y<0$, we take the C and D terms in 10.7 and 10.8. at $y=0$ we have

10.14
$$\begin{bmatrix}\bar{S}_y\\ \bar{S}_x\\ \bar{\tau}_{yy}\\ \bar{\tau}_{yx}\end{bmatrix} = \begin{bmatrix}\nu_{\alpha_2}\\ \xi\\ (\nu_{\beta_2}^2-\xi^2)\mu_2\\ 2\xi\nu_{\alpha_2}\mu_2\end{bmatrix} C_1 + \begin{bmatrix}-\xi\\ \nu_{\beta_2}\\ -2\xi\nu_{\beta_2}\mu_2\\ (\nu_{\beta_2}^2-\xi^2)\mu_2\end{bmatrix} C_2 = C_1\underline{f}_1^0 + C_2\underline{f}_2^0 \quad y=0$$

where we have written C_1 for C and C_2 for D to avoid confusion with the waveguide thickness denoted by D. Let us represent the leftmost column vector in 10.14 as $\underline{f}(y)$. Then

10.15
$$d_y\,\underline{f}(y) = \underline{\underline{A}}(y)\,\underline{f}(y) + \underline{g}(y)$$

where $\underline{\underline{A}}$ is the constant coefficient matrix in 10.6 whose propagator from zero is 10.13. The term $\underline{g}(y)$ is the rightmost vector in 10.6

From 9,12 we have

$$f_i(y) = \int_0^y P_{ij}(y,\xi)\, g_j(\xi)\, d\xi + P_{ij}(y,0)\left(C_1 f^{\circ}_{1j} + C_2 f^{\circ}_{2j}\right) \qquad 10,16$$

In a homogeneous region $\underline{P}(y,\xi) = \underline{P}(y-\xi, 0)$

$$f_i(y) = \int_0^y d\xi \left(P_{i3}(y-\xi,0)\, g_3(\xi) + P_{i4}(y-\xi,0)\, g_4(\xi) \right) + P_{ij}(y,0)\left(C_1 f^{\circ}_{1j} + C_2 f^{\circ}_{2j}\right) \qquad 10,17$$

For the source we have chosen (10,6). For $y > h$

$$f_i(y) = P_{i3}(y-h,0)\, \bar{f}(s)\, \xi - d_h P_{i4}(y-h)\, \bar{f}(s) + P_{ij}(y,0)\left(C_1 f^{\circ}_{1j} + C_2 f^{\circ}_{2j}\right) \qquad 10,18$$

When $y = D$ $f_3(D) = f_4(D) = 0$ and these two equations can be used to find C_1 and C_2

$$0 = P_{33}(D-h,0)\, \bar{f}(s)\, \xi - d_h P_{34}(D-h)\, \bar{f}(s) + P_{3j}(D,0)\left(C_1 f^{\circ}_{1j} + C_2 f^{\circ}_{2j}\right) \qquad 10,19$$

$$0 = P_{43}(D-h,0)\, \bar{f}(s)\, \xi - d_h P_{44}(D-h,0)\, \bar{f}(s) + P_{4j}(D,0)\left(C_1 f^{\circ}_{1j} + C_2 f^{\circ}_{2j}\right)$$

From 10.19 we find the secular function

10.20 $$\mathcal{S}(\xi, s) = \begin{vmatrix} P_{3j}(D,0)\, f^{\circ}_{1j} & P_{3j}(D,0)\, f^{\circ}_{2j} \\ P_{4j}(D,0)\, f^{\circ}_{1j} & P_{4j}(D,0)\, f^{\circ}_{2j} \end{vmatrix}$$

It is clearly a rather elaborate expression when written out in detail. When $D = 0$ in 10.20 (Lamb's problem, ch. 5) we have

10.21 $$\mathcal{S}(\xi, s) = f^{\circ}_{13}\, f^{\circ}_{24} - f^{\circ}_{23}\, f^{\circ}_{14}$$

which is $\mu^2 R(\xi)$ in 5.32 for the halfspace $y < 0$.

Since the algebra involved in the generalized ray expansion is quite involved we will not give the expansion in detail. It should be clear that the integrand of the ξ inversion integral will have a numerator and denominator each of which is the sum of products of algebraic factors and hyperbolic functions. The hyperbolic functions can be written in terms of exponentials. The largest exponential can be factored out of the denominator and the reciprocal of the remaining expression can be expanded in a geometrical series. This expansion gives the integrand as an infinite sum of exponentials

Each term in the sum can be evaluated by the Cagniard-de Hoop formalism. The expansion has been given by Newlands (1952) and Gilbert (1956).

M. Newlands, 1952, The disturbance due to a line source in a semi-infinite elastic medium with a single surface layer: Phil. Trans. Roy. Soc., London, A, 245, 213-308.

F. Gilbert, 1956, Seismic wave propagation in a two-layer half-space: Ph.D. thesis, M.I.T.

The first motion approximation has been used by Knopoff, Gilbert, and Pilant (1960) to investigate the early arrivals.

L. Knopoff, F. Gilbert, and W. L. Pilant, 1960, Wave propagation in a medium with a single layer: J.G.R. 65, 265-278.

Once the generalized transmission and reflection coefficients are known for a single interface the ray method is easily extended to multiple layers (Spencer, 1960; Dunkin, 1963).

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[3] T. W. Spencer, 1960, The method of generalized reflection and transmission coefficients: Geophysics, 25, 625-641.

J. W. Dunkin, 1963, A study of two dimensional head waves in fluid and solid systems: Geophysics, 28, 563-581.

In the waveguide mode approach to this problem the function $S(\xi, s)$ in 10.20 is written as $S(k, \omega)$, $\xi = ik$, $s = -i\omega$. It appears in the denominator of double inversion integrals such as

10.22 $$u(x, y, t) = \int_{-\infty}^{\infty} dk\, e^{ikx} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, N(k,\omega)/S(k,\omega)$$

and the zeros of S are the poles of the integrand. Since $\underline{P}$ has no branch points the only branch points of the integrand are those of $\underline{F}_1^0$ and $\underline{F}_2^0$ in 10.1[illegible] which are $\omega = \pm \alpha_2 k$, $\pm \beta_2 k$. In other words the branch points of the integrand are those that belong to the half space radicals ν_{α}, ν_{β}.

Let us examine $S(k, \omega)$ for its zeros. Because both ν_{α_2} and ν_{β_2} appear in $S(k, \omega)$

it must have four Riemann sheets according as $\mathrm{Re}\, \nu_{\alpha_2} \gtrless 0$, $\mathrm{Re}\, \nu_{\beta_2} \gtrless 0$. We anticipate that there are roots of Δ on all four sheets. First let $\star(s) = 0$ $\omega(s) \neq 0$. Then P_{3i} and P_{4i} are

$$P_{31} = -\mu_1 \omega^2 \beta_1^{-2} \nu_{\alpha_1}^{-1} \, sh\, \nu_{\alpha_1} D \qquad P_{41} = 0$$

10.23

$$P_{32} = 0 \qquad P_{42} = -\mu_1 \omega^2 \beta_1^{-2} \nu_{\beta_1}^{-1} \, sh\, \nu_{\beta_1} D$$

$$P_{33} = ch\, \nu_{\alpha_1} D \qquad P_{43} = 0$$

$$P_{34} = 0 \qquad P_{44} = ch\, \nu_{\beta_1} D$$

and f_1^0 and f_2^0 are

$$f_{11}^0 = \nu_{\alpha_2} \qquad f_{21}^0 = 0$$

10.24

$$f_{12}^0 = 0 \qquad f_{22}^0 = \nu_{\beta_2}$$

$$f_{13}^0 = -\mu_2 \omega^2/\beta_2^2 \qquad f_{23}^0 = 0$$

$$f_{14}^0 = 0 \qquad f_{24}^0 = -\mu_2 \omega^2/\beta_2^2$$

so that 10.20 becomes

10.25
$$S(0,\omega) = \omega^2\left(\rho_1 \nu_{\alpha_1}^{-1} \nu_{\alpha_2}\, \mathrm{sh}\, \nu_{\alpha_1} D + \rho_2\, \mathrm{ch}\, \nu_{\alpha_1} D\right) \times$$
$$\times \left(\rho_1 \nu_{\beta_1}^{-1} \nu_{\beta_2}\, \mathrm{sh}\, \nu_{\beta_1} D + \rho_2\, \mathrm{ch}\, \nu_{\beta_1} D\right)$$

In 10.25 $k = 0$ so $\nu = \mp i \bar{\nu}$

10.26
$$S(0,\omega) = \frac{-\omega^2}{\alpha_2\beta_2}\left(\rho_1 \alpha_1 \sin \omega D/\alpha_1 \mp i \rho_2 \alpha_2 \cos \omega D/\alpha_1\right) \times$$
$$\times \left(\rho_1 \beta_1 \sin \omega D/\beta_1 \mp i \rho_2 \beta_2 \cos \omega D/\beta_1\right)$$

In 10.25 the root at $\omega = 0$ is independent of the sign of $\nu_{\alpha_2}, \nu_{\beta_2}$. In 10.26 the first term in parentheses has zeros on the sheet where $\mathrm{Re}\, \nu_{\alpha_2} \leq 0$, and the second term has zeros on the sheet where $\mathrm{Re}\, \nu_{\beta_2} \leq 0$. We designate the sheets as follows:

10.27

$\mathrm{Re}\, \nu_{\alpha_2}$	$\mathrm{Re}\, \nu_{\beta_2}$	sheet
≥ 0	≥ 0	$(+,+)$
≥ 0	≤ 0	$(+,-)$
≤ 0	≥ 0	$(-,+)$
≤ 0	≤ 0	$(-,-)$

We shall assume $\rho_1\alpha_1 < \rho_2\alpha_2$, $\rho_1\beta_1 < \rho_2\beta_2$. Then the roots of $S(0,\omega)$ are

10.28 $\omega = 0$ all sheets

$$\omega D/\alpha_1 = (n+1/2)\pi \pm i\, \frac{1}{2} \log \left(\frac{\rho_2\alpha_2 - \rho_1\alpha_1}{\rho_2\alpha_2 + \rho_1\alpha_1} \right) \quad ; \quad (-,+),\ (-,-)$$

10.28

$$\omega D/\beta_1 = (n+1/2)\pi \pm i\, \frac{1}{2} \log \left(\frac{\rho_2\beta_2 - \rho_1\beta_1}{\rho_2\beta_2 + \rho_1\beta_1} \right) \quad ; \quad (+,-),\ (-,-)$$

On the $(-,+)$ and $(-,-)$ sheets there are compressional organ pipe roots. On the $(+,-)$ and $(-,-)$ there are shear organ pipe roots.

Now let both k and ω go to zero but keep the ratio k/ω fixed. Then $P_{ij} = \delta_{ij}$ and we have 10.21. Thus we have Rayleigh roots (Lamb's problem) on all four sheets.

To discuss the root loci as a function of k requires the consideration of special cases. The reader is referred to the literature (Rosenbaum, 1960; Phinney, 1961; Gilbert, 1964; Laster, Foreman and Linville, 1965)

J. H. Rosenbaum, 1960, The long time response of a layered elastic medium to explosive sound; J.G.R., 65, 1577-1613.

R.A. Phinney, 1961, Leaking modes in the crustal waveguide. Part 1. The oceanic PL wave: J.G.R., 66, 1445-1469.

F. Gilbert, 1964, Propagation of transient leaking modes in a stratified elastic waveguide: Revs. Geoph., 2, 123-153.

S. J. Laster, J. G. Foreman and A. F. Linville, 1965, Theoretical investigation of modal seismograms for a layer over a half space: Geophysics, 30, 571-596.

The fact that the organ pipe modes 10,25 are decoupled at $k=0$ can be demonstrated for arbitrary stratification. When $k(\xi)=0$ in 10.6 we have, for the homogeneous part,

$$d_y \begin{bmatrix} \bar{S}_y \\ \bar{\tau}_{yy} \end{bmatrix} = \begin{bmatrix} 0 & \sigma^{-1} \\ -\rho\omega^2 & 0 \end{bmatrix} \times \begin{bmatrix} \bar{S}_y \\ \bar{\tau}_{yy} \end{bmatrix}$$

10,29

$$d_y \begin{bmatrix} \bar{S}_x \\ \bar{\tau}_{yx} \end{bmatrix} \begin{bmatrix} 0 & \mu^{-1} \\ -\rho\omega^2 & 0 \end{bmatrix} \begin{bmatrix} \bar{S}_x \\ \bar{\tau}_{yx} \end{bmatrix}$$

which clearly exhibit the decoupling.

The roots of Δ on the $(-,+)$ sheet do not contribute to 10,22 (Gilbert, 1964).

In 10.17 let

10.30 $F_{i1}(0) = C_1 \delta_{1i}^{0}$, $F_{i2}(0) = C_2 \delta_{2i}^{0}$

Then $F_{ij}(y)$, $i = 1, \ldots, 4$, $j = 1, 2$, are the elements of a 4×2 integral matrix of 9.6

9.6 $d_y \underline{f}(y) = \underline{\underline{A}}(y) \cdot \underline{f}$

where $\underline{\underline{A}}$ is given by 10.6. We can think of $\underline{\underline{F}}(y)$ as a 4×4 integral matrix whose third and fourth columns are zero. Then $\underline{\underline{F}}$ is <u>not</u> a fundamental matrix. But $\underline{\underline{F}}(y)$ is a solution of

10.31 $d_y \underline{\underline{F}}(y) = \underline{\underline{A}}(y) \cdot \underline{\underline{F}}(y)$

Our choice of $\underline{\underline{F}}(y)$ satisfies the "radiation condition" for $y < 0$. When the propagator is known $\underline{\underline{F}}(y)$ is given by

10.32 $\underline{\underline{F}}(y) = \underline{\underline{P}}(y, 0)\, \underline{\underline{F}}(0)$

The secular function 10.20 is

10.33 $$S = \begin{vmatrix} F_{31}(D) & F_{32}(D) \\ F_{41}(D) & F_{42}(D) \end{vmatrix} = F_{31}(D) F_{42}(D) - F_{32}(D) F_{41}(D)$$

Thus S is one of the second order minors of $\underline{F}$. The $\omega(s)$ eigenvalues for a given $k(\xi)$ are the zeros of S. A difficulty that frequently arises when 10.31 is integrated is that even when the minor, S, is of moderate size (remember that we seek its zeros), the elements of $\underline{F}$ which enter into its composition, 10.33, may be so large that the (small) minor is the difference of two very large numbers. In a digital computer this can lead to a severe loss of numerical accuracy. A way to circumvent this difficulty is to compute the minor, S, directly rather than from the elements of $\underline{F}$.

A square matrix $\underline{F}$ of order n has $\binom{n}{m}^2$ minors of order $m \leq n$, where

10,34 $$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

When the minors of order m are arranged in a square array in some definite order, the array is called the <u>m^{th} minor matrix</u> of $\underline{F}$ which we denote by $\mathcal{F}^{(m)}$. Gantmacher (1959, 1, p 19-20) calls $\mathcal{F}^{(m)}$ the m^{th} compound matrix.

An m^{th} order minor of $\underline{F}$ is

10,35 $$F\binom{i_1 \ldots i_m}{k_1 \ldots k_m} = F_{i_1 j_1} \cdots F_{i_m j_m} \epsilon^{j_1 \ldots j_m}_{k_1 \ldots k_m}$$

where the summation convention is used and $\epsilon^{j_1 \cdots j_m}_{k_1 \cdots k_m}$ is defined to be 1 if all the k's are different and the j's are some even permutation of the k's; is defined to be -1 if all the k's are different and the j's are some odd permutation of the k's; and is defined to be 0 otherwise. For example when $n=4$, $m=2$

10,36 $$F\binom{3\,4}{1\,2} = F_{3j_1} F_{4j_2} \epsilon^{j_1 j_2}_{1\,2}$$

In 10,36 ϵ is zero unless $j_1 j_2 = 12$ or 21 so

10,37 $$F\binom{3\,4}{1\,2} = F_{31} F_{42} \epsilon^{12}_{12} + F_{32} F_{41} \epsilon^{21}_{12} = F_{31} F_{42} - F_{32} F_{41}$$

which is the expression for S in 10.33. Since $\underline{F}$ satisfies 10,31 we can find $dy\, F\binom{i_1 \cdots i_m}{k_1 \cdots k_m}$

10,38 $$dy\, \dot{F}\binom{i_1 \cdots i_m}{k_1 \cdots k_m} = A_{i_1 l}\, F\binom{l \cdots i_m}{k_1 \cdots k_m} + \cdots + A_{i_m l}\, F\binom{i_1 \cdots l}{k_1 \cdots k_m}$$

which has the form of a system of linear equations. Therefore $\mathcal{F}^{(m)}(y)$, the m^{th} minor matrix of $\underline{F}$, is a solution of

10,39 $$dy\, \mathcal{F}^{(m)}(y) = \mathcal{A}_m(y)\, \mathcal{F}^{(m)}(y)$$

where $\mathcal{A}_m(y)$ is a square coefficient matrix of order $\binom{n}{m}$ each of whose elements is some linear combination of the elements of $\underline{A}(y)$ in 10.31. For example when $n=4$ $m=2$ we arrange the second order minors of $\underline{F}$ in the square array of order $\binom{n}{m} = \binom{4}{2} = 6$

10.40 $$\mathcal{F}^{(2)} = \begin{bmatrix} F\binom{12}{12} & F\binom{12}{13} & F\binom{12}{14} & F\binom{12}{23} & F\binom{12}{24} & F\binom{12}{34} \\ F\binom{13}{12} & & & & & \cdot \\ F\binom{14}{12} & & & & & \cdot \\ F\binom{23}{12} & & \text{etcetera} & & & \cdot \\ F\binom{24}{12} & & & & & \cdot \\ F\binom{34}{12} & \cdot & \cdot & \cdot & \cdot & F\binom{34}{34} \end{bmatrix}$$

Then $\mathcal{A}_2$ is, in terms of the elements of $\underline{A}$ in 10.31,

10.41 $$\mathcal{A}_2 = \begin{bmatrix} A_{11}+A_{22} & A_{23} & A_{24} & -A_{13} & -A_{14} & 0 \\ A_{32} & A_{11}+A_{33} & A_{34} & A_{12} & 0 & -A_{14} \\ A_{42} & A_{43} & A_{11}+A_{44} & 0 & A_{12} & A_{13} \\ -A_{31} & A_{21} & 0 & A_{22}+A_{33} & A_{34} & -A_{24} \\ -A_{41} & 0 & A_{21} & A_{43} & A_{22}+A_{44} & A_{23} \\ 0 & -A_{41} & A_{31} & -A_{42} & A_{32} & A_{33}+A_{44} \end{bmatrix}$$

Notice that $tr(\mathcal{A}_2) = 3\, tr(\underline{A})$. In general

10.42 $tr(\mathcal{A}_m) = \binom{n-1}{m-1} tr(\underline{A})$

It is important to remember that $\mathcal{A}_m$ is not the m^{th} minor matrix of $\underline{A}$. When $tr(\underline{A}) = 0$ then $tr(\mathcal{A}_m) = 0$, $m \neq n$. When $m = n$

10.43 $\mathcal{F}^{(n)} = |\underline{E}|$, $\binom{n}{m} = 1$, $\mathcal{A}_n = tr(\underline{A})$

and 10.39 reduces to 9.15

In 10.35 the upper indices of $F\binom{i_1 \ldots i_m}{k_1 \ldots k_m}$ are row indices of $\underline{F}$ and the lower indices are column indices. In the P-SV problem $\underline{F}$ has zeros for its third and fourth columns. Then all columns of $\mathcal{F}^{(2)}$ in 10.40 are zero except the first. Eq. 10.31 is a 4X2 system equivalent to 8 first order equations and 10.39, $n=4$, $m=2$, is a 6X1 system equivalent to 6 first order equations. Thus not only do we get S more accurately from 10.39, but also we get S more quickly.

11. Free Oscillations of the Earth

We begin with the continuity equation 1.1

1.1 $$\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

and the momentum equation 1.3

1.3 $$\rho \, d_t \underline{v} = \underline{f} + \underline{\nabla} \cdot \underline{\underline{T}}$$

We write $\underline{\underline{T}}$ as

11.1 $$T_{ij} = -P \delta_{ij} + \tau_{ij} \quad ; \quad P = -\tfrac{1}{3} T_{ii} = -\tfrac{1}{3} \operatorname{tr}(\underline{\underline{T}})$$

If $\underline{\underline{\tau}}$ were zero $\underline{\underline{T}}$ would be an isotropic second order tensor. Thus $\underline{\underline{\tau}}$ represents how $\underline{\underline{T}}$ deviates from isotropy and $\underline{\underline{\tau}}$ is called the <u>stress deviator</u>.

Our model of the earth is a sphere. The density and Lamé parameters are considered to be functions only of the radius. When the earth is at rest we assume that it is in hydrostatic equilibrium. Then $\underline{\underline{\tau}} = 0$ and $\underline{\nabla} \cdot \underline{\underline{T}} = -\underline{\nabla} P$. Let P_0 be the hydrostatic pressure. Then 1.3 becomes

11.2 $$\underline{f}_0 = \underline{\nabla} P_0$$

when the earth is at rest. For the moment we neglect the rotation of the earth.

Now we define $\phi_0(r)$ as the gravitational potential of the spherically symmetric mass distribution $\rho_0(r)$, and we define $g_0(r) = \partial_r \phi_0(r)$. Then the gravitational field is $-\hat{r}\, g_0(r)$ and Poisson's equation is

11.3 $$(\partial_r + r^{-1})\, g_0(r) = 4\pi G\, \rho_0(r)$$

where G is Newton's universal constant of gravitation. A particle of density $\rho_0(r)$ feels a force per unit volume of $-\hat{r}\, \rho_0(r)\, g_0(r) = \underline{f}_0$. Thus

11.4 $$\underline{f}_0 = -\hat{r}\, \rho_0(r)\, g_0(r) = -\rho_0(r)\, \underline{\nabla}\, \phi_0(r)$$

When there is motion the density will change and so will the potential. In spherical coordinates (r, θ, φ)

11.5
$$\rho = \rho_0(r) + \rho_1(r, \theta, \varphi)$$
$$\phi = \phi_0(r) + \phi_1(r, \theta, \varphi)$$

where

11.6 $$\nabla^2 \phi_1 = 4\pi G\, \rho_1$$

We regard $\underline{f}$ in 1.3 as two terms

11.7 $\underline{f} = -\rho \nabla \varphi + \underline{f}_s$

where the first term is the body force due to the density distribution and the second term, $\underline{f}_s$, is the "source" due to explosions, ruptures, etc. For the present we neglect $\underline{f}_s$.

Eq. 11.4 and 11.2 give the equilibrium equation

11.8 $\nabla P_0 + \rho_0 \nabla \phi_0 = 0$

When there is motion, $\underline{v}$,

$$\rho = \rho_0 + \rho_1$$
$$\underline{f} = -\rho \nabla \phi$$
11.9 $$\phi = \phi_0 + \phi_1$$
$$\underline{\underline{T}} = -P \underline{\underline{I}} + \underline{\underline{\tau}}$$
$$P = P_0 + P_1$$

If we substitute 11.9 into 11.3 we have the expression. conservation of momentum.

11.10 $(\rho_0 + \rho_1)\, d_t \underline{v} = -(\rho_0 + \rho_1)\underline{\nabla}(\phi_0 + \phi_1) - \underline{\nabla}(P_0 + P_1) + \underline{\nabla} \cdot \underline{\underline{\tau}}$

We use 11.8 in 11.10 and then we linearize the result

11.11 $\rho_0 \partial_t \underline{v} = -\rho_0 \underline{\nabla} \phi_1 - \rho_1 \underline{\nabla} \phi_0 - \nabla P_1 + \underline{\nabla} \cdot \underline{\underline{\tau}}$

Since the stress deviator $\underline{\underline{\tau}}$ has zero trace it causes no volume change. The volume (or density) change is caused by P and we assume the linear relation

11.12 $$\frac{dP}{dt} = \frac{K}{\rho} \frac{d\rho}{dt}$$

We linearize 11.12

11.13 $$\frac{\partial P_1}{\partial t} + \underline{v} \cdot \underline{\nabla} P_0 = \frac{K}{\rho_0} \left(\frac{\partial \rho_1}{\partial t} + \underline{v} \cdot \nabla \rho_0 \right)$$

and we linearize 1.1

11.14 $$\frac{\partial}{\partial t} \rho_1 + \rho_0 \underline{\nabla} \cdot \underline{v} + \underline{v} \cdot \underline{\nabla} \rho_0 = 0$$

Using 11.14 in 11.13 we get

11.15 $$\frac{\partial P_1}{\partial t} + \underline{v} \cdot \nabla P_0 = - K \underline{\nabla} \cdot \underline{v}$$

Using 11.8 in 11.15 gives

11.16 $$\frac{\partial P_1}{\partial t} = \rho_0 \underline{v} \cdot \nabla \phi_0 - k \nabla \cdot \underline{v}$$

In terms of the displacement $\underline{s}$ 11.16 is

11.17 $$P_1 = \rho_0 \underline{s} \cdot \nabla \phi_0 - k \underline{\nabla} \cdot \underline{s} = P_1^g + P_1^e$$

and 11.14 is

11.18 $$\rho_1 = -\rho_0 \underline{\nabla} \cdot \underline{s} - \underline{s} \cdot \nabla \rho_0$$

We have all of the first order quantities except the stress deviator, $\underline{\underline{\tau}}$. Let $\underline{\underline{e}}$ be the strain

11.19 $$e_{ij} = \frac{1}{2} (s_{i,j} + s_{j,i})$$

and write the strain as the sum of an isotropic part and a deviatoric part

11.20 $$e_{ij} = \frac{1}{3} e_{kk} \delta_{ij} + \epsilon_{ij} = \frac{1}{3} \underline{\nabla} \cdot \underline{s} \, \delta_{ij} + \epsilon_{ij}$$

where $\underline{\epsilon}$ is the strain deviator. The total elastic stress is $-P_1^e \delta_{ij} + \tau_{ij}$

11.21 $$\tau_{ij}^e = -P_1^e \delta_{ij} + \tau_{ij} = \lambda \underline{\nabla} \cdot \underline{s} \, \delta_{ij} + 2\mu \, e_{ij}$$

The trace of 11,21 is

11,22 $$-3 P_1^e = (3\lambda + 2\mu)\, \underline{\nabla} \cdot \underline{s}$$

and we have $\kappa = \lambda + 2/3\,\mu$, and κ is the bulk modulus or incompressibility of the material. Then

11,23 $$\underline{\underline{\tau}} \overset{?}{=} \mu\, \underline{\underline{\epsilon}} \quad \text{Should}\quad \tau_{ij} = 2\mu\epsilon_{ij} \quad (\text{P.G.})$$

The stress deviator is linearly proportional to the strain deviator. The factor of proportionality is the rigidity.

Substituting 11,17, 11,18 and 11,21 into 11,11 gives

11,24 $$\rho_0 \partial_t^2 \underline{s} = -\rho_0 \nabla \phi_1 + (\rho_0 \nabla \cdot \underline{s} + \underline{s} \cdot \underline{\nabla} \rho_0)\, \hat{r} g_0 - \underline{\nabla}(\rho_0\, \underline{s} \cdot \hat{r} g_0) + \underline{\nabla} \cdot \underline{\underline{\tau}}^e$$

We must solve 11,24 and 11,6 along with the appropriate boundary conditions. We have already seen in Ch. 1 that $\underline{s}$ and $\hat{n} \cdot \underline{\underline{T}}$ must be continuous on a deformed boundary. Now consider

11,25 $$\nabla^2 \phi = 4\pi G \rho$$

We integrate 11,25 over a volume cleft by the interface between two continua.

11,26 $$\int dv\, \underline{\nabla} \cdot \underline{\nabla} \phi = \int dv\, 4\pi G \rho$$

We use Gauss's divergence theorem to write 11.26 as

11.27 $\int dA\, \hat{n}\cdot\underline{\nabla}\phi = \int dV\, 4\pi G \rho$

Keeping the area fixed and letting the volume collapse onto the interface we see that

11.28 $$\int dA\,(\hat{n}_1\cdot\underline{\nabla}\phi_1 - \hat{n}_2\cdot\underline{\nabla}\phi_2) = 0$$

Thus $\hat{n}\cdot\nabla\phi$ must be continuous. Hence ϕ must be continuous. Since ϕ_0 and $\hat{r}\cdot\underline{\nabla}\phi_0$ are already continuous we must require continuity of ϕ_1 and $\hat{n}\cdot\underline{\nabla}\phi_1$. A boundary with position $\underline{r}$ is deformed to have position $\underline{r}+\underline{s}$. Then

$$\underline{\underline{T}}(\underline{r}+\underline{s}) = -P(\underline{r}+\underline{s})\,\underline{\underline{I}} + \underline{\underline{\tau}}(\underline{r}+\underline{s})$$

11.29 $$= -\left(P_0(r) + P_1^g(r) + P_1^e(r) + \underline{s}\cdot\underline{\nabla}P_0(r)\right)\underline{\underline{I}} + \underline{\underline{\tau}}(\underline{r})$$

correct to first order in $\underline{s}$. Using 11.8 and 11.17 we have for 11.29

11.30 $$\underline{\underline{T}}(\underline{r}+\underline{s}) = -\left(P_0(r) + P_1^e(r)\right)\underline{\underline{I}} + \underline{\underline{\tau}}(\underline{r})$$

$$= -P_0(r)\,\underline{\underline{I}} + \underline{\underline{\tau}}^e(\underline{r})$$

Since $P_0(r)$ is continuous we have the condition that $\hat{n} \cdot \underline{\underline{\tau}}^e(r)$ must be continuous. Since $\underline{\underline{\tau}}^e$ is already first order in $\underline{s}$ we require $\hat{r} \cdot \underline{\underline{\tau}}^e(\underline{r})$ to be continuous.

Now

11.31
$$\phi(\underline{r}+\underline{s}) = \varphi_0(\underline{r}) + \varphi_1(r) + \underline{s}\cdot\underline{\nabla}\varphi_0(r)$$
$$= \varphi_0(r) + \varphi_1(r) + \underline{s}\cdot\hat{r}\, g_0(r)$$

Since φ_0, g_0 and s are continuous we require $\varphi_1(\underline{r})$ to be continuous. Also

$$\underline{\nabla}\varphi = \underline{\nabla}\varphi_0 + \underline{\nabla}\varphi_1 = \hat{r}\, g_0 + \underline{\nabla}\varphi_1$$

11.32
$$= \hat{r}\,(g_0(\underline{r}+\underline{s}) + \underline{\nabla}\varphi_1(\underline{r})$$

$$= \hat{r}\,(g_0(\underline{r}) + \underline{s}\cdot\underline{\nabla}g_0(\underline{r})) + \underline{\nabla}\varphi_1(\underline{r})$$

and

11.33 $$\underline{\nabla}g_0(r) = \hat{r}\, \partial_r g_0(r)$$

Using 11.3, 11.33 becomes

11.34 $$\underline{\nabla}g_0(r) = \hat{r}\,(4\pi G\rho_0(r) - r^{-1}g_0(r))$$

and 11.32 is

11.35 $$\underline{\nabla}\varphi = \hat{r}\left(g_0(r) + \underline{s}\cdot\hat{r}\,(4\pi G\rho_0(r) - r^{-1}g_0(r))\right) + \underline{\nabla}\varphi_1(r)$$

Since $g_0(r)$ is continuous we require

11.36 $$\hat{n} \cdot \left[\underline{\nabla} \varphi_1(\underline{r}) + \hat{r} \left(\underline{s} \cdot \hat{r}\, 4\pi G \rho_0(r) \right) \right]$$

to be continuous. Since the quantity in brackets is first order in $\underline{s}$ we require

11.37 $$\hat{r} \cdot \left[\nabla \varphi_1(\underline{r}) + \hat{r} (\underline{s} \cdot \hat{r}) 4\pi G \rho_0(r) \right]$$

to be continuous. Thus the linearization of the boundary conditions on the deformed boundary leads to the following equations on the undeformed boundary

11.38
$$\underline{s}(\underline{r}) \text{ continuous}$$

$$\hat{r} \cdot \underline{\underline{\tau}}^e(\underline{r}) \text{ continuous}$$

$$\varphi_1(\underline{r}) \text{ continuous}$$

$$\hat{r} \cdot \nabla \varphi_1(\underline{r}) + \underline{s} \cdot \hat{r}\, 4\pi G \rho_0(\underline{r}) \text{ continuous}$$

The boundary conditions 11.38 must be satisfied by the solutions of 11.24 and 11.6

In this problem we use spherical coordinates (r, θ, φ). Boundary conditions are applied on surfaces of constant radius. We wish to represent a vector, such as displacement, as the sum of three orthogonal vectors, one radial.

Suppose the curvilinear coordinates ξ_1, ξ_2, ξ_3 with scale factors h_1, h_2, h_3 are appropriate for the boundary surface under consideration. Let that surface be $\xi_1 =$ constant.

A vector normal to the surface is

11.39 $$\underline{P} = \hat{\xi}_1 U(\xi_1, \xi_2, \xi_3)$$

A vector normal to $\underline{P}$ and therefore tangential to the boundary surface is

11.40 $$\underline{C} = \underline{\nabla} \times \left(\hat{\xi}_1 c(\xi_1, \xi_2, \xi_3) \right)$$

A vector normal to both $\underline{P}$ and $\underline{C}$ and tangential to the boundary surface is

11.41 $$\underline{B} = \hat{\xi}_1 \times \underline{\nabla} \times \left(\hat{\xi}_1 b(\xi_1, \xi_2, \xi_3) \right)$$

$\underline{C}$ and $\underline{B}$ are expressed as

$$\underline{C} = \frac{\hat{\xi}_2}{h_1 h_3}\frac{\partial}{\partial \xi_3}(h_1 c) - \frac{\hat{\xi}_3}{h_1 h_2}\frac{\partial}{\partial \xi_2}(h_1 c)$$

11.42
$$\underline{B} = \frac{\hat{\xi}_2}{h_1 h_2}\frac{\partial}{\partial \xi_2}(h_1 b) + \frac{\hat{\xi}_3}{h_1 h_3}\frac{\partial}{\partial \xi_3}(h_1 b)$$

In spherical coordinates (r, θ, φ)

11.43
$$\begin{aligned} \xi_1 &= r & h_1 &= 1 \\ \xi_2 &= \theta & h_2 &= r \\ \xi_3 &= \varphi & h_3 &= r \sin\theta \end{aligned}$$

11.42 is

$$\underline{C} = \left(\frac{\hat{\theta}}{\sin\theta}\frac{\partial c}{\partial \varphi} - \hat{\varphi}\frac{\partial c}{\partial \theta} \right) r^{-1}$$

11.44
$$\underline{B} = \left(\hat{\theta}\frac{\partial b}{\partial \theta} + \hat{\varphi}\sin^{-1}\theta \frac{\partial b}{\partial \varphi} \right) r^{-1}$$

Let

11.45 $$\underline{\nabla}_1 = \hat{\theta}\frac{\partial}{\partial \theta} + \hat{\varphi}\sin^{-1}\theta\frac{\partial}{\partial \varphi}, \quad c = r W, \quad b = r V$$

Then we can represent a vector, Ξ, as

11.46 $\underset{\sim}{S} = \hat{r}\, U(r,\theta,\varphi) + \underline{\nabla}_1 V(r,\theta,\varphi) - \hat{r} \times \underline{\nabla}_1 W(r,\theta,\varphi)$

Eq. 11.46 is called the vector representation theorem on a sphere. Expanding U, V, W in spherical harmonics gives

11.47 $$U(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} U_l^m(r)\, Y_l^m(\theta,\varphi)$$

with similar expressions for V, W. In 11.47

11.48 $$Y_l^m(\theta,\varphi) = (-)^m \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(\cos\theta)\, e^{im\varphi}$$

Thus the Y_l^m are normalized spherical harmonics. If we write

$$\underline{P}_l^m(\theta,\varphi) = \hat{r}\, Y_l^m(\theta,\varphi)$$

11.49 $$\underline{B}_l^m(\theta,\varphi) = \underline{\nabla}_1 Y_l^m(\theta,\varphi) = r \underline{\nabla} Y_l^m(\theta,\varphi) = \hat{r} \times \underline{C}_l^m(\theta,\varphi)$$

$$\underline{C}_l^m(\theta,\varphi) = \underline{\nabla}_1 \times (\hat{r}\, Y_l^m(\theta,\varphi)) = \underline{\nabla} \times (\underline{r}\, Y_l^m(\theta,\varphi)) = -\hat{r} \times \underline{B}_l^m(\theta,\varphi)$$

Then 11.46 becomes

11.50 $$\underline{S} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ U_l^m(r)\, \underline{P}_l^m(\theta,\varphi) + V_l^m(r)\, \underline{B}_l^m(\theta,\varphi) + W_l^m(r)\, \underline{C}_l^m(\theta,\varphi) \right\}$$

In 11.49 the vectors are called vector spherical harmonics (Morse and Feshbach, 1953, 2)

In 11.50 V_0° and W_0° are arbitrary. Without loss of generality we assume $V_0^\circ = W_0^\circ = 0$.

To use the expansion in 11.6, 11.24, and 11.38 we need expressions for $\underline{\nabla}\cdot\underline{\underline{\tau}}^e$ and $\hat{r}\cdot\underline{\underline{\tau}}^e$. We are interested in the isotropic constitutive relation 1.63 (page 19)

11.51 $$\tau^e_{ij} = \lambda s_{kk}\delta_{ij} + \mu(s_{i,j} + s_{j,i})$$

where λ and μ are functions of r. For $\hat{r}\cdot\underline{\underline{\tau}}^e$ we write $\hat{r}$ in component form x_i/r. Then

11.52 $$\hat{r}\cdot\underline{\underline{\tau}}^e = \lambda\,\underline{\nabla}\cdot\underline{s}\,\hat{r} + \mu\left[(\hat{r}\cdot\underline{\nabla})\,\underline{s} + r^{-1}\underline{\nabla}(\underline{r}\cdot\underline{s}) - r^{-1}\underline{s}\right]$$

and

11.53 $$\underline{\nabla}\cdot\underline{\underline{\tau}}^e = (\lambda+2\mu)\underline{\nabla}\,\underline{\nabla}\cdot\underline{s} - \mu\,\underline{\nabla}\times\underline{\nabla}\times\underline{s} + \hat{r}\lambda'\,\underline{\nabla}\cdot\underline{s} + \\ + \mu'\left[(\hat{r}\cdot\nabla)\underline{s} + r^{-1}\underline{\nabla}(\underline{r}\cdot\underline{s}) - r^{-1}\underline{s}\right]$$

Next we need expressions for $\underline{\nabla}\cdot\underline{s}$ and $\underline{\nabla}\times\underline{s}$. First we note that $\underline{C}_l^m(\theta,\varphi) = -\hat{r}\times\underline{B}_l^m(\theta,\varphi)$; $\underline{B}_l^m(\theta,\varphi) = \hat{r}\times\underline{C}_l^m(\theta,\varphi)$. Then

11.54 $$\underline{\nabla}\cdot\underline{s} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} D_l^m(r)\,Y_l^m(\theta,\varphi)$$

where

11.55 $$D_\ell^m(r) = \dot{U}_\ell^m(r) + 2r^{-1} U_\ell^m(r) - \ell(\ell+1) r^{-1} V_\ell^m(r)$$

In deriving 11.55 we have used $(r^2\nabla^2 + \ell(\ell+1))Y_\ell^m(\theta,\varphi) = 0$.

For $\underline{\nabla}\times\underline{s}$ we get

11.56 $$\underline{\nabla}\times\underline{s} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big\{ \underline{P}_\ell^m(\theta,\varphi)\left[\ell(\ell+1) r^{-1} W_\ell^m(r)\right] + \underline{B}_\ell^m(\theta,\varphi)\left[\dot{W}_\ell^m(r) + 3r^{-1} W_\ell^m(r)\right] + \underline{C}_\ell^m(\theta,\varphi)\left[r^{-1}U_\ell^m(r) - r^{-1}V_\ell^m(r) - \dot{V}_\ell^m(r)\right]\Big\}$$

also

11.57 $$(\hat{r}\cdot\nabla)\underline{s} + r^{-1}\underline{\nabla}(\underline{r}\cdot\underline{s}) - r^{-1}\underline{s} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big\{ \underline{P}_\ell^m(\theta,\varphi)\, 2\dot{U}_\ell^m(r) + \underline{B}_\ell^m(\theta,\varphi)\left[\dot{V}_\ell^m(r) - r^{-1}V_\ell^m(r) + r^{-1}U_\ell^m(r)\right] + \underline{C}_\ell^m(\theta,\varphi)\left[\dot{W}_\ell^m(r) - r^{-1}W_\ell^m(r)\right]$$

In 11.52 we write

11.58 $$\hat{r}\cdot\underline{\underline{\tau}}^e = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big\{ \underline{P}_\ell^m(\theta,\varphi) R_\ell^m(r) + \underline{B}_\ell^m(\theta,\varphi) S_\ell^m(r) + \underline{C}_\ell^m(\theta,\varphi) T_\ell^m(r)\Big\}$$

where

$$R_l^m(r) = (\lambda+2\mu)\, \dot{U}_l^m(r) + \lambda\, 2r^{-1} U_l^m(r) - \lambda\, l(l+1) r^{-1} V_l^m(r)$$

11.59 $$S_l^m(r) = \mu\left[\dot{V}_l^m(r) - r^{-1} V_l^m(r) + r^{-1} U_l^m(r) \right]$$

$$T_l^m(r) = \mu\left[\dot{W}_l^m(r) - r^{-1} W_l^m(r) \right]$$

Boundary conditions require continuity of U, V, W, R, S, T. For the potential φ_1 in 11.6 we use the expansion

11.60 $$\varphi_1(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi_l^m(r)\, Y_l^m(\theta, \varphi)$$

Let: $g_1 = \hat{r}\cdot\nabla\varphi_1 + \underline{s}\cdot\hat{r}\, 4\pi G \rho_0$. Then

11.61 $$g_1(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\theta, \varphi)\left[\dot{\Phi}_l^m(r) + 4\pi G \rho_0\, U_l^m(r) \right]$$

Boundary conditions require continuity of φ_1 and g_1. We seek eight first order equations in U, V, W, R, S, T, Φ and Q where, from 11.61,

11.62 $$Q_l^m(r) = \dot{\Phi}_l^m(r) + 4\pi G \rho_0\, U_l^m(r)$$

Eqs. 11.59 and 11.61 give four of the equations. We get the other four from 11.6 and 11.24 by substituting 11.60 into 11.6 and 11.46 into 11.24 and by using the definitions 11.59. The algebra is heavy so we quote the result. From 11.24 we get

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Big\{ \underline{P}_l^m(\theta,\varphi)\Big[\partial_r R_l^m(r) - (4r^{-2}\gamma + \rho_0\partial_t^2 - 4\rho_0 g_0 r^{-1})\cdot$$

$$\cdot U_l^m(r) - \overset{-l(l+1)(2r^{-2}\gamma+\rho_0 g_0 r^{-1})}{\cancel{(\ r^{-2}\gamma + \ l(l+1)\rho_0 g_0 r^{-1})}}\, V_l^m(r) -$$

$$- 2r^{-1}(\lambda\sigma^{-1} - 1) R_l^m(r) - l(l+1)r^{-1}S_l^m(r) - \rho_0 Q_l^m(r)\Big] +$$

$$+ \underline{B}_l(\theta,\varphi)\Big[\partial_r S_l^m - (\rho_0 g_0 r^{-1} - 2r^{-2}\gamma)U_l^m(r) - (\rho_0\partial_t^2 -$$

11.63

$$- 2\mu r^{-2} + (\gamma+\mu)l(l+1)r^{-2})V_l^m(r) + \lambda\sigma^{-1}r^{-1}R_l^m(r) +$$

$$+ 3r^{-1}S_l^m(r) - \rho_0 r^{-1}\Phi_l^m(r)\Big] + \underline{C}_l(\theta,\varphi)\Big[\partial_r T_l^m(r) - [\rho_0\partial_t^2 +$$

$$+ r^{-2}\mu(2 - l(l+1))]W_l^m(r) + 3r^{-1}T_l^m(r)\Big]\Big\} = 0$$

$\gamma = \lambda + \mu - \lambda^2\sigma^{-1}$, $\sigma = \lambda + 2\mu$

From 11.6 we get

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\theta,\varphi)\Big[\partial_r Q_l^m(r) + 2r^{-1}Q_l^m(r) - r^{-2}l(l+1)\Phi_l^m(r) +$$

11.64

$$- 4\pi G\rho_0 r^{-1} l(l+1) V_l^m(r)\Big] = 0$$

The harmonics are orthogonal

$$\int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin\theta \, Y_l^m(\theta,\varphi) Y_n^{k*}(\theta,\varphi) = \delta_{mk}\delta_{ln}$$

$$\int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin\theta \, \underline{P}_l^m(\theta,\varphi) \cdot \underline{P}_n^{k*}(\theta,\varphi) = \delta_{mk}\delta_{ln}$$

11.65 $$\int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin\theta \, \underline{C}_l^m(\theta,\varphi) \cdot \underline{C}_n^{k*}(\theta,\varphi) =$$

$$= \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin\theta \, \underline{B}_l^m(\theta,\varphi) \cdot \underline{B}_n^{k*}(\theta,\varphi) = l(l+1)\,\delta_{mk}\delta_{ln}$$

$$\int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin\theta \left[\underline{P}_l^m \cdot \underline{B}_n^{k*},\ \underline{P}_l^m \cdot \underline{C}_n^{k*},\ \underline{B}_l^m \cdot \underline{C}_n^{k*} \right] = 0$$

Therefore each bracketed terms in 11.63 and 11.64 must vanish separately for each m and l. Eqs. 11.59, 11.61, 11.63, 11.64 then give, for each m and l, a system of first order ordinary linear differential equations of the form 9.5. When we take the Fourier transform on t ($e^{-i\omega t}$) we replace ∂_t^2 by $-\omega^2$.

Notice that T and W appear together in the third part of 11.59 and 11.63. Thus the eighth order system decouples into a sixth order system and a second order system. The second order system represents motion parallel to surfaces of constant radius and is analogous to the Love wave problem. The sixth order system is analogous to the

P-SV problem. The gravitational terms make the matrix system sixth order rather than fourth order. In matrix form we have

$$
11.66 \quad \partial_r \begin{bmatrix} U \\ V \\ R \\ S \\ \Phi \\ Q \end{bmatrix} = \begin{bmatrix} -2\lambda\sigma^{-1}r^{-1} & \lambda\sigma^{-1}\ell(\ell+1)r^{-1} & \sigma^{-1} & 0 & 0 & 0 \\ -r^{-1} & r^{-1} & 0 & \mu^{-1} & 0 & 0 \\ -\rho\omega^2-4\rho g r^{-1}+4\gamma r^{-2} & \ell(\ell+1)\left[\rho g/r-2\gamma r^{-2}\right] & 2(\lambda\sigma^{-1}-1)r^{-1} & \ell(\ell+1)r^{-1} & 0 & \rho \\ \rho g r^{-1}-2\gamma r^{-2} & -\rho\omega^2-2\mu r^{-2}+(\gamma+\mu)\ell(\ell+1)r^{-2} & -\lambda\sigma^{-1}r^{-1} & -3r^{-1} & \rho r^{-1} & 0 \\ -4\pi G\rho & 0 & 0 & 0 & 0 & 1 \\ 0 & 4\pi G\rho\ell(\ell+1)r^{-1} & 0 & 0 & \ell(\ell+1)r^{-2} & -2r^{-1} \end{bmatrix} \times \begin{bmatrix} U \\ V \\ R \\ S \\ \Phi \\ Q \end{bmatrix}
$$

$$
11.67 \quad \partial_r \begin{bmatrix} W \\ T \end{bmatrix} = \begin{bmatrix} r^{-1} & \mu^{-1} \\ -\rho\omega^2+\mu(\ell-1)(\ell+2)r^{-2} & -3r^{-1} \end{bmatrix} \times \begin{bmatrix} W \\ T \end{bmatrix}
$$

The second order systems in 11.67 is analogous to the Love wave problem. The modes ($\underline{C}$ vector) of 11.67 are called <u>toroidal</u> or <u>torsional</u> modes

The sixth order system in 11.66 is analogous to the P-SV wave problem. The modes ($\underline{P}$ and $\underline{B}$ vectors) of 11.66 are called poloidal or spheroidal modes. Eqs. 11.66 and 11.67 must be solved for each l and m in the harmonic expansion. The solutions must be regular at $r=0$. At the surface, $r=a$, R, S, and T must vanish. Q must equal its external value. If $\Phi_l^m(a)$ is the value of Φ at $r=a$ then for $r>a$

11.68 $$\Phi_l^m(r) = \Phi_l^m(a)\left(\frac{a}{r}\right)^{l+1} \qquad r>a$$

and

11.69 $$Q_l^m(r) = -(l+1)\,\Phi_l^m(a)\,a^{l+1}/r^{l+2} \qquad r>a$$

Thus at $r=a$ $Q_l^m = -(l+1)\,\Phi_l^m/a$. To get a simpler boundary condition we introduce the function

11.70 $$\Psi_l^m(r) = Q_l^m(r) + (l+1)\,\Phi_l^m(r)/r$$

Ψ must be regular at $r=0$, continuous, and zero at $r=a$. Using Ψ rather than Q 11.66 becomes

$$11.71\quad \partial_r \begin{bmatrix} U \\ V \\ R \\ S \\ \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} \multicolumn{4}{l}{\text{the upper left } 4\times 4 \text{ here is the same}} & 0 & 0 \\ \multicolumn{4}{l}{\text{as in } 11.66.} & 0 & 0 \\ & & & & -(l+1)\rho r^{-1} & g \\ & & & & \rho r^{-1} & 0 \\ -4\pi G\rho & 0 & 0 & 0 & -(l+1)r^{-1} & 1 \\ -4\pi G\rho(l+1)r^{-1} & 4\pi G\rho l(l+1)r^{-1} & 0 & 0 & 0 & (l-1)r^{-1} \end{bmatrix} \times \begin{bmatrix} U \\ V \\ R \\ S \\ \Phi \\ \Psi \end{bmatrix}$$

In 11.71 R, S and Ψ must vanish at $r=a$.

When $l=0$ in 11.66, $V=S\equiv 0$ and we have the fourth order system

$$11.72\quad \partial_r \begin{bmatrix} U \\ R \\ \Phi \\ Q \end{bmatrix} = \begin{bmatrix} -2\lambda\sigma^{-1}r^{-1} & \sigma^{-1} & 0 & 0 \\ -\rho\omega^2-4\rho g r^{-1}+4\gamma r^{-2} & 2(\lambda\sigma^{-1}-1)r^{-1} & 0 & g_0 \\ -4\pi G\rho & 0 & 0 & 1 \\ 0 & 0 & 0 & -2r^{-1} \end{bmatrix} \times \begin{bmatrix} U \\ R \\ \Phi \\ Q \end{bmatrix}$$

In 11.72 the fourth equation is $\partial_r Q = -2Qr^{-1}$. Thus

11.73 $$Q(r) = Q(r_0)(r_0/r)^2$$

From 11.68 we see that $\Phi(r>a) = \text{const}$. Then $Q(a)=0$. From 11.73 $Q \equiv 0$. Then the third equation of 11.72 is $\partial_r \Phi = -4\pi G \rho U$ so

11.74 $$\Phi = -4\pi G \int^r dr\, \rho(r) U(r) + \text{const.}$$

Thus when $l=0$ 11.66 reduces to 11.72. Because $Q\equiv 0$ when $l=0$ 11.72 reduces to

11.75 $$\partial_r \begin{bmatrix} U \\ R \end{bmatrix} = \begin{bmatrix} -2\lambda\sigma^{-1}r^{-1} & \sigma^{-1} \\ -\rho\omega^2 - 4\rho g r^{-1} + 4\gamma r^{-2} & 2(\lambda\sigma^{-1}-1)r^{-1} \end{bmatrix} \times \begin{bmatrix} U \\ R \end{bmatrix}$$

When $l=0$ the motion is purely radial and the modes of 11.75 are called <u>radial</u> modes. Radial modes are a special case of poloidal or spheroidal modes.

A much more elegant derivation of the first order equations has been given by Backus (1966)

G. E. Backus, 1966, Converting vector and tensor equations to scalar equations in spherical coordinates: in preparation Geophys Jour 14 1967

Notice that the sum index, m, does not appear in the coefficient matrices. For each l, $-l \le m \le l$, there are $2l+1$ modes with the same eigenfrequency. We speak of spectral degeneracy in this case. Only the radial modes are non degenerate. All other modes are degenerate. If we take account of earth rotation the degeneracy is completely removed. The effect of rotation is discussed in ch. 12.

In 11.66 the trace of the coefficient matrix is $-6r^{-1}$. Thus the propagator from r_0, $\underline{P}(r, r_0)$, has the determinant (see 9, 17)

11.76 $$|\underline{P}(r, r_0)| = (r_0/r)^6$$

Therefore the propagator from zero is singular. The reason for this is that of the six independent solutions of 11.66 three are regular at $r=0$ and three are not. We must "start" the solution of 11.66 with a 6×3 integral matrix near $r=0$. The same difficulty arises in 11.67 and 11.75

As one example of starting the solution we assume that λ, μ, ρ are constant near $r=0$. We return to 11.24

11.77 $$\rho\omega^2\underline{s} - \rho\nabla\varphi_1 + \rho\underline{\nabla}\cdot\underline{s}\,\hat{r}g - \rho\underline{\nabla}(\underline{s}\cdot\hat{r}g) + (\lambda+2\mu)\underline{\nabla}\,\underline{\nabla}\cdot\underline{s} - \mu\underline{\nabla}\times\underline{\nabla}\times\underline{s} = 0$$

We divide 11.77 by ρ

11.78 $\omega^2 \underline{s} - \nabla\varphi_1 + \underline{\nabla}\cdot \underline{s}\, \hat{r} g - \underline{\nabla}(\underline{s}\cdot\hat{r} g) + \alpha^2 \underline{\nabla}\,\underline{\nabla}\cdot\underline{s} - \beta^2 \underline{\nabla}\times\underline{\nabla}\times\underline{s} = 0$

and rewrite 11.6.

11.79 $\nabla^2\varphi_1 = -4\pi G \rho \underline{\nabla}\cdot\underline{s}$

The divergence of 11.78 is

11.80 $\omega^2 \underline{\nabla}\cdot\underline{s} - \nabla^2\varphi_1 + \underline{\nabla}\cdot(\underline{\nabla}\cdot\underline{s}\,\hat{r} g) - \nabla^2(\underline{s}\cdot\hat{r} g) + \alpha^2\nabla^2\underline{\nabla}\cdot\underline{s} = 0$

The curl of 11.78 is

11.81. $\omega^2 \underline{\nabla}\times\underline{s} + \underline{\nabla}\times(\underline{\nabla}\cdot\underline{s}\,\hat{r} g) + \beta^2\nabla^2\underline{\nabla}\times\underline{s} = 0$

First we consider toroidal oscillations. Then

11.82 $$\underline{s} = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \underline{C}_l^m(\theta,\varphi)\, W_l^m(r)$$

and $\underline{\nabla}\cdot\underline{s} = 0$, $\varphi_1 = 0$, $\hat{r}\cdot\underline{s} = 0$. Eq 11.80 is identically zero. In 11.81 $\nabla^2\underline{\nabla}\times\underline{s} = \underline{\nabla}\times(\nabla^2\underline{s})$
Then we can write 11.81 as

11.83 $\omega^2\underline{s} + \beta^2\nabla^2\underline{s} = 0$

Using 11.82, 11.56, and 11.54 we find

11.84 $$\nabla^2 \underline{S} = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \underline{C}_l^m(\theta,\varphi)\left[\nabla_r^2 - l(l+1)r^{-2}\right] W_l^m(r)$$

so 11.83 is

11.85 $$\sum_{l=1}^{\infty} \sum_{m=-l}^{l} \underline{C}_l^m(\theta,\varphi)\left[\nabla_r^2 - l(l+1)r^{-2} + \omega^2/\beta^2\right] W_l^m(r) = 0$$

Since the $\underline{C}_l^m$ are orthogonal each term in 11.85 must vanish

11.86 $$\left[\nabla_r^2 - l(l+1)r^{-2}\ \omega^2/\beta^2\right] W_l^m(r) = 0$$

where $\nabla_r^2 = \partial_r(r^2 \partial_r)$. Solutions of 11.86 are the spherical Bessel functions (Morse and Feshbach, 1953, 2)

11.87 $$W = A\, j_l(\omega r \beta^{-1}) + B\, y_l(\omega r \beta^{-1})$$

The j_l are regular at $r=0$. The y_l diverge at $r=0$. Thus we must take $B=0$ in 11.87. Then, using 11.59

11.88 $$\begin{bmatrix} W \\ T \end{bmatrix} = A \begin{bmatrix} j_l(\omega r \beta^{-1}) \\ \mu\left(\omega\beta^{-1} j_l'(\omega r\beta^{-1}) - r^{-1} j_l(\omega r \beta^{-1})\right) \end{bmatrix}$$

and 11.88 is the "starting" solution for 11.67

In terms of the propagator of 11.67

11.89 $$\begin{bmatrix} W(a) \\ \\ T(a) \end{bmatrix} = \begin{bmatrix} P_{11}(a, r_0) & P_{12}(a, r_0) \\ \\ P_{21}(a, r_0) & P_{22}(a, r_0) \end{bmatrix} \times \begin{bmatrix} W(r_0) \\ \\ T(r_0) \end{bmatrix}$$

Since $T(a) =$ the secular equation is

11.90 $$P_{21}(a, r_0)\, W(r_0) + P_{22}(a, r_0)\, T(r_0) = 0$$

Values of ω that satisfy 11.90 for each l, m are the toroidal eigen frequencies of the earth. For the Gutenberg model the smallest ω root of 11.90 belongs to $l = 2$ and is $\omega = 2.3778 \times 10^{-3}$ rad./sec. This represents a frequency of 1.362 cycles/hour and a period of 44.04 minutes.

Since there is ample seismic evidence that the earth has a fluid core with radius $r_0 = 3473$ km. we do not have to start the solution at $r = 0$. The rigidity, μ, is zero for $r < r_0$ and $T \equiv 0$ for $r \le r_0$. Then 11.90 becomes

11.91 $$P_{21}(a, r_0) = 0$$

for a fluid core.

The toroidal ω, l diagram for the Gutenberg model is shown in Fig 11.1. The roots are those of 11.91.

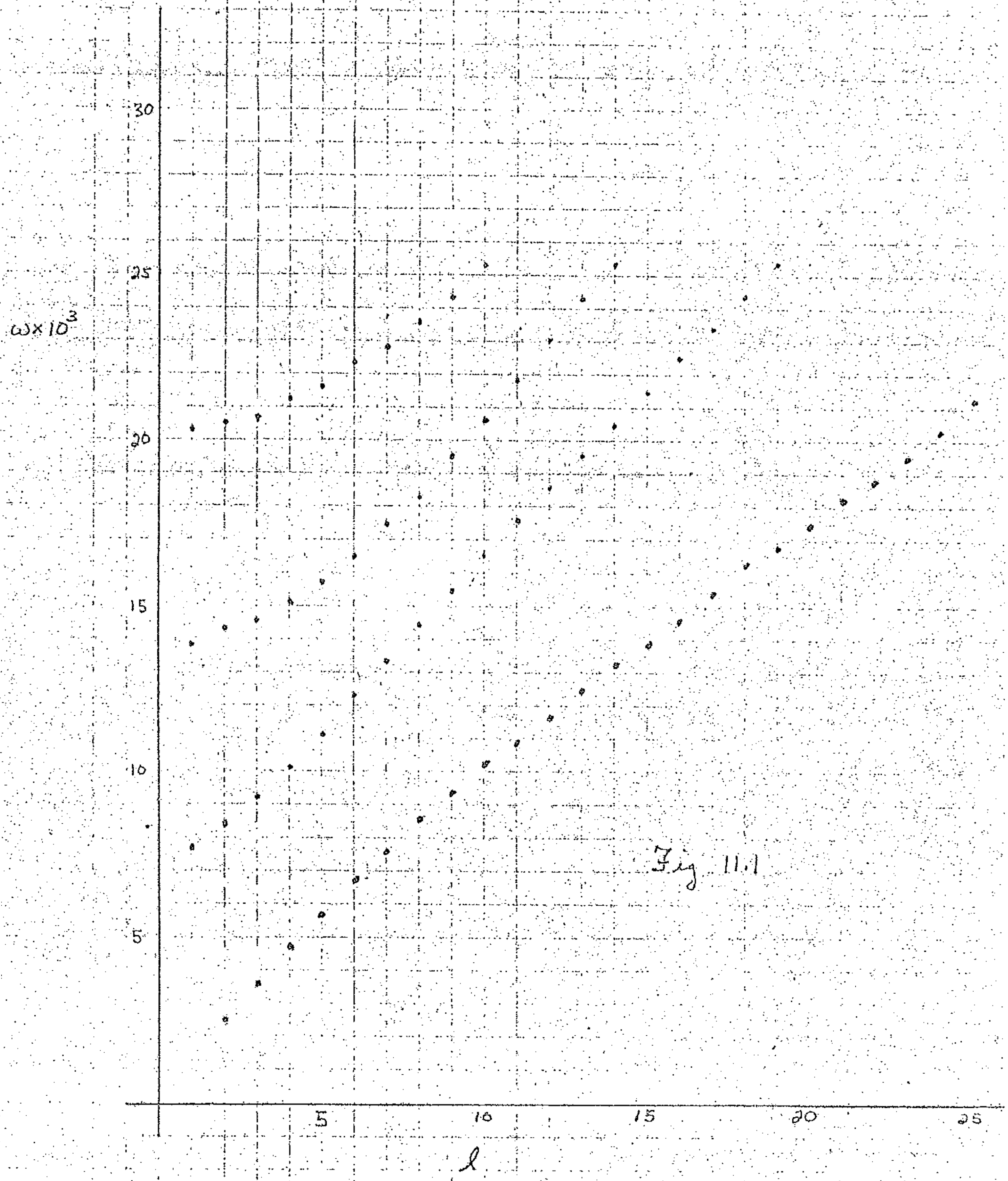

Fig 11.1

Notice in Fig 11.1 that the first set of modes has no entry for $l=1$. This triplet $(2l+1=3)$ represents the three orthogonal rigid rotations of the earth. Also there are no $l=0$ terms $(\underline{C}_0^0(\theta,\varphi)\equiv 0)$.

To investigate the excitation of the toroidal modes we consider a point source of force density pointing in the direction $\hat{a}$

11.92 $$\underline{f}^s = \hat{a}\,\delta(t)\,\delta(r-r_s)\,\delta(\theta-\theta_s)\,\delta(\varphi-\varphi_s)/r^2\sin\theta$$

We expand the Fourier transform of 11.92 in a harmonic series

11.93 $$\bar{\underline{f}}^s = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\alpha_l^m\,\underline{P}_l^m(\theta,\varphi)+\beta_l^m\,\underline{B}_l^m(\theta,\varphi)+\gamma_l^m\,\underline{C}_l^m(\theta,\varphi)\right]$$

and we see that only the γ_l^m terms excite the toroidal modes

11.94 $$\gamma_l^m = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi\,\sin\theta\,\bar{\underline{f}}^s\cdot\underline{C}_l^{m*}(\theta,\varphi) = F_l^m\,\delta(r-r_s)/r^2$$

where $F_l^m = \hat{a}\cdot\underline{C}_l^{m*}(\theta_s,\varphi_s)$. Then 11.67 is

11.95 $$\partial_r\begin{bmatrix} W \\ T \end{bmatrix} = \begin{bmatrix} r^{-1} & \mu^{-1} \\ -\rho\omega^2+(l-1)(l+2)\mu r^{-2} & -3r^{-1} \end{bmatrix} \times \begin{bmatrix} W \\ T \end{bmatrix} + \begin{bmatrix} 0 \\ -F\,\delta(r-r_s)/r^2 \end{bmatrix}$$

which is a special case of 9.5. In terms of the propagator the solution is given by 9.12

$$\begin{bmatrix} W(a) \\ \\ T(a) \end{bmatrix} = \begin{bmatrix} \underline{\underline{P}}(a, r_0) \end{bmatrix} \times \begin{bmatrix} W(r_0) \\ \\ T(r_0) \end{bmatrix} - F r_s^{-2} \begin{bmatrix} P_{12}(a, r_s) \\ \\ P_{22}(a, r_s) \end{bmatrix} \tag{11.96}$$

Applying the boundary conditions $T(a) = T(r_0) = 0$ we get

$$W(a) = \left[\frac{P_{11}(a, r_0) P_{22}(a, r_s) - P_{21}(a, r_0) P_{22}(a, r_s)}{P_{21}(a, r_0)} \right] \frac{F}{r_s^2} \tag{11.97}$$

for each l, m. $W(a)$ is an even function of ω as is clear from 11.95.

The Fourier inversion of 11.97 is

$$W(a, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, W(a, \omega) \, e^{-i\omega t} = \frac{Re}{\pi} \int_{0}^{\infty} d\omega \, W(a, \omega) \, e^{-i\omega t} \tag{11.98}$$

and the poles are on the real ω axis (Fig 11.2)

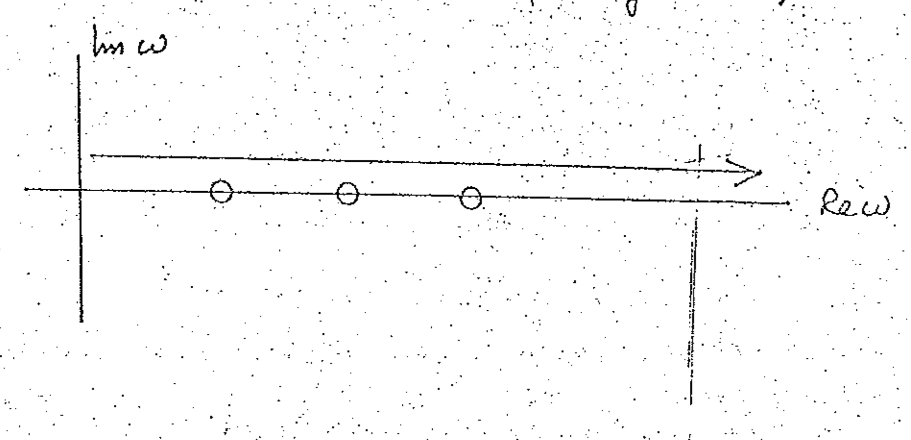

Fig 11.2

$W(a,\omega)$ has no branch points in the ω plane. We close the path in the fourth quadrant to get

$$W(a,t) = \text{Re}\left[-2i\, e^{-i\omega_n t} \ \text{Res}\ W(a,\omega)\right] \tag{11.99}$$

Thus the mode superposition is

$$\underline{S}(a,\theta,\varphi,t) = -2 r_s^{2} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \sum_{n=0}^{\infty} \hat{a}\cdot \underline{C}_l^{m*}(\theta_s,\varphi_s) \left[\frac{P_{11}(a,r_0;\omega_{ln})\,P_{22}(a,r_s;\omega_{ln})}{P_{21}'(a,r_0;\omega_{ln})} \right] \underline{C}_l^{m}(\theta,\varphi) \sin \omega_{ln} t \tag{11.100}$$

We shall not give a detailed discussion of the poloidal modes. Notice that each mode begins at $t=0$. The sum of the modes obeys the causality condition but each mode does not.

12. Rotational Splitting of the Free Oscillations of the Earth

In ch. 11 we considered a spherical non rotating earth. We found that the eigenfrequencies of order l are degenerate of order $2l+1$. The two most obvious perturbations that we should consider are rotation and ellipticity. We expect an effect from ellipticity of about 1/300 and from rotation of about $44^m/1\,day = .0305$. For long periods rotation appears to be roughly ten times more important than ellipticity as a perturbation. The earth's angular velocity $\Omega = 7.2921 \times 10^{-5}$ rad sec^{-1} is small compared to the smallest toroidal (and poloidal) eigen frequency ω_0. We consider, then, $\Omega/\omega_0 \ll 1$.

The Coriolis acceleration is $O(\Omega)$ and the centrifugal acceleration is $O(\Omega^2)$. We neglect the latter. For the toroidal modes the momentum equation becomes

12.1 $$\rho\, \partial_t^2 \underline{s} + \rho\, 2\, \underline{\Omega} \times \partial_t \underline{s} - \underline{\nabla} \cdot \underline{\underline{\tau}}^e = 0$$

Let $\underline{I} = \underline{\nabla} \cdot \underline{\underline{\tau}}^e$ and take the Fourier transform 12.1

12.2 $$\rho \omega^2 \underline{s} + i \rho \omega \underline{\Omega} \times \underline{s} + \underline{I} = 0$$

For a particular value of l let $\omega = \omega_0$ be the unperturbed eigenfrequency and let $\underline{s}_0^m$ the the m^{th} $(-l \le m \le l)$ unperturbed member of the multiplet. Assuming $\underline{s}$ to be analytic in Ω for small Ω we make the power series expansions in the small quantity Ω/ω_0

$$\omega/\omega_0 = 1 + \sigma_1 (\Omega/\omega_0)^1 + \sigma_2 (\Omega/\omega_0)^2 + \cdots$$

12.3 $$\underline{s} = \underline{s}_0 + \underline{s}_1 (\Omega/\omega_0) + \underline{s}_2 (\Omega/\omega_0)^2 + \cdots$$

$$\underline{T} = \underline{T}_0 + \underline{T}_1 (\Omega/\omega_0) + \underline{T}_2 (\Omega/\omega_0)^2 + \cdots$$

Let $\underline{\Omega} = \Omega \hat{z}$. We substitute 12.3 into 12.2 and equate powers of (Ω/ω_0)

$$\rho \omega_0^2 \underline{s}_0 + \underline{T}_0 = 0$$

12.4

$$\rho \omega_0^2 \underline{s}_1 + \underline{T}_1 + 2 \rho \omega_0^2 (\sigma_1 \underline{s}_0 + i \hat{z} \times \underline{s}_0) = 0$$

Now $\underline{s}_1$ must be orthogonal to $\underline{s}_0$, Otherwise that part of $\underline{s}_1$ proportional to $\underline{s}_0$ would be a solution to the first of 12.4 with eigenfrequency ω_0, and we could then redefine $\underline{s}_0$ so that $\underline{s}_1$ is orthogonal to it. Since $\underline{T}_1$ has the same harmonic vector as $\underline{s}_1$ it is also orthogonal to $\underline{s}_0$. Thus if we dot the second of 12.4 by $\overset{*}{\underline{s}}{}_0^m$ and integrate over the volume we get

12.5 $$\int dV \rho \left[\overset{*}{\underline{S}}{}_0^m \cdot \underline{S}_0 \, \sigma_1 + i \, \overset{*}{\underline{S}}{}_0^m \cdot (\hat{z} \times \underline{S}_0) \right] = 0$$

Thus

12.6 $$\sigma_1 = -i \int dV \rho \left(\overset{*}{\underline{S}}{}_0^m \cdot (\hat{z} \times \underline{S}_0) \right) \Big/ \int dV \rho \, \overset{*}{\underline{S}}{}_0^m \cdot \underline{S}$$

Eq. 12.6 is also valid for poloidal oscillations (Backus and Gilbert, 1961; Gilbert and Backus, 1965)

G. E. Backus and F. Gilbert, 1961, The rotational splitting of the free oscillations of the earth: Proc. Nat. Acad. Sci. U.S. 47, 362-371.

F. Gilbert and G. E. Backus, 1965, The rotational splitting of the free oscillations of the earth, 2: Revs. Geophy. 3, 1-9.

For the toroidal modes

$$\underline{S}_0 = \sum_{m=-l}^{l} C_l^m(\theta,\varphi) \, W_l(r)$$

12.7 $$\overset{*}{\underline{S}}{}_0^m = \overset{*}{C}{}_l^m(\theta,\varphi) \, W_l(r)$$

and

12.8 $$\sigma_1 = -\frac{m}{l(l+1)} = -m\beta_l$$

Therefore the first order splitting of the toroidal modes is independent of earth structure. In the absence of degeneracy we see from 12.6 that $\sigma_1 = 0$.

From 12.3 we have

12.9 $$\omega_l^m = \omega_{l0}^m - \frac{m}{l(l+1)}\Omega \quad ; \quad -l \le m \le l$$

The members of the rotationally split multiplet are evenly spaced and are symmetric about the central ($m=0$) member of the multiplet which is unperturbed.

For some source the motion in the absence of rotation, for a given l, can be written as a linear vector operation

12.10 $$\underline{s} = \underline{\mathcal{L}}\ P_l\left[\cos\theta\cos\theta_s + \sin\theta\sin\theta_s\cos(\varphi-\varphi_s)\right]\sin\omega_0 t$$

Using the Legendre addition theorem we can write 12.10 as

12.11 $$\underline{S} = -\mathcal{L}\, \mathrm{Im} \sum_{m=-l}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta) P_l^m(\cos\theta_s)\, e^{im(\varphi-\varphi_s) - i\omega_0 t}$$

When there is rotation we replace ω_0 by $\omega_0 - m\beta\Omega$ and 12.11 is

12.12 $$\underline{S} = -\mathcal{L}\, \mathrm{Im} \sum_{m=-l}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta) P_l^m(\cos\theta_s)\, e^{im[(\varphi-\varphi_s) + \beta\Omega t] - i\omega_0 t}$$

which reduces to

12.13 $$\underline{S} = \mathcal{L}\, P_l\left[\cos\theta\cos\theta_s + \sin\theta\sin\theta_s\cos(\varphi-\varphi_s + \beta_l\Omega t)\right]\sin\omega_0 t$$

Therefore, the geographical amplitude pattern, for a given l, is stationary on a non rotating earth, but drifts to the west on a rotating earth at the rate $\beta\Omega$ radians/sec.

13. The Inverse Problem for the Free Oscillations of the Earth

To simplify the discussion we restrict our attention to the toroidal modes. For a given l we start with 11.67

11.67 $$\partial_r \begin{bmatrix} W \\ T \end{bmatrix} = \begin{bmatrix} r^{-1} & \mu^{-1} \\ -\rho\omega^2 + \mu(l-1)(l+2)r^{-2} & -3r^{-1} \end{bmatrix} \times \begin{bmatrix} W \\ T \end{bmatrix}$$

We multiply the second of 11.67 by W, the first by T, add and integrate with respect to r^2 from the core-mantle boundary, r_0, to the surface, a

$$\int_{r_0}^{a} dr\, r^2 \left\{ \dot{W} T + W \dot{T} \right\} = \int_{r_0}^{a} dr\, r^2 \left\{ -2r^{-1} W T + \mu^{-1} T^2 + \right.$$

13.1

$$\left. + \left(-\rho\omega^2 + \mu(l-1)(l+2)r^{-2}\right) W^2 \right\}$$

Integrating the left integral by parts and using the boundary condition ($T(a) = T(r_0) = 0$) we get

13.2 $$0 = \int_{r_0}^{a} dr\, r^2 \left\{ \mu^{-1} T^2 + \left(-\rho\omega^2 + \mu(l-1)(l+2)r^{-2}\right) W^2 \right\}$$

Eq. 13.2 is an identity for each mode and is also a variational integral for ω^2. Using the third of 11.59 we write 13.2 as

13.3 $$\int_{r_0}^{a} dr\, r^2 \left\{ \rho\omega^2 W^2 - \mu \left[(\dot{W} - W r^{-1})^2 + (l-1)(l+2) W^2 r^{-2} \right] \right\} = 0$$

The first term in braces in 13.3 is the kinetic energy density and the second is the elastic energy density. Eq. 13.3 tells us that the kinetic energy per cycle is equal to the elastic energy per cycle.

Suppose we have a "good" earth model, ρ_1, μ_1, whose frequencies are close to those measured for the earth. We seek a "better" model, ρ_2, μ_2, that is close to ρ_1, μ_1

13.4 $$\int_{r_0}^{a} dr\, r^2 \left[(\rho_2 - \rho_1)^2 + (\mu_2 - \mu_1)^2 \right] = \min.$$

We want the better model to have better frequencies. Let us write 13.3 as

13.5 $$\int_{r_0}^{a} dr\, r^2 \left[\rho_1 \omega_{1i}^2 K_i - \mu_1 M_i \right] = 0$$

for the i^{th} frequency. For our good model we know K, M, ω_1. We do not know K, M for the better model because we do not yet have the better model. But we want the better model to have frequencies ω_{2i} that have been observed. So we know the ω_2.

For the better model we approximate 13.5. We use the K, M of the good (known) model and ω_2 as observed. We demand

13.6 $$\int_{r_0}^{a} dr\, r^2 \left[\rho_2 \omega_{2i}^2 K_i - \mu_2 M_i \right] = 0$$

for the better model. Thus we wish to minimise 13.4 in the presence of the constraints 13.6. This problem is an isoperimetric problem in the calculus of variations (Courant and Hilbert, 1953, 1)

R. Courant and D. Hilbert, 1953, Methods of Mathematical Physics: Interscience, New York.

Let $\omega_{2i}^2 K_i = R_i$. We introduce Lagrange multipliers ν_i and form the functional

13.7 $$\mathcal{F} = (\rho_2 - \rho_1)^2 + (\mu_2 - \mu_1)^2 + 2 \sum_{i=1}^{n} \nu_i (\rho_2 R_i - \mu_2 M_i)$$

When 13.4 is a minimum so is

13.8 $$\int_{r_0}^{a} dr\, r^2 \mathcal{F}$$

To minimize 13.8 we want to make $\mathcal{F}$ in 13.7 and 13.8 as small as possible. As a function of ρ_2, μ_2 $\mathcal{F}$ will have a minimum when

13.9 $\frac{\partial F}{\partial \rho_2} = \frac{\partial F}{\partial \mu_2} = 0$

or when

$$\rho_2 = \rho_1 - \sum_{i=1}^{N} \nu_i R_i$$

13.10

$$\mu_2 = \mu_1 + \sum_{i=1}^{N} \nu_i M_i$$

To determine the Lagrange multipliers, ν_i, we appeal to the contstraints, 13.6,

13.11 $\int_{r_0}^{a} dr\, r^2 \left(\rho_2 R_i - \mu_2 M_i \right) = 0 \quad , i = 1, \ldots, N$

Substituting 13.10 into 13.11 we get

$$\sum_{i=1}^{N} \int_{r_0}^{a} dr\, r^2 \left[R_j R_i + M_j M_i \right] \nu_i =$$

13.13

$$= \left(\omega_{2j}^2 - \omega_{1j}^2 \right) \int_{r_0}^{a} dr\, r^2 \rho_1 W_j^2$$

Therefore we determine the ν_i from 13.13 and build the better model 13.10. If we regard 13.6 as a variational integral then the better model has variationally estimated frequencies that agree with the observations.

We can now repeat the process. Using the new model we return to 11.67 and calculate its exact frequencies and its wave functions, W. Then 13.2 is true and we call the new model ρ_1, μ_1. Then we seek a better ρ_2, μ_2 that minimises 13.4 and satisfies 13.6. This iterative process is continued until we get as close as we can to the observed frequencies.

To preserve the correct mass and moment of inertia we merely add two more ρ constraints to 13.6.

Notice that there is no proof that the final model is unique.

The general problem has been treated by Backus and Gilbert (1966)

G. E. Backus and F. Gilbert, 1966, Earth models constructed from normal mode data: in preparation*.

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*see Geophys. Jour. 14 1967

In 11.72 the fourth equation is $\partial_r Q = -2Qr^{-1}$. Thus

11.73 $$Q(r) = Q(r_0)(r_0/r)^2$$

From 11.68 we see that $\Phi(r > a) = \text{const}$. Then $Q(a) = 0$. From 11.73 $Q \equiv 0$. Then the third equation of 11.72 is $\partial_r \Phi = -4\pi G \rho U$ so

11.74 $$\Phi = -4\pi G \int^r dr\, \rho(r) U(r) + \text{const.}$$

Thus when $l = 0$ 11.66 reduces to 11.72. Because $Q \equiv 0$ when $l = 0$ 11.72 reduces to

11.75 $$\partial_r \begin{bmatrix} U \\ R \end{bmatrix} = \begin{bmatrix} -2\lambda\sigma^{-1}r^{-1} & \sigma^{-1} \\ -\rho\omega^2 - 4\rho g r^{-1} + 4\gamma r^{-2} & 2(\lambda\sigma^{-1} - 1)r^{-1} \end{bmatrix} \times \begin{bmatrix} U \\ R \end{bmatrix}$$

When $l = 0$ the motion is purely radial and the modes of 11.75 are called <u>radial</u> modes. Radial modes are a special case of poloidal or spheroidal modes.

A much more elegant derivation of the first order equations has been given by Backus (1966)

G. E. Backus, 1966, Converting vector and tensor equations to scalar equations in spherical coordinates: in preparation
Geophys Journ 14 1967