No classes next two weeks

References:

Morse + Feshbach - Methods of Theoretical Physics - 2 vol McGraw Hill

Jeffreys + Jeffreys - Mathematical Physics - Cambridge.

Dettmann - Mathematical Methods in Physics + Engineering McGraw Hill

Sneddon - Elements of P.D.E.

Moon + Spencer - Field Theory for Engineers, Van Nostrand.

Kelloggs - Foundation of Potential Theory } Dover

McMillan's - Theory of Potential

Sternberg + Smith - Theory of Potential + Spherical Harmonics. Univ. of Toronto

Ramsey - Theory of Newtonian Attraction, Cambridge.

Grant + West - Interpretation Theory of Applied Geophysics.

1. Newtonian field of force law for particles.
   inverse square law.

2. Field intensity - Force on unit hypothetical body.

3. Conditions for existence of potential -
   necessary + sufficient that $\nabla \times \vec{f} = 0$
   or $\int_A^B \vec{f} \cdot d\vec{r}$ is independent of path.
   $\oint \vec{f} \cdot d\vec{r} = 0$

4. Newtonian potential $\phi = \frac{KM}{r}$ for a particle.

$$\phi = K \int_{vol} \frac{\sigma\, dV}{r}$$

5. Logarithmic potential. 2-D point - infinite line.

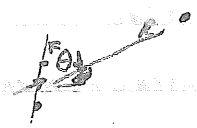
$$\phi_2 = 2K\sigma_L \ln \frac{1}{r}$$

6. Dipoles potential. $\phi_{dipole} = \frac{KM}{r^2}\frac{dr}{ds} = \frac{KM}{r^2}\cos\theta.$

 double layer.

$$\phi_{\text{DIPOLE DISTRIBUTION}} = K\int_V \frac{M_v}{r^2}\cos\theta\, dV = K\int M_v\left[\nabla\left(\tfrac{1}{r}\right)\cdot\hat{e}_s\right]dV$$

$$\phi_{\text{2-D DIPOLE}} = \frac{2K M_L \cos\theta}{R}$$



FIELD INTENSITY = $F = \nabla\phi$

PROPERTIES OF NEWTONIAN POTENTIAL:

a. VALUES AT INFINITY $\lim_{r\to\infty}\phi = 0$

$\lim_{r\to\infty} r\phi = Km$

b. ϕ and its first derivatives are always finite and continuous for volume distribution of matter.

c. $\nabla^2\phi = 0$ is satisfied at all matter free points = points where there are no sources of field.

d. $\nabla^2\phi = 4\pi K\sigma$ is satisfied at all interior points of matter.

NEITHER ARE SATISFIED ON THE BOUNDARY

DEPENDENCE ON BOUNDARY CONDITIONS, HAVE FOR $\nabla^2\phi = 0$.

Sneddon pg 151 ff.

(a) DIRICHLET PROBLEM

(i) INTERIOR – IF f is a continuous function prescribed on boundary S of some finite region V, determine a function ϕ such that $\nabla^2\phi = 0$ within V and $\phi = f$ on S.

Torus, [illegible] is multiply connected

(ii) EXTERIOR – IF f is a continuous function prescribed on a boundary S of a finite simply connected region V, determine a function ϕ which satisfies $\nabla^2\phi = 0$ outside the region and is such that $\phi = f$ on S.

simply connected if [illegible] where a boundary lies within a region

connect two points + make curve connecting points go to zero + does it stay in region

P151 Sneddons

To make simply connected put a barrier

solution not unique unless restriction placed on limiting value of ϕ as $r \to \infty$.

UNIQUE IF $|\phi| < \frac{c}{r}$ for 3 space.

ϕ BOUNDED AT ∞ for 2 space.

(b) i. INTERIOR NEUMANN PROBLEM.

If f is a continuous function which is prescribed at each point of the boundary S of a finite region V, determine a function ϕ such that $\nabla^2\phi = 0$ within V and its normal derivative $\frac{\partial\phi}{\partial n}$ coincides with f at every point of S.

If $\nabla^2\phi = 0$ [illegible] no [illegible] sources + theorem says the flux of $= [illegible]$ $S\left[\int_S \frac{\partial\phi}{\partial n}\,dS\right] = 0$

Necessary condition for existence of a solution is $\int_{\text{Surface}} f\,dS = 0$

ii Exterior Neuman Problem $\nabla^2\phi = 0$ a[illegible] must be bounded simply connected

$\frac{\partial\phi}{\partial n} = f$ on S

(c) i Interior Mixed (Churchill) problem.

If f is a continuous function prescribed on the boundary S of a finite region V determine a function ϕ such that $\nabla^2\phi = 0$ within V and $\frac{\partial \phi}{\partial n} + (K+1)\phi = f$ at every point of S.

need not be a constant but must be a continuous function for the body. same fun.

ii Exterior Churchill problem.

If f is a . . . .

. . . . $\nabla^2\phi = 0$ outside V . . . etc.

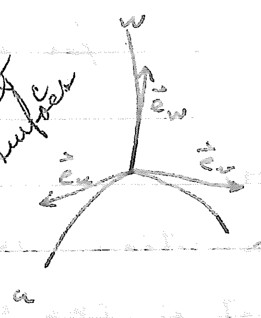
INSULATED $\leftarrow \frac{\partial T}{\partial n} = 0$

SOURCE $T = c$

MIXED CONDITION...

Curvilinear Coordinates – Dettmann p 137 ff

u, v, w curves - ortho-normal (orthogonal system)...



must surfaces

$\hat{e}_u \perp \hat{e}_v \perp \hat{e}_w$

I. $\hat{e}_u \cdot \hat{e}_v = \hat{e}_v \cdot \hat{e}_w = \hat{e}_u \cdot \hat{e}_w = 0$ orthogonal coordinates

element of distance along u curve is not in general, du...

element of length will be ds_u, ds_v, ds_w since u, v, w change position in space

3 ways of expressing orthogonality

II NORMAL VECTORS. $\hat{e}_u = \frac{\vec{\nabla} u}{|\vec{\nabla} u|}$ $\hat{e}_v = \frac{\vec{\nabla} v}{|\nabla v|}$ $\vec{e}_w$

$$\vec{\nabla} u = \frac{\partial u}{\partial x}\hat{e}_x + \frac{\partial u}{\partial y}\hat{e}_y + \frac{\partial u}{\partial z}\hat{e}_z$$

$$\left.\begin{aligned} \nabla u \cdot \nabla v = 0 &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} \\ 0 &= \frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} \\ 0 &= \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \end{aligned}\right\} \text{orthogonality...}$$

$\textcircled{0}$ $du = |\vec\nabla u|\, \vec e_u = \frac{\partial u}{\partial x}\vec e_x + \frac{\partial u}{\partial y}\vec e_y + \frac{\partial u}{\partial z}\vec e_z$

unit normal
$$\vec e_u = \frac{\frac{\partial u}{\partial x}\vec e_x + \frac{\partial u}{\partial y}\vec e_y + \frac{\partial u}{\partial z}\vec e_z}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2}}$$

Also,

$\frac{\partial \vec v}{\partial u}$ unit ~~tangent~~
$$\vec e_u = \frac{\frac{\partial x}{\partial u}\vec e_x + \frac{\partial y}{\partial u}\vec e_y + \frac{\partial z}{\partial u}\vec e_z}{\sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}}$$

$d\vec r = dx\,\vec i + dy\,\vec j + dz\,\vec k$

$d\vec r = h_1 du\,\vec e_u + h_2 dv\,\vec e_v + h_3 dw\,\vec e_w$

$ds = \sqrt{dx^2 + dy^2 + dz^2}$

$= \sqrt{h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dw^2}$

$h_1 = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right]^{1/2}$

$h_2 = \left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]^{1/2}$

$h_3 = \left[\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2\right]^{1/2}$

$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw$

$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw$

$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw$

$dx^2 = \left(\frac{\partial x}{\partial u}\right)^2 du^2 + \left(\frac{\partial x}{\partial v}\right)^2 dv^2 + \left(\frac{\partial x}{\partial w}\right)^2 dw^2 + 2\frac{\partial x \partial x}{\partial v \partial u}du\,dv + 2\frac{\partial x \partial x}{\partial u \partial w} + 2\frac{\partial x \partial x}{\partial v \partial w}$

III Parametric Equation of u-coordinate curve:

u curve is normal to vw plane

$x = x(u, v, w)$

$y = y(u, v, w)$

$z = z(u, v, w)$.

~~normal to u plane...~~

$\vec e_u$ = (unit tangent to u-coordinate curve)

s_u = distance along u-curve

$= \frac{\partial x}{\partial u}\frac{du}{ds_u}\vec e_x + \frac{\partial y}{\partial u}\frac{du}{ds_u}\vec e_y + \frac{\partial z}{\partial u}\frac{du}{ds_u}\vec e_z$

$e_x\ e_y\ e_z$ are unit vector

$\vec r + d\vec r$, ds_u, $\vec r$, v, w const

$d\vec r = dx\,\vec e_x + dy\,\vec e_y + dz\,\vec e_z = \left(\frac{\partial x}{\partial u}\vec e_x + \frac{\partial y}{\partial u}\vec e_y + \frac{\partial z}{\partial u}\vec e_z\right)\frac{du}{ds_u}$

$dx = \frac{\partial x}{\partial u}du + \cancel{\frac{\partial x}{\partial v}dv} + \cancel{\frac{\partial x}{\partial w}dw}$

e_u unit vector i.

$\frac{du}{ds}\sqrt{x_u^2 + y_u^2 + z_u^2} = 1$

other momentum

$\vec e_u \cdot \vec e_w = 0 \qquad \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 0$

$\vec e_v \cdot \vec e_w = 0 \qquad \frac{\partial x}{\partial v}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial v}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial v}\frac{\partial z}{\partial w} = 0$

$\vec e_w \cdot \vec e_u = 0 \qquad \frac{\partial x}{\partial w}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial w}\frac{\partial y}{\partial u} + \frac{\partial z}{\partial w}\frac{\partial z}{\partial u} = 0$

Element of arc length.

$\textcircled{0}$ RECTANGULAR $(ds^2) = (dx)^2 + (dy)^2 + (dz)^2$

$(ds^2) = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right]du^2 + \left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]dv^2$

$+ \left[\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2\right]dw^2 \ldots$

$$2\left[\frac{\partial x}{\partial u}\frac{\partial x}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial y}{\partial u} + \cdots\right] + 2[\quad] + 2[\quad]$$

but these last three are zero by orthogonality condition.

$$ds^2 = h_1^2 (du)^2 + h_2^2 (dv)^2 + h_3^2 (dw^2)$$

$$h_1^2 = x_u^2 + y_u^2 + z_u^2 \quad \text{etc.}$$
$$h_2^2 = x_v^2 + y_v^2 + z_v^2$$
$$h_3 = x_w^2 + y_w^2 + z_w^2$$

very important

$$ds_u = h_1 du$$
$$ds_v = h_2 dv$$
$$ds_w = h_3 dw$$

Elements of area.

$$dS_1 = h_2 h_3 \, dv \, dw$$
$$dS_2 = h_1 h_3 \, du \, dw$$
$$dS_3 = h_1 h_2 \, du \, dv$$

Element of volume

$$dV = h_1 h_2 h_3 \, du_1 du_2 du_3$$

Fri Sept 22.

GRADIENT OF A SCALAR $\phi(u,v,w)$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{e}_x + \frac{\partial\phi}{\partial y}\vec{e}_y + \frac{\partial\phi}{\partial z}\vec{e}_z$$

$$= \left(\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial x}\right)\vec{e}_x +$$

$$\left(\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial y}\right)\vec{e}_y +$$

$$\left(\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial z}\right)\vec{e}_z$$

$h_1 = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right]^{1/2}$

$$\nabla\phi = \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}\vec{e}_x + \frac{\partial u}{\partial y}\vec{e}_y + \frac{\partial u}{\partial z}\vec{e}_z\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}\vec{e}_x + \frac{\partial v}{\partial y}\vec{e}_y + \frac{\partial v}{\partial z}\vec{e}_z\right)$$

$$+ \frac{\partial\phi}{\partial w}\left(\frac{\partial w}{\partial x}\vec{e}_x + \frac{\partial w}{\partial y}\vec{e}_y + \frac{\partial w}{\partial z}\vec{e}_z\right)$$

$$\nabla\phi = \frac{\partial\phi}{\partial u}\nabla u + \frac{\partial\phi}{\partial v}\nabla v + \frac{\partial\phi}{\partial w}\nabla w$$

$$= \frac{\partial\phi}{\partial u}|\nabla u|\,\vec{e}_u + \frac{\partial\phi}{\partial v}|\nabla v|\,\vec{e}_v + \frac{\partial\phi}{\partial w}|\nabla w|\,\vec{e}_w$$

better $\nabla u \cdot ds = du = |\nabla u|\,\vec{e}_u \cdot (h_1 du\,\vec{e}_u + h_2 dv\,\vec{e}_v + h_3 dw\,\vec{e}_w)$

$= |\nabla u|\, h_1\, du \quad \therefore \quad |\nabla u| = 1/h_1 \Rightarrow$ avoids the ds_u

$$|\nabla u| = \frac{du}{ds_u} = \frac{1}{h_1} \quad |\nabla v| = \frac{dv}{ds_v} = \frac{1}{h_2} \quad |\nabla w| = \frac{dw}{ds_w} = \frac{1}{h_3}$$

$$\nabla\phi = \frac{\partial\phi}{\partial u}\frac{du}{ds_u}\vec{e}_u + \frac{\partial\phi}{\partial v}\frac{dv}{ds_v}\vec{e}_v + \frac{\partial\phi}{\partial w}\frac{dw}{ds_w}\vec{e}_w$$

ds_u = element of length in direction of u-coordinate ~~coor~~ curve

Curvilinear Gradient

$$\boxed{\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u}\vec{e}_u + \frac{1}{h_2}\frac{\partial\phi}{\partial v}\vec{e}_v + \frac{1}{h_3}\frac{\partial\phi}{\partial w}\vec{e}_w = \sum \frac{1}{h_i}\frac{\partial\phi}{\partial u_i}\vec{e}_i}$$

DIVERGENCE: $\vec{\nabla}\cdot\vec{F}$

$$\vec{F} = F_u\vec{e}_u + F_v\vec{e}_v + F_w\vec{e}_w = F_x\vec{e}_x + F_y\vec{e}_y + F_z\vec{e}_z$$

can use vector identities now as in x, y, z coordinate system

$$\vec{e}_u = \vec{e}_v \times \vec{e}_w, \quad \vec{e}_v = \vec{e}_w \times \vec{e}_u, \quad \vec{e}_w = \vec{e}_u \times \vec{e}_v$$

$$= h_2 h_3 \vec{\nabla}v \times \vec{\nabla}w; \quad = h_3 h_1 \nabla w \times \nabla u; \quad h_1 h_2 \vec{\nabla}u \times \vec{\nabla}v$$

$$\vec{F} = h_2 h_3 F_u (\nabla v \times \nabla w) + h_1 h_3 F_v (\nabla w \times \nabla u) + h_1 h_2 F_w (\nabla u \times \nabla v)$$

$\nabla\cdot\phi\vec{u} = \nabla\phi\cdot\vec{u} + \phi\nabla\cdot\vec{u}$

$$\vec{\nabla}\cdot\vec{F} = \nabla(h_2 h_3 F_u)\cdot(\nabla v \times \nabla w) + \nabla(h_3 h_1 F_v)\cdot(\nabla w \times \nabla u) + \nabla(h_1 h_2 F_w)\cdot(\nabla u \times \nabla v)$$

$$+ h_2 h_3 F_u \nabla\cdot(\nabla v \times \nabla w) + h_3 h_1 F_v \nabla\cdot(\nabla w \times \nabla u) + h_1 h_2 F_w \nabla\cdot(\nabla u \times \nabla v)$$

$$\nabla \cdot (\nabla v \times \nabla w) = \nabla w \cdot (\nabla \times \nabla v) - \nabla v \cdot (\nabla \times \nabla w)$$

$\nabla \times \nabla \phi = 0$ curl of gradient $= 0$.

So last three terms drop out.

$$\nabla (h_2 h_3 F_u) \cdot (\nabla v \times \nabla w)$$

using expression for gradient in curvilinear coordinates –

$$\nabla (h_2 h_3 F_u) = \frac{1}{h_1}\frac{\partial (h_2 h_3 F_u)}{\partial u}\vec{e}_u + \frac{1}{h_2}\frac{\partial (h_2 h_3 F_u)}{\partial v}\vec{e}_v + \frac{1}{h_3}\frac{\partial (h_2 h_3 F_u)}{\partial w}\vec{e}_w$$

$\vec{e}_v = \dfrac{\nabla v}{|\nabla v|}$

but $\frac{1}{h_1}\vec{e}_u = \nabla u$ $\quad \| \; 0 \quad \| \; 0$

when dotted.

$$\nabla (h_2 h_3 F_u) \overset{\vec{e}}{\underset{}{\cdot}} (\nabla v \times \nabla w) = \frac{1}{h_1}\frac{\partial (h_2 h_3 F_u)}{\partial u}\vec{e}_u \cdot (\nabla v \times \nabla w) = \frac{1}{h_1}\frac{\partial}{\partial u}(h_2 h_3 F_u)\, \nabla u \cdot (\nabla v \times \nabla w)$$

$\nabla u \cdot (\nabla v \times \nabla w) = \dfrac{\vec{e}_u}{h_2 h_3}$

$$= \frac{1}{h_1}\frac{\partial (h_2 h_3 F_u)}{\partial u}\vec{e}_u \cdot \frac{\vec{e}_u}{h_2 h_3}$$

use $1 = \vec{e}_u \cdot (\vec{e}_v \times \vec{e}_w)$

So

Curvilinear Divergence.

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial (h_2 h_3 F_u)}{\partial u} + \frac{\partial (h_1 h_3 F_v)}{\partial v} + \frac{\partial (h_1 h_2 F_w)}{\partial w}\right]$$

$$= \frac{1}{h_1 h_2 h_3}\sum_{i=1}^{3}\frac{\partial}{\partial u_i}\left(\frac{h_1 h_2 h_3}{h_i}F_i\right)$$

LAPLACIAN $\quad \nabla^2 \phi = \nabla \cdot \nabla \phi$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_1 h_3}{h_2}\frac{\partial \phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial w}\right)\right]$$

CURL: $\vec{\nabla} \times \vec{F}$

$$\vec{F} = F_u \vec{e}_u + F_v \vec{e}_v + F_w \vec{e}_w = h_1 F_u \vec{\nabla} u + h_2 F_v \vec{\nabla} v + h_3 F_w \vec{\nabla} w$$

$\nabla \times \psi \vec{E} = \psi \nabla \times \vec{E} + \nabla\psi \times \vec{E}$

$$\nabla \times \vec{F} = \nabla(h_1 F_u) \times \nabla u + \nabla(h_2 F_v) \times \nabla v + \nabla(h_3 F_w) \times \nabla w$$
$$h_1 F_u \nabla \times \nabla u + h_2 F_v \nabla \times \nabla v + h_3 F_w \nabla \times \nabla w$$

first term,

$$\nabla(h_1 F_u) \times \nabla u = \nabla(h_1 F_u) \times \frac{\vec{e}_u}{h_1}$$

$$= \left\{ \frac{1}{h_1} \frac{\partial(h_1 F_u)}{\partial u} e_u + \frac{1}{h_2} \frac{\partial(h_1 F_u)}{\partial v} \vec{e}_v + \frac{1}{h_3} \frac{\partial h_1 F_u}{\partial w} \vec{e}_w \right\} \times \frac{\vec{e}_u}{h_1}$$

$$= -\frac{1}{h_2 h_1} \frac{\partial(h_1 F_u)}{\partial v} \vec{e}_w + \frac{1}{h_3 h_1} \frac{\partial (h_1 F_u)}{\partial w} \vec{e}_v$$

second term,

$$\nabla(h_2 F_v) \times \nabla v = \nabla(h_2 F_v) \times \frac{\vec{e}_v}{h_2}$$

$$= \frac{1}{h_1 h_2} \frac{\partial(h_2 F_v)}{\partial u} \vec{e}_w - \frac{1}{h_2 h_3} \frac{\partial(h_2 F_v)}{\partial w} \vec{e}_u$$

third term,

$$\nabla(h_3 F_w) \times \nabla w = \nabla(h_3 F_w) \times \frac{\vec{e}_w}{h_3}$$

$$= -\frac{1}{h_1 h_3} \frac{\partial(h_3 F_w)}{\partial u} \vec{e}_v + \frac{1}{h_2 h_3} \frac{\partial(h_3 F_w)}{\partial v} e_u$$

$$\begin{array}{ccc} e_u & e_v & e_w \\ \frac{1}{h_1}\frac{\partial h_1 F_u}{\partial u_1} & \frac{1}{h_2}\frac{\partial h_1 F_u}{\partial u_2} & \frac{1}{h_3}\frac{\partial h_1 F_u}{\partial u_3} \\ \frac{1}{h_1} & 0 & 0 \end{array}$$

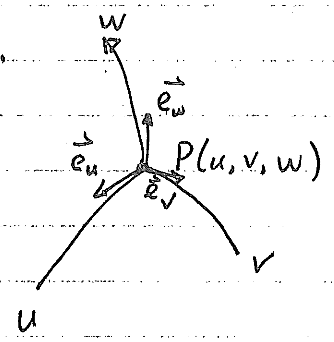
$$\frac{1}{h_1 h_2 h_3}\left\{ \begin{array}{ccc} h_1 \vec{e}_u & h_2 \vec{e}_v & h_3 \vec{e}_w \\ \frac{1}{h_1}\frac{\partial}{\partial u_1} & \frac{1}{h_2}\frac{\partial}{\partial u_2} & \frac{1}{h_3}\frac{\partial}{\partial u_3} \\ h_1 F_u & h_2 F_v & h_3 F_w \end{array} \right.$$

$$\nabla \times \vec{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{e}_u & h_2 \vec{e}_v & h_3 \vec{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 F_u & h_2 F_v & h_3 F_w \end{vmatrix}$$

8/29/96

Curvilinear Coordinates

Consider a 3-D system with coordinates, u, v and w. This means that a point in space is described using these coordinates and that locally we may consider directions (vectors) of increasing u, v and w.

I. 

The local system is orthogonal if $\vec{e}_u \perp \vec{e}_v \perp \vec{e}_w$, or

$\vec{e}_u \cdot \vec{e}_v = 0$ $\quad \vec{e}_u \cdot \vec{e}_w = 0$ and $\vec{e}_v \cdot \vec{e}_w =$

Note that these vectors may not be in a fixed direction, e.g. they will change as P changes. In addition we may not require that u, v and w have units of length.

How do we define these vectors.

II. Under suitable circumstances

$$u = u(x, y, z)$$
$$v = v(x, y, z)$$
$$w = w(x, y, z)$$

$u(x, y, z)$ is a surface, and we can define the unit vectors in terms of gradients to the surface.

$$\vec{e}_u = \vec{\nabla} u / |\vec{\nabla} u| \qquad \vec{e}_v = \vec{\nabla} v / |\vec{\nabla} v| \qquad \vec{e}_w = \vec{\nabla} w / |\vec{\nabla} w|$$

8/29/96

Orthogonality is satisfied if

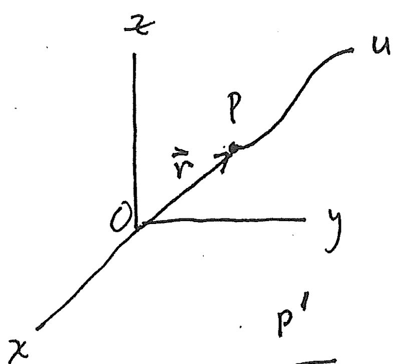
$$\vec{\nabla} u \cdot \vec{\nabla} v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} = 0$$

$$\vec{\nabla} u \cdot \nabla w = \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} = 0$$

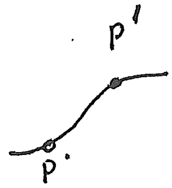
$$\nabla v \cdot \nabla w = \frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} = 0$$

III. Parametric equation of coordinate system.

Consider
$$x = x(u, v, w)$$
$$y = y(u, v, w)$$
$$z = z(u, v, w)$$



The point $P(x, y, z)$ can be used together with the origin $(0,0,0)$ to define a vector
$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$= x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$$



If we change u to $u + \Delta u$, keeping v and w fixed, we move from P to P' in the direction of increasing u

$$\overrightarrow{OP} = x(u,v,w)\vec{e}_x + y(u,v,w)\vec{e}_y + z(u,v,w)\vec{e}_z$$
$$\overrightarrow{OP'} = x(u+\Delta u, v, w)\vec{e}_x + y(u+\Delta u, v, w)e_y + z(u+\Delta u, v, w)\vec{e}_z$$

As $\Delta u \to 0$ $\overrightarrow{OP'} - \overrightarrow{OP}$ is proportional to $\vec{e}_u$

or

$$\overrightarrow{OP'} - \overrightarrow{OP} \cong \left(\frac{\partial x}{\partial u}\vec{e}_x + \frac{\partial y}{\partial u}\vec{e}_y + \frac{\partial z}{\partial u}\vec{e}_z\right)\Delta u$$

and after normalizing

Thus $$\vec{e}_u = \frac{\frac{\partial x}{\partial u}\vec{e}_x + \frac{\partial y}{\partial u}\vec{e}_y + \frac{\partial z}{\partial u}\vec{e}_z}{\sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}}$$

orthogonality requires

$$\vec{e}_u \cdot \vec{e}_v = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 0$$

$$\vec{e}_u \cdot \vec{e}_w = \frac{\partial x}{\partial u}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial w} = 0$$

$$\vec{e}_v \cdot \vec{e}_w = \frac{\partial x}{\partial v}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial v}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial v}\frac{\partial z}{\partial w} = 0$$

IV. Units of length.

Since the coordinates may not have dimensions of length, e.g. the coordinate may be an angle θ, we need a general way to define an element of length along a curve. ~~f con~~ In cartesian coordinates

$$d\vec{r} = dx\,\vec{e}_x + dy\,\vec{e}_y + dz\,\vec{e}_z$$

For curvilinear coordinates we introduce scale factors h_u, h_v and h_w to define

$$d\vec{r} = du\,h_u\,\vec{e}_u + dv\,h_v\,\vec{e}_v + h_w\,\vec{e}_w\,dw$$

The length of this element is $ds = \sqrt{d\vec{r}\cdot d\vec{r}}$

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

$$= \sqrt{h_u^2 du^2 + h_v^2 dv^2 + h_w^2 dw^2}$$

but $dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw$

$$dx^2 = \left(\frac{\partial x}{\partial u}\right)^2 du^2 + \left(\frac{\partial x}{\partial v}\right)^2 dv^2 + \left(\frac{\partial x}{\partial w}\right)^2 dw^2$$
$$+ 2\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right) du\,dv + 2\left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x}{\partial v}\right) du\,dw + 2\left(\frac{\partial x}{\partial v}\right)\left(\frac{\partial x}{\partial w}\right) dv\,d$$

$$dy^2 = \left(\frac{\partial y}{\partial u}\right)^2 du^2 + \left(\frac{\partial y}{\partial v}\right)^2 (dv)^2 + \left(\frac{\partial y}{\partial w}\right)^2 dw^2$$
$$+ 2(\quad) du\,dv + 2(\quad) du\,dw + 2(\quad) dv\,dw$$

$$dz^2 = \left(\frac{\partial z}{\partial u}\right)^2 du^2 + \left(\frac{\partial z}{\partial v}\right)^2 (dv)^2 + \left(\frac{\partial z}{\partial w}\right)^2 dw^2$$
$$+ 2\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} du\,dv + 2\frac{\partial z}{\partial u}\frac{\partial z}{\partial w} du\,dw + 2\frac{\partial z}{\partial v}\frac{\partial z}{\partial w} dw\,d$$

combining terms

$$dx^2 + dy^2 + dz^2 = \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right] du^2$$
$$+ \left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] dv^2$$
$$+ \left[\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2\right] dw^2$$
$$+ 2\left[\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right] du\,dv$$
$$+ 2\left[\frac{\partial x}{\partial u}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial w}\right] du\,dw$$
$$+ 2\left[\frac{\partial x}{\partial v}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial v}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial v}\frac{\partial z}{\partial w}\right] dv\,dw$$

Note that the factors of $du\,dv$, $du\,dw$ and $dv\,dw$ are 0 if the u, v, w system is orthogonal.

Thus for orthogonal coordinates, we immediately see that

$$h_u^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \quad ✓$$

$$h_v^2 = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

$$h_w^2 = \left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2$$

Therefore the elements of length are

$$ds_u = h_u\,du$$
$$ds_v = h_v\,dv$$
$$ds_w = h_w\,dw$$

The basic elements of area ~~are~~ are

$$dS_u = h_v h_w\,dv\,dw$$

$$dS_v = h_u h_w\,du\,dw$$

$$dS_w = h_u h_v\,du\,dv$$

The element of volume is

$$dV = h_u h_v h_w\,du\,dv\,dw \qquad \text{or } dV = h_1 h_2 h_3\,du_1\,du_2\,d$$

Gradient

$$\phi = \phi(x,y,z) = \phi(u(x,y,z), v(x,y,z), w(x,y,z))$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{e}_x + \frac{\partial\phi}{\partial y}\vec{e}_y + \frac{\partial\phi}{\partial z}\vec{e}_z$$

$$= \left(\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial x}\right)\vec{e}_x$$

$$+ \left(\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial y}\right)\vec{e}_y$$

$$+ \left(\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial z}\right)\vec{e}_z$$

$$= \left(\frac{\partial\phi}{\partial u}\right)\left(\frac{\partial u}{\partial x}\vec{e}_x + \frac{\partial u}{\partial y}\vec{e}_y + \frac{\partial u}{\partial z}\vec{e}_z\right)$$

$$+ \left(\frac{\partial\phi}{\partial v}\right)\left(\frac{\partial v}{\partial x}\vec{e}_x + \frac{\partial v}{\partial y}\vec{e}_y + \frac{\partial v}{\partial z}\vec{e}_z\right)$$

$$+ \left(\frac{\partial\phi}{\partial w}\right)\left(\frac{\partial w}{\partial x}\vec{e}_x + \frac{\partial w}{\partial y}\vec{e}_y + \frac{\partial w}{\partial z}\vec{e}_z\right)$$

$$= \frac{\partial\phi}{\partial u}\nabla u + \frac{\partial\phi}{\partial v}\nabla v + \frac{\partial\phi}{\partial w}\nabla w$$

$$= \frac{\partial\phi}{\partial u}|\nabla u|\vec{e}_u + \frac{\partial\phi}{\partial v}|\nabla v|\vec{e}_v + \frac{\partial\phi}{\partial w}|\nabla w|\vec{e}_w$$

but $|\nabla u| = \frac{1}{h_u}$, $|\nabla v| = \frac{1}{h_v}$, $|\nabla w| = \frac{1}{h_w}$

$$du = \nabla\vec{u}\cdot d\vec{r} = |\nabla u|\vec{e}_u \cdot (h_u du\,\vec{e}_u + h_v dv\,\vec{e}_v + h_w dw\,\vec{e}_w) = h_u|\nabla u|\,du$$

$$\therefore h_u = \frac{1}{|\nabla u|}$$

$$\nabla\phi = \frac{1}{h_u}\frac{\partial\phi}{\partial u}\vec{e}_u + \frac{1}{h_v}\frac{\partial\phi}{\partial v}\vec{e}_v + \frac{1}{h_w}\frac{\partial\phi}{\partial w}\vec{e}_w$$

or shorthand $\nabla\phi = \sum \frac{1}{h_i}\frac{\partial\phi}{\partial u_i}\vec{e}_{u_i}$

Note

$$du = \nabla u \circ ds$$

$$= |\nabla u| \vec{e}_u \circ (h_u\, du\, \vec{e}_u + h_v\, dv\, \vec{e}_v + h_w\, dw\, \vec{e}_w)$$

$$= |\nabla u|\, h_u\, du \qquad \text{if } e_u \circ e_v \circ e_w = 0$$

$$\therefore \quad |\nabla u|\, h_u = 1$$

Divergence

$$\vec{\nabla} \circ \vec{F}$$

$$\vec{F} = F_x \vec{e}_x + F_y \vec{e}_y + F_w \vec{e}_w$$

$$= F_u \vec{e}_u + F_v \vec{e}_v + F_w \vec{e}_w$$

Now use orthogonality and right hand coordinate system

$$\vec{e}_u = \vec{e}_v \times \vec{e}_w \qquad \vec{e}_v = \vec{e}_w \times \vec{e}_u \qquad \vec{e}_w = \vec{e}_u \times \vec{e}_v$$

$$= h_v h_w \vec{\nabla} v \times \nabla w \qquad = h_w h_u \vec{\nabla} w \times \nabla u \qquad = h_u h_v \vec{\nabla} u \times \vec{\nabla} v$$

$$\therefore \quad \vec{F} = h_v h_w F_u (\vec{\nabla} v \times \vec{\nabla} w) + h_w h_u F_v (\vec{\nabla} w \times \vec{\nabla} u) + h_u h_v F_w (\vec{\nabla} u \times \vec{\nabla} v)$$

Recall $\vec{\nabla} \circ \phi \vec{u} = \nabla \phi \circ \vec{u} + \phi \nabla \circ \vec{u}$

$$\vec{\nabla} \circ \vec{F} = \nabla (h_v h_w F_u) \circ (\vec{\nabla} v \times \vec{\nabla} w) + \nabla (h_w h_u F_v) \circ (\vec{\nabla} w \times \vec{\nabla} u) + \nabla (h_u h_v F_w) \circ (\vec{\nabla} u \times \vec{\nabla} v)$$

$$+ h_v h_w F_u \vec{\nabla} \circ (\vec{\nabla} v \times \vec{\nabla} w)$$

$$+ h_w h_u F_v \vec{\nabla} \circ (\vec{\nabla} w \times \vec{\nabla} u)$$

$$+ h_u h_v F_w \vec{\nabla} \circ (\vec{\nabla} u \times \vec{\nabla} v)$$

but

$$\nabla\cdot(\nabla v \times \nabla w) = \nabla w \cdot (\nabla\times\nabla v) - \nabla v \cdot (\nabla \times \nabla w)$$

and

$\nabla \times \nabla \phi = 0$ always, so last terms drop

Recall from definition of gradient,

$$\nabla(h_v h_w F_u) = \frac{1}{h_u}\frac{\partial(h_v h_w F_u)}{\partial u}\vec{e}_u + \frac{1}{h_v}\frac{\partial(h_v h_w F_u)}{\partial v}\vec{e}_v + \frac{1}{h_w}\frac{\partial(h_v h_w F_u)}{\partial w}\vec{e}_w$$

and $\frac{1}{h_u}\vec{e}_u = \vec{\nabla} u$

and $A \times (A \times B) = 0$

we have.

$$\nabla(h_v h_w F_u)\cdot(\vec{\nabla} v \times \vec{\nabla} w)$$

$$= \frac{1}{h_u}\frac{\partial(h_v h_w F_u)}{\partial u}\vec{e}_u \cdot (\vec{\nabla} v \times \vec{\nabla} w)$$

$$\boxed{\frac{1}{h_v h_w}\vec{e}_v \times \vec{e}_w = \frac{1}{h_v h_w}\vec{e}_u}$$

$$= \frac{1}{h_u}\frac{\partial(h_v h_w F_u)}{\partial u}\vec{e}_u \cdot \frac{\vec{e}_u}{h_v h_w}$$

Putting all together

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial(h_v h_w F_u)}{\partial u} + \frac{\partial(h_u h_w F_v)}{\partial v} + \frac{\partial(h_u h_v F_w)}{\partial w}\right]$$

or $$\vec{\nabla}\cdot\vec{F} = \frac{1}{h_1 h_2 h_3}\sum_{i=1}^{3}\frac{\partial}{\partial u_i}\left(\frac{h_1 h_2 h_3}{h_i}F_i\right)$$

Laplacian:

$$\nabla^2\phi = \nabla\cdot\nabla\phi$$

$$= \frac{1}{h_1h_2h_3}\left[\sum_{i=1}^{3}\frac{\partial}{\partial u_i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial\phi}{\partial u_i}\right)\right]$$

Curl:

$$\nabla\times\vec{F}$$

$$F = F_u\vec{e}_u + F_v\vec{e}_v + F_w\vec{e}_w$$

$$= h_1F_u\vec{\nabla}u + h_2F_v\vec{\nabla}v + h_3F_w\vec{\nabla}w$$

~~$\nabla\times\vec{F} =$~~

Now use

$$\nabla\times(\psi\vec{E}) = \psi\,\vec{\nabla}\times\vec{E} + \vec{\nabla}\psi\times\vec{E}$$, which gives

$$\nabla\times\vec{F} = \nabla(h_1F_u)\times\vec{\nabla}u + \nabla(h_2F_v)\times\vec{\nabla}v + \nabla(h_3F_w)\times\vec{\nabla}w$$
$$+ h_1F_u\,\vec{\nabla}\times\underline{\vec{\nabla}u} + h_2F_v\,\nabla\times\nabla v + h_3F_w\,\nabla\times\nabla w$$

Examine First term

$$\nabla(h_1F_u)\times\vec{\nabla}u = \nabla(h_1F_u)\times\frac{\vec{e}_u}{h_1}$$

$$= \left\{\frac{1}{h_1}\frac{\partial(h_1F_u)}{\partial u}\vec{e}_u + \frac{1}{h_2}\frac{\partial(h_1F_u)}{\partial v}\vec{e}_v + \frac{1}{h_3}\frac{\partial(h_1F_u)}{\partial w}\vec{e}_w\right\}\times\frac{\vec{e}_u}{h_1}$$

$$= -\frac{1}{h_2h_1}\frac{\partial(h_1F_u)}{\partial v}\vec{e}_w + \frac{1}{h_3h_1}\frac{\partial(h_1F_u)}{\partial w}\vec{e}_v$$

Second term, third term, combine into determinant

$$\nabla \times F = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{e}_u & h_2 \vec{e}_v & h_3 \vec{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 F_u & h_2 F_v & h_3 F_w \end{vmatrix}$$

Now different coordinate systems.

Morse + Feshbach, Methods of Theoretical Physics

10-9-67 Sec 8 chap 1.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

i. $\phi = \left.\begin{matrix} e^{px} \\ e^{-px} \end{matrix}\right\} \left.\begin{matrix} e^{ipy} \\ e^{-ipy} \end{matrix}\right\}$

ii $\phi = \left.\begin{matrix} e^{px} \\ e^{-px} \end{matrix}\right\} \left.\begin{matrix} \cos \\ \sin \end{matrix}\, py \right\}$

iii $\phi = \left.\begin{matrix} \cosh \\ \sinh \end{matrix}\, px \right\} \left.\begin{matrix} e^{ipy} \\ e^{-ipy} \end{matrix}\right\}$

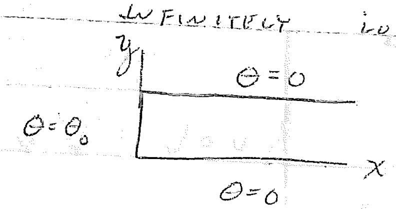
iv $\phi = \left.\begin{matrix} \cosh \\ \sinh \end{matrix}\, px \right\} \left.\begin{matrix} \cos \\ \sin \end{matrix}\, py \right\}$

v $\phi = \sum_{i=0}^{\infty} \left(A_i e^{\pm p_i x} \cos p_i y + B_i e^{\pm p_i x} \sin p_i y \right) + \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} y \\ 1 \end{matrix}\right\}$

etc.

INTERIOR DIRICHLET PROBLEM.

INFINITELY LONG RECTANGULAR PLATE.



FIND T AT SOME INTERIOR POINT.

$\theta = \theta(x, y) \qquad x \geq 0$

$b > y > 0$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \quad \text{IN REGION...} \qquad \theta \to 0, \; x \to \infty.$$

$\theta = X(x)\, Y(y)$

$X(x) = A_1 e^{-px} = A_m e^{-px}$

$Y(y) = A_2 \sin(py + \phi)$

$y = 0 \quad \theta = 0, \quad \phi = 0$

$y = b \quad \theta = 0 \qquad pb = n\pi \qquad \to \; p = \dfrac{n\pi}{b}$

$$\theta = \sum_{n=0}^{\infty} A_n e^{-px} \sin\left(\frac{n\pi}{b}y\right)$$

$$= \sum_{n=0}^{\infty} A_n e^{-\frac{n\pi}{b}x} \sin\left(\frac{n\pi}{b}y\right).$$

$$\theta(0,y) = \theta_0$$

$$\theta_0 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{b}y\right)$$

$$A_n = \frac{2}{b}\int_0^b \sin\left(\frac{n\pi y}{b}\right) dy$$

$$= \frac{2}{b}\left[-\frac{b}{n\pi}\theta_0 \cos\left(\frac{n\pi y}{b}\right)\right]_0^b$$

$$= \frac{-2\theta_0}{n\pi}\left[\cos n\pi - \cos 0\right]$$

for n even $A_n = 0$

n odd $A_n = \dfrac{4\theta_0}{n\pi}$

$$A_{2r+1} = \frac{4\theta_0}{\pi(2r+1)}$$

$$A_{2r} = 0$$

$$\theta = \frac{4\theta_0}{\pi}\sum_{r=0}^{\infty} \frac{e^{-\frac{(2r+1)\pi x}{b}}}{(2r+1)} \sin\{(2r+1)\pi y/b\}$$

10-11-67 Laplaces equation in Polar coordinates.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

$$V = R(r)\,\Theta(\theta)$$

$$\frac{r}{R}\left(\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right) = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = n^2$$

$R = r^l$

$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0 \qquad \frac{d^2\Theta}{d\theta^2} + n^2\Theta = 0$

$l(l-1) + l - n^2 = 0$

$l^2 = n^2$

$l = \pm n$

$R = r^n$ or r^{-n} $\quad n \neq 0$

$1, \ln r \quad n = 0$

$(n+1)(n) \quad -(n+1)$

$n^2 + n - n + 1 - n^2 + 1$

Let $r = e^s$

$$\frac{dR}{dr} = \frac{dR}{ds}\frac{ds}{dr} = \frac{1}{r}\frac{dR}{ds}$$

$$\frac{d^2R}{dr^2} = \frac{-1}{r^2}\frac{dR}{ds} + \frac{1}{r^2}\frac{d^2R}{ds^2}$$

$$\frac{d^2R}{ds^2} - n^2R = 0$$

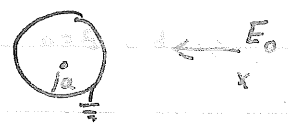
$$R = A_1 e^{-ns} + B_1 e^{ns}$$

$$= A_1 r^{-n} + B_1 r^{+n}$$

If $n = 0 \qquad r\frac{dR}{dr} = \text{const} \qquad R = A\ln r + B$

$\Theta = \text{const}$

or $A_0\theta + B_0$



DIRICHLET EXTERNAL PROBLEM.

POTENTIAL CONTINUOUS ACROSS BOUNDARY, 1ST DERIVATIVE HAS DISCONTINUITY...

$$\nabla^2\phi = 0 \quad \text{for} \quad r > a$$

BOUNDARY CONDITIONS i $\phi = 0$, $r = a$ for all θ

ii $\phi \doteq E_0 x = E_0 r\cos\theta$ for large x or r.

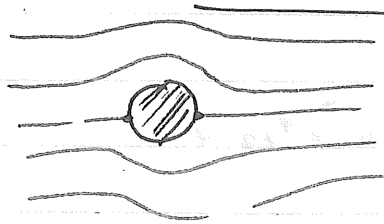
or $\frac{\partial\phi}{\partial x} = \text{const}\ldots$ ϕ goes at r^2 not $\underline{\log}$

$$\phi = E_0 r\cos\theta + \sum_{n=1}^{\infty} r^{-n}\left(C_n \cos n\theta + D_n \sin n\theta\right)$$

r^{-n} used rather than r^n for condition #2...

$C_1 = -E_0 a^2 \quad D_1 = 0$

10-13-67



2) at $r = a$ $\frac{\partial\phi}{\partial r} = 0$

1) Velocity = const for large r

Exterior Neuman problem.

$$\phi = \left(1 + \frac{a^2}{r^2}\right) U r\cos\theta \qquad u = -\nabla\phi$$

$$u_r = -\left(1 - a^2/r^2\right) U\cos\theta$$

$$u_\theta = \left(1 + \frac{a^2}{r^2}\right) U\sin\theta$$

$$\nabla^2\phi = 0$$

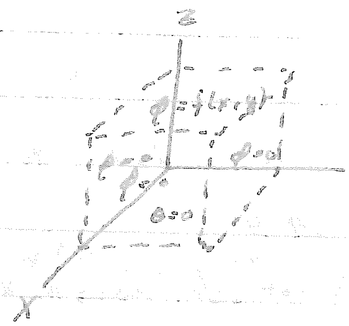
$$\left.\begin{aligned} &\frac{d^2X}{dx^2} + p^2 X = 0 \\ &\frac{d^2Y}{dy^2} + q^2 Y = 0 \\ &\frac{d^2Z}{dz^2} - (p^2 + q^2) Z = 0 \end{aligned}\right\}$$

Given

$\phi = 0$ on $x=0,\ x=a,\ y=0,\ y=b,\ z=0$

$\phi = \phi(x,y)$ on $z=c$

Find ϕ in the region bounded by the plane surfaces.



$$\phi = \begin{matrix}\cos\\ \sin\end{matrix} px \Big\} \begin{matrix}\cos\\ \sin\end{matrix} qy \Big\} \begin{matrix}\cosh\\ \sinh\end{matrix} \sqrt{p^2+q^2}\, z$$

Coefficient of $\cosh\sqrt{p^2+q^2}\, z = 0$ since $\phi = 0$ $z=0$

$\cos px$ $\phi=0$ $x=0$

$\cos py$ $\phi=0$ $y=0$

$$\phi = \sin px \ \sin qy \ \sinh\sqrt{p^2+q^2}\, z$$

$\phi = 0$ $x = a$

$\sin pa = 0$ $pa = n\pi$ $p = \frac{n\pi}{a}$

$q = \frac{m\pi}{b}$

$$\phi = A_n \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sinh \pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\, z$$

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sinh \pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\, c$$

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} F_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

$$F_{mn} = A_{mn} \sinh \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\, c$$

$$F_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}\, dx\, dy$$

$$\phi = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{F_{mn}}{\sinh \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\, c} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sinh \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\, z$$

Cylindrical Polar:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\phi = R(r)\, \Theta(\theta)\, Z(z)$$

$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{r^2 \Theta} \frac{d^2 \Theta}{d\theta^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0$$

$$\frac{1}{Z} \frac{d^2 Z}{dz^2} = +p^2$$

$$\frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = -q^2$$

$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{r^2 \Theta} \frac{d^2 \Theta}{d\theta^2} + p^2 = 0$$

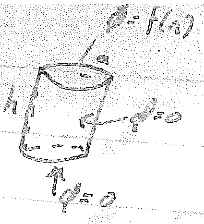
$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + \left(p^2 r^2 - q^2 \right) R = 0$$

$$R = J_q(pr) + Y_q(pr)$$

$$\phi = \left.\begin{matrix} e^{pz} \\ e^{-pz} \end{matrix}\right\} \left.\begin{matrix} \cos q\theta \\ \sin q\theta \end{matrix}\right\} \left.\begin{matrix} J_q(pr) \\ Y_q(pr) \end{matrix}\right\}$$

$$\left.\begin{matrix} \cos pz \\ \sin pz \end{matrix}\right\} \left.\begin{matrix} \cos q\theta \\ \sin q\theta \end{matrix}\right\} \left.\begin{matrix} I_q(pr) \\ \; \end{matrix}\right\}$$

10-16-67



$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = 0$$

$$r\frac{\partial \phi}{\partial r} = c$$

$$\phi = a_1 \ln r + b_1$$

$$\phi = \left.\begin{matrix}\sinh pz \\ \cosh pz\end{matrix}\right\} J_0(pr)$$

$\phi = 0 \quad$ at $z = 0$

$\phi = 0 \quad r = a$

$$J_0(p_n a) = 0 \qquad p_n = \frac{c_n}{a}$$

$$\phi = \sum_1^\infty A_n \sinh p_n z \; J_0(p_n r)$$

$$f(r) = \sum_1^\infty A_n \sinh(p_n h) \; J_0(p_n r)$$

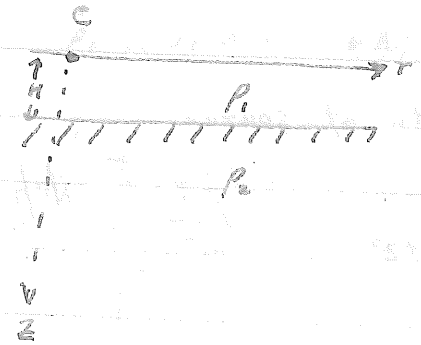
162-163
Churchill

$$f(r) = \sum_1^\infty K_n \; J_0(p_n r)$$

$$K_n = \frac{2\int_0^a r f(r) \, J_0(p_n r)\, dr}{a^2 [J_1(p_n a)]^2}$$

PROBLEM IN ELECTRICAL RESISTIVITY.

Proc. of Amer. Soc. of Testing Materials. Spec. Tech Publications #122, 1952
I. Roman "Resistivity Reconnaissance" pp 171-220



HOMOGENEOUS LAYER OVER INFINITE HALFSPACE.
HAVE CURRENT SOURCE AT FREE SURFACE..

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{except at } r=0,\ z=0$$

$i = \frac{1}{\rho}\nabla\phi$ = current density
ϕ = potential
ρ = resistivity ...

Boundary conditions:

i) $\frac{\partial \phi_1}{\partial z} = 0$ at $z=0$ (except at $r=0, z=0$)

ii) continuity of potential $\phi_1 = \phi_2$ at $z=H$

iii) $\frac{1}{\rho_1}\frac{\partial \phi_1}{\partial z} = \frac{1}{\rho_2}\frac{\partial \phi_2}{\partial z}$ at $z=H$

Properties of potential

(iv) $\phi_1 = \phi_2 = 0$ at $r=\infty$

(v) $\phi_2 = 0$ at $z=\infty$

(vi) ϕ = finite everywhere except at source ($r=0, z=0$)

Solutions from separation of variables.

$$\phi = \left.\begin{matrix} e^{\lambda z} \\ e^{-\lambda z} \end{matrix}\right\} \begin{matrix} J_0(\lambda r) \\ Y_0(\lambda r) \end{matrix}$$

$Y_0(\lambda r)$ gives a potential anywhere on z axis.

For just halfspace.

Websters P.D.E. paperback p. 365 eq. 133...

I = strength of source.

$$\varphi_{H.S.} = \frac{\rho I}{2\pi R} = \frac{\rho I}{2\pi\sqrt{r^2+z^2}} = \frac{\rho I}{2\pi}\int_0^\infty e^{-\lambda z} J_0(\lambda r)\,d\lambda$$

f_1, f_2, $g_1\ g_2$ correction factors and are solutions to Laplace's equation...

$$\varphi_1 = \frac{\rho_1 I}{2\pi}\int_0^\infty \left[e^{-\lambda z} + f_1(\lambda) e^{-\lambda z} + g_1(\lambda) e^{\lambda z}\right] J_0(\lambda r)\,d\lambda$$

$$\varphi_2 = \frac{\rho_2 I}{2\pi}\int_0^\infty \left[\cancel{e^{-\lambda z}} + f_2(\lambda) e^{-\lambda z} + g_2(\lambda) e^{\lambda z}\right] J_0(\lambda r)\,d\lambda$$

OK

(vii) $\varphi_2 = 0$ at $z = \infty$, $g_2(\lambda) = 0$

i. $\frac{\partial \varphi_1}{\partial z} = 0$ at $z = 0$ $\quad \frac{\partial}{\partial z}\left\{ \frac{\rho_1 I}{2\pi}\int_0^\infty \left[e^{-\lambda z} + f_1(\lambda) e^{-\lambda z} + g_1(\lambda) e^{\lambda z}\right] J_0(\lambda r)\,d\lambda \right\}\Big|_{z=0,\ r\neq 0}$

$$\frac{\partial}{\partial z}\left\{ \frac{\rho_1 I}{2\pi}\int_0^\infty e^{-\lambda z} J_0(\lambda r)\,d\lambda \right\}\Big|_{r\neq 0,\ z=0} = \frac{\partial}{\partial z}\left\{ \frac{\rho_1 I}{2\pi\sqrt{r^2+z^2}}\right\}\Big|_{z=0,\ r\neq 0} = 0$$

other two terms.

$$\frac{\rho_1 I}{2\pi}\int_0^\infty \left[-\lambda f_1(\lambda) + \lambda g_1(\lambda)\right] J_0(\lambda r)\,d\lambda \} = 0$$

$$f_1(\lambda) = g_1(\lambda)$$

ii $\varphi_1 = \varphi_2$ at $z = H$

$$e^{-\lambda H} f_2(\lambda) = f_1(\lambda) e^{-\lambda H} + f_1(\lambda) e^{\lambda H} + e^{-\lambda H}$$

iii

$$\frac{1}{\rho_1}\left[-\lambda f_1(\lambda) e^{-\lambda H} + \lambda f_1(\lambda) e^{\lambda H} - \lambda e^{-\lambda H}\right] = \frac{1}{\rho_2}\left[-\lambda f_2(\lambda) e^{-\lambda H} - \lambda e^{-\lambda H}\right]$$

$$f_2(\lambda) = (1+e^{2\lambda H}) f_1(\lambda)$$

$$\frac{1}{\rho_1}\left[f_1(\lambda) e^{-\lambda H} - f_1 e^{\lambda H} + e^{-\lambda H}\right] = \frac{1}{\rho_2}\left[f_2(\lambda) e^{-\lambda H} + e^{-\lambda H}\right]$$

$$\frac{1}{\rho_1}\left[f_1(\lambda)(1 - e^{2\lambda H}) + 1\right] = \frac{1}{\rho_2}\left[f_2(\lambda) + 1\right]$$

10-18-67 $$\frac{1}{\rho_1}\left[f_1(\lambda)(1 - e^{2\lambda H}) + 1\right] = \frac{1}{\rho_2}\left[(1+e^{2\lambda H}) f_1(\lambda) + 1\right]$$

$$f_1(\lambda) = \frac{(\rho_2-\rho_1)/(\rho_2+\rho_1)}{\{-(\rho_2-\rho_1)/(\rho_1+\rho_2)\} + e^{2\lambda H}}$$

define $$K \equiv \frac{\rho_2-\rho_1}{\rho_1+\rho_2}$$

$$f_1(\lambda) = \frac{K}{e^{2\lambda H} - K}$$

$$f_2(\lambda) = \frac{K}{e^{2\lambda H} - K}(1+e^{2\lambda H})$$

$$\phi_1 = \frac{\rho_1 I}{2\pi}\int_0^\infty \left[e^{-\lambda |z|} + (e^{-\lambda z} + e^{\lambda z})\left(\frac{K}{e^{2\lambda H} - K}\right)\right] J_0(\lambda r)\, d\lambda \quad \text{for } 0 \le z \le H$$

$$\phi_2 = \frac{\rho_1 I}{2\pi}\int_0^\infty e^{-\lambda z}\left[1 + \frac{K}{(e^{\lambda H} - K)}(1+e^{2\lambda H})\right] J_0(\lambda r)\, d\lambda \quad H \le z$$

separating

$$\phi_1 = \frac{\rho_1 I}{2\pi}\left\{ \frac{1}{R} + K\int_0^\infty \frac{e^{-2\lambda H}(e^{-\lambda z} + e^{\lambda z})}{1 - K e^{-2\lambda H}} J_0(\lambda r)\, d\lambda \right\}$$

$$\phi_2 = \frac{\rho_1 I}{2\pi}\left\{ \frac{1}{R} + K\int_0^\infty \frac{(1+e^{-2\lambda H})e^{-\lambda z}}{1 - K e^{-2\lambda H}} J_0(\lambda r)\, d\lambda \right\}$$

$$\frac{1}{1 - Ke^{-2\lambda H}} = \sum_{n=0}^{\infty} K^n e^{-2n\lambda H}$$

$$\phi_1 = \frac{\rho I}{2\pi} \left\{ \frac{1}{R} + K \sum_{n=0}^{\infty} K^n \int_0^{\infty} e^{-2\lambda(n+1)H} (e^{-\lambda z} + e^{\lambda z}) J_0(\lambda r) d\lambda \right.$$

$$\phi_2 = \frac{\rho_1 I}{2\pi} \left\{ \frac{1}{R} + K \sum_{n=0}^{\infty} K^n \int_0^{\infty} e^{-2n\lambda H} (1 + e^{-2\lambda H}) e^{-\lambda z} J_0(\lambda r) d\lambda \right\}$$

To simplify integrals:

$$\frac{1}{R} = \frac{1}{\sqrt{r^2 + z^2}} = \int_0^{\infty} e^{-\lambda z} J_0(\lambda r) d\lambda$$

Look at

$$\int_0^{\infty} e^{-\lambda z} e^{-2\lambda(n+1)H} J_0(\lambda r) d\lambda$$

Since z is constant of integration

$$= \int_0^{\infty} e^{-\lambda(z + 2(n+1)H)} J_0(\lambda r) d\lambda = \frac{1}{\sqrt{r^2 + [z + 2(n+1)H]^2}}$$

$$\phi_1 = \frac{\rho_1 I}{2\pi} \left\{ \frac{1}{R} + K \sum_{n=0}^{\infty} \left[\frac{K^n}{\sqrt{r^2 + [z + 2(n+1)H]^2}} + \frac{K^n}{\sqrt{r^2 + [2(n+1)H - z]^2}} \right] \right\}$$

$$\phi_2 = \frac{\rho_1 I}{2\pi} \left\{ \frac{1}{R} + K \sum_{n=0}^{\infty} K^n \left[\frac{1}{\sqrt{r^2 + (z + 2nH)^2}} + \frac{1}{\sqrt{r^2 + [z + 2(n+1)H]^2}} \right] \right\}$$

SPHERICAL POLAR.

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \phi}{\partial \psi^2} = 0$$

Special cases:

i) Spherical symmetry $\phi = f(r)$ only.

$$\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 0$$

General solution : $\phi = \frac{A_0}{R} + B_0$

ii) Axial symmetry $\phi \neq f(\psi)$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = \lambda = n(n+1)$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - n(n+1)R = 0$$

$$\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + n(n+1)\sin\theta\,\Theta = 0 \qquad \mu = \cos\theta$$

or $\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0$

$$R = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} \qquad \Theta = \left.\begin{matrix} P_n(\mu) \\ Q_n(\mu) \end{matrix}\right\}$$

iii

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0$$

$$\frac{d^2\Psi}{d\psi^2} + m^2\Psi = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}\Theta = 0$$

$$\phi = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} \left.\begin{matrix} \cos m\psi \\ \sin m\psi \end{matrix}\right\} \left.\begin{matrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{matrix}\right\}$$

m = positive integer $\quad P_n^m = (\mu^2-1)^{m/2}\frac{d^m P_n(\mu)}{d\mu^m}$, $Q_n^m(\mu) = (\mu^2-1)^{m/2}\frac{d^m Q_n(\mu)}{d\mu^m}$

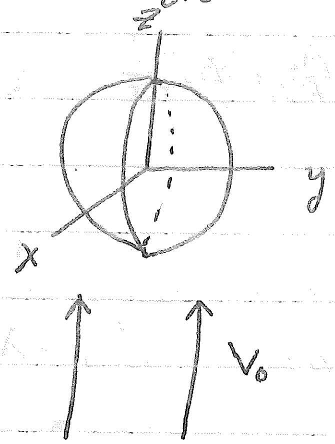
10-20-67

Sphere in uniform moving fluid.

$\nabla^2\phi = 0$

$\frac{\partial\phi}{\partial r} = 0 \quad r=a \qquad \phi = -V_0 r\cos\theta \text{ as } r\to\infty$

$= -V_0 r P_1(\cos\theta)$



$\phi = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} P_n \cos\theta$

$\phi = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n \cos\theta$

$\lim_{r\to 0}\phi = -\lim_{r\to\infty}\left[V_0 r P_1(\cos\theta)\right]$

$= \lim_{r\to\infty} \sum \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n \cos\theta$

$= \lim_{r\to 0} \left[A_1 r P_1 \cos\theta\right] \qquad \therefore A_1 = -V_0$

$\int_{-1}^{1} P_n(\mu) P_m(\mu)\, d\mu = 0 \text{ if } n\neq m$

so can equate like powers of r and n.

Also, it is seen that $A_0 = 0$, $A_n = 0$ for $n \geq 2$.

$\phi = -V_0 r P_1(\cos\theta) + \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta)$

$\frac{\partial\phi}{\partial r} = -V_0 P_1(\cos\theta) + \sum_{n=0}^{\infty} \frac{-(n+1)B_n}{r^{n+2}} P_n(\cos\theta)\Big|_{r=a} = 0$

$\left(-V_0^{P_1\cos\theta} - \frac{B_0}{a^2}\right) B_0 P_0 \cos\theta + \sum_{n=1}^{\infty} \frac{-(n+1)B_n}{a^{n+2}} P_n \cos\theta = 0$

So

$$V_0 P_1(\cos\theta) = \frac{-2B_1}{a^3} P_1(\cos\theta)$$

$$B_1 = -\frac{a^3 V_0}{2}$$

$$B_0 = 0, \quad B_n = 0 \text{ for } n \geq 2$$

$$\phi = \left[-V_0 r - \frac{V_0 a^3}{2r^2}\right] P_1(\cos\theta)$$

$$= -V_0\left[r + \frac{a^3}{2r^2}\right]\cos\theta$$

$$\nabla\phi = \frac{1}{h_\rho}\frac{\partial\phi}{\partial\rho}\hat{e}_\rho + \frac{1}{h_\theta}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta$$

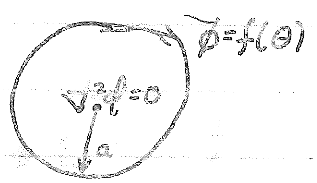
$$= \frac{\partial\phi}{\partial\rho}\bar{e}_\rho + \frac{1}{\rho}\frac{\partial\phi}{\partial\theta}\bar{e}_\theta$$

$$\nabla\phi = -V_0\left[1 - \frac{a^3}{r^3}\right]\cos\theta\,\hat{e}_\rho + V_0\left[1 + \frac{a^3}{2r^3}\right]\sin\theta\,\bar{e}_\theta$$

$$V_r = V_0\left[1 - \frac{a^3}{r^3}\right]\cos\theta; \quad V_\theta = -V_0\left(1 + \frac{a^3}{2r^3}\right)\sin\theta$$

Radial velocity is zero for (a, θ), $(a, \frac{\pi}{2})$

Tangential velocity is zero for $(r, n\pi)$



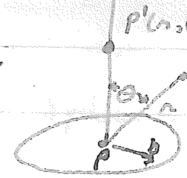
$$\phi = r^n \left\{ \begin{matrix} P_n\cos\theta \\ Q_n\cos\theta \end{matrix} \right.$$

$$\phi = \sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta)$$

at $r=a$ $\quad f(\theta) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta)$

$$A_n = \frac{2n+1}{2}\int_0^\pi f(\theta) P_n(\cos\theta)\sin\theta\, d\theta$$

AXIAL SYMMETRY PROBLEM



$$\phi = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta)$$

$$\phi(r,0) = G\sigma_s \int_0^b \int_0^{2\pi} \frac{\rho \, d\psi \, d\rho}{\sqrt{\rho^2 + r^2}}$$

$$= 2\pi G\sigma_s \left[\sqrt{r^2+b^2} - r \right] \quad \text{for } \theta = 0,\ r > 0$$

for $0 \le r \le b$ $\quad \sqrt{r^2+b^2} = b\left(1 + \frac{1}{2}\frac{r^2}{b^2} - \frac{1}{2\cdot4}\frac{r^4}{b^4} + \frac{1\cdot3}{2\cdot4\cdot6}\frac{r^6}{b^6} + \cdots\right)$

for $r > b$ $\quad \sqrt{r^2+b^2} = r\left(1 + \frac{1}{2}\frac{b^2}{r^2} - \frac{1}{2\cdot4}\frac{b^4}{r^4} + \frac{1\cdot3}{2\cdot4\cdot6}\frac{b^6}{r^6} + \cdots\right)$

$$\phi(r,0) = 2\pi G\sigma_s b \left[1 - \frac{r}{b} + \frac{1}{2}\frac{r^2}{b^2} - \frac{1}{2\cdot4}\frac{r^4}{b^4} + \frac{1\cdot3}{2\cdot4\cdot6}\frac{r^6}{b^6} \right] \text{ for } 0 \le r \le b$$

$$\phi(r,0) = 2\pi G\sigma_s b \left[\frac{1}{2}\frac{b}{r} - \frac{1}{2\cdot4}\frac{b^3}{r^3} + \frac{1\cdot3}{2\cdot4\cdot6}\frac{b^5}{r^5} - \cdots \right] \quad r \ge b$$

For $0 \le r \le b$, all $B_n = 0$

$A_0 = 2\pi G\sigma_s b \qquad A_4 = -2\pi G\sigma_s / 8b^3$

$A_1 = -2\pi G\sigma_s$

$A_2 = 2\pi G\sigma_s / 2b$

$A_3 = 0$

For $r \ge b$, $A_n = 0$

$B_0 = 2\pi G\sigma_s b^2/2 \qquad B_3 = 0$

$B_1 = 0 \qquad B_4 = 6\pi G\sigma_s b^6/48$

$B_2 = -2\pi G\sigma_s b^4/8$

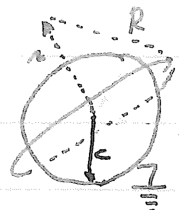
For $0 \leq r \leq b$,

$$\phi(r,\theta) = 2\pi G \sigma_s b \left[1 - \frac{r}{b} P_1(\cos\theta) + \frac{r^2}{2b^2} P_2(\cos\theta) - \frac{r^4}{2\cdot 4 b^4} P_4(\cos\theta) \right.$$
$$\left. + \frac{1\cdot 3}{2\cdot 4\cdot 6} \frac{r^6}{b^6} P_6(\cos\theta) + \ldots \right]$$

For $r > b$

$$\phi(r,\theta) = 2\pi G \sigma_s b \left[\frac{1}{2} \frac{b}{r} P_0(\cos\theta) - \frac{1}{2\cdot 4} \frac{b^3}{r^3} P_2(\cos\theta) + \frac{1\cdot 3}{2\cdot 4\cdot 6} \frac{b^5}{r^5} P_4(\cos\theta) - \ldots \right.$$

P162-163 SNEDDON



SPHERICAL CONDUCTOR SURROUNDED BY CIRCULAR WIRE OF RADIUS a CARRYING UNIFORM CHARGE

FIND POTENTIAL AT POINTS OUTSIDE OF SPHERICAL CONDUCTOR

Potential due to wire on axis of wire.

$1, \frac{1}{2}, \frac{3}{8}, \frac{15}{16},$

$$\phi_w(r,0) = \int_c \frac{e\,ds}{R} = \frac{2\pi e a}{\sqrt{a^2+r^2}}$$

For $r \leq a$ $\quad \phi_w(r,0) = 2\pi e \left\{ 1 - \frac{1}{2}\left(\frac{r}{a}\right)^2 + \frac{3}{8}\left(\frac{r}{a}\right)^4 - \ldots \right\}$

$$= 2\pi e \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2})_n}{n!} \left(\frac{r}{a}\right)^{2n}$$

where

$(\frac{1}{2})_0 = 1 \qquad (\frac{1}{2})_n = (\frac{1}{2})(\frac{1}{2}+1)(\frac{1}{2}+2) \times \ldots \times (\frac{1}{2}+n-1)$

For $r \geq a$ $\quad \phi_w(r,0) = 2\pi e \frac{a}{r} \left\{ 1 - \frac{1}{2}\left(\frac{a^2}{r^2}\right) + \frac{1}{8}\left(\frac{a^4}{r^4}\right) \right\} = 2\pi e \frac{a}{r} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} (-1)^n \left(\frac{a}{r}\right.$

$$= 2\pi e \left\{ \frac{a}{r} - \frac{1}{2}\left(\frac{a^3}{r^3}\right) + \frac{1}{8}\left(\frac{a^5}{r^5}\right) + \ldots \right\} =$$

$$= 2\pi e \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})_n}{n!} \left(\frac{a}{r}\right)^{2n+1}$$

In general,

$$\phi_w(r,\theta) = 2\pi e \sum_{n=0}^{\infty} \left[A_n r^n P_n(\cos\theta) + B_n \frac{1}{r^{n+1}} P_n(\cos\theta) \right]$$

for $r \le a$ $\quad \phi(r,\theta) = 2\pi e \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (-1)^n}{n!} \frac{r^{2n}}{a^{2n}} P_{2n}(\cos\theta)$

for $r \ge a$ $\quad \phi(r,\theta) = 2\pi e \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (-1)^n}{n!} \frac{a^{2n+1}}{r^{2n+1}} P_{2n}(\cos\theta)$

10-25-67 A) $\phi = 0$, $r = c$

B) $\phi_1 = \phi_2$ for $r = a$ $\qquad \phi_1$ $r < a$
ϕ_2 $r > a$

C) $\frac{\partial\phi_1}{\partial r} = \frac{\partial\phi_2}{\partial r}$ at $r = a$ except for points on the wire...

For $c \le r \le a$

$$\phi_1 = 2\pi e \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n (\frac{1}{2})_n}{n!} \left(\frac{r}{a}\right)^{2n} \right\} P_{2n}\cos\theta$$

$$+ \; 2\pi e \sum_{n=0}^{\infty} \left\{ A_n \left(\frac{r}{a}\right)^{2n} + B_n \left(\frac{c}{r}\right)^{2n+1} \right\} P_{2n}(\cos\theta)$$

Physically r^{2n} symmetry of top + bottom $P_n(\;)$ is odd
P_{2n} = even.

$r \ge a$

$$\phi_2 = 2\pi e \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n (\frac{1}{2})_n}{n!} \left(\frac{a}{r}\right)^{2n+1} + C_n \left(\frac{a}{r}\right)^{2n+1} \right\} P_{2n}(\cos\theta)$$

From (A) $\quad 0 = \frac{(-1)^n (\frac{1}{2})_n}{n!} \left(\frac{c}{a}\right)^{2n} + A_n \left(\frac{c}{a}\right)^{2n} + B_n$

(B) $\quad \frac{(\frac{1}{2})_n (-1)^n}{n!} + C_n = \frac{(-1)^n (\frac{1}{2})_n}{n!} + A_n + B_n \left(\frac{c}{a}\right)^{2n+1}$

$P_0 =$
$P_1 = \frac{1}{2}\cos\theta$
P_2
$x^4 - 2x^2+1$ (x^2-1)
$4x^3-4x$
$12x^2-4$

(c) Series are not uniformly convergent at $r=a$, so not ~~d~~ justified in termwise differentiation...

Terms causing trouble are $\frac{(\frac{1}{2})_m(-1)^m}{m!}(\;)^{2m}$

Let $\phi_1' = \phi_1 - \phi_{w,1}$
$\phi_2' = \phi_2 - \phi_{w,2}$
and ϕ_1' and ϕ_2' are uniformly convergent.
Then $\frac{\partial\phi_1'}{\partial r} = \frac{\partial\phi_2'}{\partial r}$ $r=a$

$$\phi_1' = 2\pi e\sum_{m=0}^{\infty}\left\{A_m\left(\frac{r}{a}\right)^{2m} + B_m\left(\frac{c}{r}\right)^{2m+1}\right\}P_{2m}(\cos\theta)$$

$$\phi_2' = 2\pi e\sum_{m=0}^{\infty} C_m\left(\frac{a}{r}\right)^{2m+1}P_{2m}\cos\theta.$$

Only sure of approach on physical terms...

$$\left.\frac{2m}{a}A_m\left(\frac{r}{a}\right)^{2m-1} + (2m+1)B_m\left(\frac{c}{r}\right)^{2m}\left(\frac{-c}{r^2}\right) - (2m+1)C_m\left(\frac{a}{r}\right)^{2m}\left(\frac{-a}{r^2}\right)\right]_{r=a} = 0$$

(c') at $r=a$: $\frac{2m}{a}A_m - (2m+1)\frac{B_m}{a}\left(\frac{c}{a}\right)^{2m+1} + \frac{C_m}{a}(2m+1) = 0$...

$$\therefore A_m = 0;\quad B_m = \frac{-(-1)^m(\frac{1}{2})_m}{m!}\left(\frac{c}{a}\right)^{2m};\quad C_m = \frac{(-1)^m(\frac{1}{2})_m}{m!}\left(\frac{c}{a}\right)^{2m+1}$$

$$\left\{\begin{aligned}\phi_1 &= 2\pi e\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{1}{2})_m}{m!}\left\{\left(\frac{r}{a}\right)^{2m} - \frac{c^{4m+1}}{a^{2m}r^{2m+1}}\right\}P_{2m}(\cos\theta)\\ \phi_2 &= 2\pi e\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{1}{2})_m}{m!}\left(\frac{a}{r}\right)^{2m+1}\left\{1+\left(\frac{c}{a}\right)^{4m+1}\right\}P_{2m}(\cos\theta)\end{aligned}\right\}$$

Nov. 8. Midsemester... open book...

10-27-67

Properties of Potential

a) Newtonian.

i) always finite at mass points (particles)

ii) $\lim\limits_{r \to \infty} \phi = 0$

iii) $\lim\limits_{r \to \infty} (r\phi) = \text{const.}$

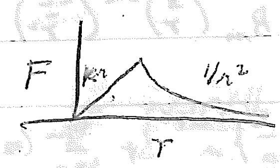
Volume distrubutions of matter,

iv) ϕ is continuous every where

v) $\nabla\phi$ is continuous everywhere

vi) $\frac{\partial^2 \phi}{\partial n^2}$ has a finite discontinuity... at places where there is a density discontinuity.

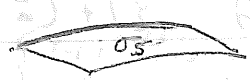
(density discontuity)



Surface distributions of matter.

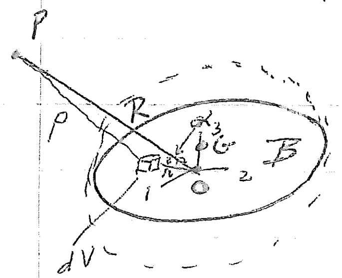
vii) ϕ continuous everywhere

$\left(\frac{\partial \phi}{\partial n}\right)$ has a finite discontinuity at places where there is a discontinuity in the surface density.



$$\nabla^2 \phi = -4\pi G \rho(x,y,z)$$

Potential of a Body at a Distant Point (MacMillan pg 81-ff)



P must be at least far away from smallest sphere that contains body

Region of space for valid P depends on choice of O

σ = volume density

$$\phi(P) = \int_B \frac{\sigma\, dV}{\rho}$$

By law of cosines,

$$\rho = (R^2 + r^2 - 2Rr\cos\alpha)^{1/2}$$

$$\phi(p) = \int \sigma (R^2 + r^2 - 2Rr\cos\alpha)^{-1/2}\, dV$$

condition on region brought about by absolute convergence.

Expanding parenthesis term and rearranging after checking for convergence (2003)

$$\frac{1}{\rho} = \frac{1}{R} + \frac{r}{R^2}\cos\alpha + \frac{r^2}{R^3}\cdot\frac{1}{2}(3\cos^2\alpha - 1) + \frac{r^3}{R^4}\cdot\frac{1}{2}(5\cos^2\alpha - 3\cos\alpha)$$

$R > r$

$$+ \frac{r^4}{R^5}\cdot\frac{1}{8}(35\cos^4\alpha - 30\cos^2\alpha + 3)$$

$$= \frac{1}{R}P_0(\cos\alpha) + \frac{r}{R^2}P_1(\cos\alpha) + \ldots$$

$$\frac{1}{\rho} = \sum_{n=0}^{\infty} \frac{r^n}{R^{n+1}} P_n(\cos\alpha)$$

$$\phi(p) = \frac{1}{R}\int_B \sigma\, dV + \frac{1}{R^2}\int \sigma r\cos\alpha\, dV + \frac{1}{2R^3}\int \sigma(3r^2\cos^2\alpha - r^2)\, dV$$

$$+ \frac{1}{2R^4}\int_B \sigma(5r^3\cos^3\alpha - 3r^3\cos\alpha)\, dV + \ldots$$

Can rearrange series requires convergence

if integrated series uniform convergence, then result is convergent.

$$\frac{1}{R}\int_B \sigma\, dV = \frac{M}{R}$$

By choice of origin of coordinates, can get rid of second term

Put origin at centroid.

g projection of OV on OG $\bar{R}$

$$\int_B \sigma r \cos\alpha\, dV = M g \quad \text{by definition of centroid.}$$

$$\bar{x} = \int \sigma x\, dV \Big/ \int \sigma\, dV$$
$$\bar{y} = \int \sigma \bar{y}\, dV \Big/ \int \sigma\, dV$$
$$\bar{z} = \int \sigma z\, dV \Big/ \int \sigma\, dV$$

~~$x = r\cos\alpha$~~

Also, $\frac{1}{2R^3}\int_B \sigma(3r^3\cos^2\alpha - r^2)\, dV$

$$= \frac{1}{2R^3}\int_B \sigma(3r^2 - 3r^2\sin^2\alpha - r^2)\, dV$$

$$= \frac{1}{R^3}\int_B \sigma r^2\, dV - \frac{3}{2R^2}\int_B \sigma r^2 \sin^2\alpha\, dV$$

Moment of inertia with respect to point O. I_0

Moment of inertia with respect to line $\underline{OP}$ I_R

$$\phi = \frac{M}{R} + \frac{M\bar{R}}{R^2} + \frac{2I_0 - 3I_R}{R^3} + \dots$$

If we put O at the centroid

$$\phi = \frac{M}{\ } + \frac{2I_0 - 3I_R}{\ }$$

Numerical it is easier to calculate by using.

Let $P(x, y, z)$ $\quad V(\xi, \eta, \zeta)$

$$r = \sqrt{\xi^2 + \eta^2 + \zeta^2}$$

$$R = \sqrt{x^2 + y^2 + z^2}$$

$$r \cos\alpha = \frac{\xi x + \eta y + \zeta z}{R}$$

$$\varphi = \frac{M}{R} + \frac{x}{R^3}\int_B \xi\, dm + \frac{y}{R^3}\int_B \eta\, dm + \frac{z}{R^3}\int \zeta\, dm$$

$$+ \frac{2x^2 - y^2 - z^2}{2R^5}\int \xi^2\, dm + \frac{(-x^2 + 2y^2 - z^2)}{2R^5}\int \eta^2\, dm$$

$$+ \frac{(-x^2 - y^2 + 2z^2)}{2R^5}\int \zeta^2\, dm + \frac{3xy}{R^5}\int \xi\eta\, dm$$

$$+ \frac{3yz}{R^5}\int \eta\zeta\, dm + \frac{3xz}{R^5}\int \zeta\xi\, dm$$

$$+ \frac{(2x^3 - 3xy^2 - 3xz^2)}{2R^7}\int \xi^3\, dm + \frac{(-3x^2y + 2y^3 - 3yz^2)}{2R^7}\int \eta^3\, dm$$

$$+ \frac{(-3x^2z - 3y^2z + 2z^3)}{2R^7}\int \zeta^3\, dm$$

$$+ \frac{(12x^2y - 3y^3 - 3yz^2)}{2R^7}\int \xi^2\eta\, dm$$

$$+ \frac{(-3x^2z + 12y^2z - 3z^3)}{2R^7}\int \zeta\eta^2\, dm$$

$$+ \frac{(-3x^3 - 3xy^2 + 12xz^2)}{2R^7}\int \zeta^2\xi\, dm$$

$$+ \frac{(12x^2z - 3y^2z - 3z^3)}{2R^7}\int \xi^2\zeta\, dm$$

Multipole expansion for Geophysics, see Grant+West

$$+\frac{(-3x^2+12xy^2-3xz^2)}{2R^7}\int \xi^2 \eta \, dm$$

$$+\frac{(-3x^2y-3y^3+12yz^2)}{2R^7}\int \eta \zeta^2 \, dm$$

$$+\frac{15xy^2}{R^7}\int \xi \eta \zeta \, dm + \text{higher order terms}$$

Once evaluate integrals for any shape of a body, are finished calculating integrals for that case.

10-30-67 DIRICHLET PROBLEM IN SPHERICAL COORDINATES:

- INTERIOR - for $r < a$

$\nabla^2 \phi = 0$

$\phi = F(\theta, \psi)$ on $r = a$

ϕ is finite at all points inside (included in $\nabla^2\phi = 0$)

Particular Solutions in spherical coordinates -

$$\phi = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} \left.\begin{matrix} \cos \\ \sin \end{matrix} m\psi \right\} \begin{matrix} P_n^m(\cos\theta) \\ \cancel{Q_n^m(\cos\theta)} \end{matrix}$$

$$\phi = \sum_n \sum_m r^n P_n^m(\cos\theta) \{ \cos m\psi \quad \sin m\psi \}$$

or

$$\phi = \sum_{n=0}^{\infty} B_{0n} \left(\frac{r}{a}\right)^n P_n(\cos\theta) + \sum_n \sum_m^{n} \left\{ B_{mn} \left(\frac{r}{a}\right)^n P_n^m(\cos\theta)\cos(m\psi) + A_{mn}\left(\frac{r}{a}\right)^n P_n^m(\cos\theta)\sin(m\psi) \right\}$$

Then at $r=a$, $\mu=\cos\theta$

$$F(\theta,\psi) = \sum_{n=0}^{\infty} B_{on} P_n(\mu) + \sum_{n=0}^{\infty}\sum_{m=1}^{n} \left\{ B_{mn} P_n^m(\mu)\cos m\psi + A_{mn} P_n^m(\mu)\sin m\psi \right\}$$

$$\int_0^{2\pi} F(\theta,\psi)\,d\psi = \sum_{n=0}^{\infty} B_{on} P_n(\mu)\int_0^{2\pi} d\psi + \sum_{n=0}^{\infty}\sum_{m=1}^{n} \left\{ B_{mn} P_n^m(\mu)\int_0^{2\pi}\cos m\psi\,d\psi + A_{mn}P_n^m(\mu)\int_0^{2\pi}\sin m\psi \right.$$

$$\frac{1}{2\pi}\int_0^{2\pi} F(\theta,\psi)\,d\psi = \sum_{n=0}^{\infty} B_{on} P_n(\mu)$$

$$\frac{1}{2\pi}\int_{-1}^{1}\int_0^{2\pi} F(\theta,\psi)\,P_k(\mu)\,d\psi\,d\mu = \sum_{n=0}^{\infty} b_{on}\int_{-1}^{1} P_k(\mu)\,P_n(\mu)\,d\mu$$

$$\therefore \quad \underline{B_{on} = \frac{2n+1}{4\pi}\int_{-1}^{1}\int_0^{2\pi} F(\theta,\psi)\,P_n(\mu)\,d\psi\,d\mu}$$

Multiply through by $\cos m\psi$, using orthogonality properties, also multiply by.

$$\int_0^{2\pi} F(\theta,\psi)\cos k\psi\,d\psi = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \left\{ B_{mn} P_n^m(\mu)\int_0^{}\cos m\psi\cos k\psi\,d\psi \right\}$$

integrals of the other terms drop out by orthogonality only one term of $\sum_{m=1}^{\infty}$ is needed,

$$\int_0^{\pi} F(\theta,\psi)\cos k\psi\,d\psi = \pi\sum_{n=0}^{\infty} B_{kn} P_n^k(\mu)$$

$$\int_{-1}^{1}\int_0^{2\pi}\cos m\psi\,P_s^m(\mu)\;F(\theta,\psi)\,d\psi\,d\mu = \pi\sum_{n=0}^{\infty} B_{mn}\int_{-1}^{1} P_n^m(\mu)\,P_s^m(\mu)\,d\mu$$

Using orthogonality of Legendre Associated Functions

$$B_{0n} = \frac{2n+1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} F(\theta,\psi)\, P_n(\mu)\, d\psi\, d\mu$$

$$\therefore\quad B_{mn} = \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \int_{-1}^{1} \int_{0}^{2\pi} P_n^m(\mu)\, F(\theta,\psi) \cos m\psi\, d\psi\, d\mu$$

Similarly

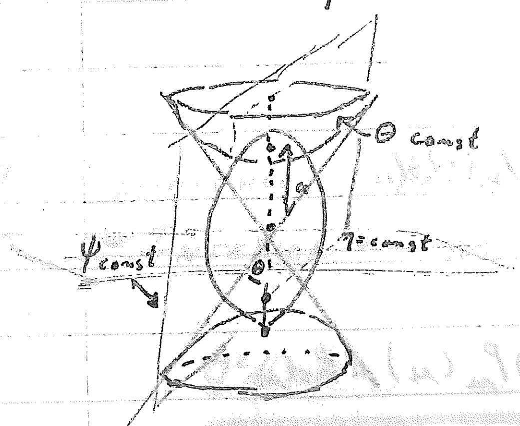
$n \geq m \geq 0$

$$A_{mn} = \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \int_{-1}^{1} \int_{0}^{2\pi} P_n^m(\mu)\, F(\theta,\psi) \sin m\psi\, d\psi\, d\mu$$

Moon + Spencer p236 ff

~~Oblate~~ Prolate Spheroidal Coordinates

a = dist to foci of ellipse.



A Asymptotic Cone

$$x = a \sinh\eta \sin\theta \cos\psi$$
$$y = a \sinh\eta \sin\theta \sin\psi$$
$$z = a \cosh\eta \cos\theta$$

Surfaces of constant η are prolate spheroids.

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $b = a \sinh\eta \quad c = a\cosh\eta$

Surfaces of constant θ are hyperboloids of two sheets

$$-\frac{x^2}{b^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $b = a\sin\theta \quad c = a\cos\theta$

For $\theta \to 0$ z axis from F to ∞

$\theta = \pi/2 \to xy$ plane.

11-1-67 $h_1 = h_2 = a\sqrt{\sinh^2\eta + \sin^2\theta}\ ,\ h_3 = a\sinh\eta\sin\theta$

$$\nabla^2\phi = \left[\frac{1}{a^2(\sinh^2\eta + \sin^2\theta)}\left\{\frac{\partial^2\phi}{\partial\eta^2} + \coth\eta\frac{\partial\phi}{\partial\eta} + \frac{\partial^2\phi}{\partial\theta^2} + \cot\theta\frac{\partial\phi}{\partial\theta}\right\}\right.$$

$$\left. + \frac{1}{a^2\sinh^2\eta\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2}\right] = 0$$

Let $\phi = H(\eta)\,\Theta(\theta)\,\Psi(\psi)$

$$\frac{1}{a^2(\sinh^2\eta + \sin^2\theta)}\left\{\frac{1}{H}\left(\frac{d^2H}{d\eta^2} + \coth\eta\frac{dH}{d\eta}\right) + \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta^2}\right)\right\}$$

$$+ \frac{1}{a^2\sinh^2\eta\sin^2\theta}\frac{1}{\Psi}\frac{d^2\Psi}{d\psi^2} = 0$$

$$\frac{\sinh^2\eta\sin^2\theta}{\sinh^2\eta + \sin^2\theta}\left\{\frac{1}{H}\left(\frac{d^2H}{d\eta^2} + \coth\eta\frac{dH}{d\eta}\right) + \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta^2}\right)\right\} =$$

$$-\frac{1}{\Psi}\frac{d^2\Psi}{d\psi^2} = q^2$$

$$\frac{q^2}{\sin^2\theta} - \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta}\right) = \frac{-q^2}{\sinh^2\eta} + \frac{1}{H}\left(\frac{d^2H}{d\eta^2} + \coth\eta\frac{dH}{d\eta}\right) = p(p+1)$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0$$

$$\frac{d^2H}{d\eta^2} + \coth\eta\frac{dH}{d\eta} - \left[p(p+1) + \frac{q^2}{\sinh^2\eta}\right]H = 0$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + \left[p(p+1) - \frac{q^2}{\sin^2\theta}\right]\Theta = 0$$

$\xi = \cancel{\cosh} \cosh\eta$

$\mu = \cos\theta$

$$(\xi^2-1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - \left[p(p+1) + \frac{q^2}{\xi^2-1}\right]H = 0$$

$$(\mu^2-1)\frac{d^2\Theta}{d\mu^2} + 2\mu\frac{d\Theta}{d\mu} - \left[p(p+1) + \frac{q^2}{\mu^2-1}\right]\Theta = 0$$

These are of Legendre form;

The solutions are,

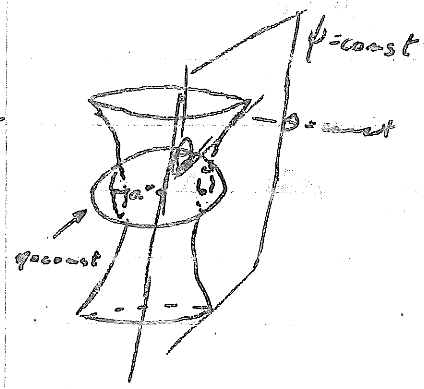
$$\phi = \left.\begin{matrix} P_p^q(\cosh\eta) \\ Q_p^q(\cosh\eta) \end{matrix}\right\} \left.\begin{matrix} P_p^q(\cos\theta) \\ Q_p^q(\cos\theta) \end{matrix}\right\} \begin{matrix} \cos \\ \sin \end{matrix} q\psi$$

For axial symmetry ($\phi \neq f(\psi)$)

$$\phi = \left.\begin{matrix} P_p(\cosh\eta) \\ Q_p(\cosh\eta) \end{matrix}\right\} \left.\begin{matrix} P_p(\cos\theta) \\ Q_p(\cos\theta) \end{matrix}\right\}$$

OBLATE SPHEROIDAL COORDINATE SYSTEM:

Mean & Spencer p 267



$$x = a \cosh\eta \sin\theta \cos\psi$$
$$y = a \cosh\eta \sin\theta \sin\psi$$
$$z = a \sinh\eta \cos\theta$$

Surfaces of constant η

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \text{where } b = a\cosh\eta, \quad c = a\sinh\eta$$

Surfaces of constant θ,

hyperboloids of one sheet

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad b = a\sin\theta, \quad c = a\cos\theta$$

z axis $\equiv \theta = 0$ $\qquad \theta = \pi/2$ entire x,y plane except

$-z$ axis $\equiv \theta = \pi$ $\qquad$ intersection of spheroid with xy plane

$$h_1 = h_2 = a\sqrt{\cosh^2\eta - \sin^2\theta} \qquad h_3 = a\cosh\eta\sin\theta$$

$$\nabla^2\phi = \frac{1}{a^2(\cosh^2\eta - \sin^2\theta)}\left\{\frac{\partial^2\phi}{\partial\eta^2} + \tanh\frac{\partial\phi}{\partial\eta} + \frac{\partial^2\phi}{\partial\theta^2} + \cot\theta\frac{\partial\phi}{\partial\theta}\right\}$$

$$+ \frac{1}{a^2\cosh^2\eta\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2} = 0$$

Let $\phi = H(\eta)\,\Theta(\theta)\,\Psi(\psi)$

$$\frac{1}{a^2(\cosh^2\eta - \sin^2\theta)}\left\{\frac{1}{H}\frac{d^2H}{d\eta^2} + \tanh\eta\frac{dH}{d\eta} + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta}\right\}$$

$$+ \frac{1}{a^2\cosh^2\eta\sin^2\theta}\frac{1}{\Psi}\frac{d^2\Psi}{d\psi^2} = 0$$

$$\frac{d^2H}{d\eta^2} + \tanh\eta \frac{dH}{d\eta} + \left[-p(p+1) + \frac{q^2}{\cosh^2\eta}\right]H = 0$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} + \left[p(p+1) - \frac{q^2}{\sin^2\theta}\right]\Theta = 0$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0$$

Let $\xi = i\sinh\eta$

$$(\xi^2-1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - \left[p(p+1) + \frac{q^2}{\xi^2-1}\right]H = 0$$

$\therefore$

$$\phi = \left.\begin{matrix} P_p^q(i\sinh\eta) \\ Q_p^q(i\sinh\eta) \end{matrix}\right\} \left.\begin{matrix} P_p^q(\cos\theta) \\ Q_p^q(\cos\theta) \end{matrix}\right\} \begin{matrix} \cos \\ \sin \end{matrix} q\psi$$

For axial symmetry $q=0$

$$\phi = \left.\begin{matrix} P_p^{\cancel{q}}(i\sinh\eta) \\ Q_p^{\cancel{q}}(i\sinh\eta) \end{matrix}\right\} \left.\begin{matrix} P_p(\cos\theta) \\ Q_p(\cos\theta) \end{matrix}\right\} \cancel{\begin{matrix} \cos \\ \sin \end{matrix} q\psi}$$

Conditions for separability of Laplace's Equation.

Found in Morse + Feshbach Chap 5. } Though for
& Moon + Spencer Chap 11 } wave equation.

Conditions on the coordinate system.

11-3-67 $$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_1 h_3}{h_2}\frac{\partial \phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial w}\right)\right] = 0$$

Let $\phi = U(u)V(v)W(w)$.

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{1}{U}\frac{d}{du}\left(\frac{h_2 h_3}{h_1}\frac{dU}{du}\right) + \frac{1}{V}\frac{d}{dv}\left(\frac{h_1 h_3}{h_2}\frac{dV}{dv}\right) + \frac{1}{W}\frac{d}{dw}\left(\frac{h_1 h_2}{h_3}\frac{dW}{dw}\right)\right] = 0$$

Necessary condition is
$$\frac{h_2 h_3}{h_1} = f_1(u)\,F_1(v,w)$$
$$\frac{h_1 h_3}{h_2} = f_2(v)\,F_2(u,w)$$
$$\frac{h_1 h_2}{h_3} = f_3(w)\,F_3(u,v)$$

$$0 = \nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{F_1(v,w)}{U}\frac{d}{du}\left(f_1(u)\frac{dU}{du}\right) + \frac{F_2(u,w)}{V}\frac{d}{dv}\left(f_2(v)\frac{dV}{dv}\right)\right] + \frac{F_3(u,v)}{W}\frac{d}{dw}\left(f_3(w)\frac{dW}{dw}\right)$$

U, V, W are functions of separation constants p^2, q^2 e.g. $P_p^q(\theta)$ etc.
but $f_1(u)$, $f_2(v)$, $f_3(w)$, $F_1(v,w)$, $F_2(u,w)$, $F_3(u,v)$ are not.

For wave equation also α_1

Call the separation constants α_2 and α_3

U, V, W are functions of $\alpha_2 + \alpha_3$, f's and F's are not.

$$F_1 \frac{\partial}{\partial \alpha_2}\left[\frac{1}{U}\frac{d}{du}\left(f_1\frac{dU}{du}\right)\right] + F_2\frac{\partial}{\partial \alpha_2}\left[\frac{1}{V}\frac{d}{dv}\left(f_2\frac{dV}{dv}\right)\right] + F_3\frac{\partial}{\partial \alpha_2}\left[\frac{1}{W}\frac{d}{dw}\left(f_3\frac{dW}{dw}\right)\right] = 0$$
$$F_1 \frac{\partial}{\partial \alpha_3}\left[\frac{1}{U}\frac{d}{du}\left(f_1\frac{dU}{du}\right)\right] + F_2\frac{\partial}{\partial \alpha_3}\left[\frac{1}{V}\frac{d}{dv}\left(f_2\frac{dV}{dv}\right)\right] + F_3\frac{\partial}{\partial \alpha_3}\left[\frac{1}{W}\frac{d}{dw}\left(f_3\frac{dW}{dw}\right)\right] = 0$$

Define $$\Phi_{ij}(u_i) = \frac{1}{f_i(u_i)}\frac{\partial}{\partial \alpha_j}\left[\frac{1}{U_i}\frac{d}{du_i}\left(f_i\frac{dU_i}{du_i}\right)\right]$$

$$F_1 f_1 \Phi_{12}(u) + F_2 f_2 \Phi_{22}(v) + F_3 f_3 \Phi_{32}(w) = 0$$

$$F_1 f_1 \Phi_{13}(u) + F_2 f_2 \Phi_{23}(v) + F_3 f_3 \Phi_{33}(w) = 0$$

for wave equation get Stäckel determinant

$$\Phi_{22} \frac{f_2 F_2}{f_1 F_1} + \Phi_{32} \frac{f_3 F_3}{f_1 F_1} = -\Phi_{12}$$

$$\Phi_{23} \frac{f_2 F_2}{F_1 f_1} + \Phi_{33} \frac{f_3 F_3}{f_1 F_1} = -\Phi_{13}$$

$$\frac{f_2 F_2}{f_1 F_1} = \frac{\begin{vmatrix} -\Phi_{12}(u) & \Phi_{32}(w) \\ -\Phi_{13}(u) & \Phi_{33}(w) \end{vmatrix}}{\begin{vmatrix} \Phi_{22}(v) & \Phi_{32}(w) \\ \Phi_{23}(v) & \Phi_{33}(w) \end{vmatrix}} = \frac{M_{21}(u,w)}{M_{11}(v,w)}$$

$$\frac{f_3 F_3}{f_1 F_1} = \frac{\begin{vmatrix} \Phi_{22}(v) & -\Phi_{12}(u) \\ \Phi_{23}(v) & -\Phi_{13}(u) \end{vmatrix}}{\begin{vmatrix} \Phi_{22}(v) & \Phi_{32}(w) \\ \Phi_{23}(v) & \Phi_{33}(w) \end{vmatrix}} = \frac{M_{31}(u,v)}{M_{11}(v,w)}$$

I. First Condition of Separability.

$$\frac{f_2 F_2}{f_1 F_1} = \frac{h_1^2}{h_2^2} \quad \text{and} \quad \frac{f_3 F_3}{f_1 F_1} = \frac{h_1^2}{h_3^2}$$

$$f_1(u)\left[\frac{F_1(v,w)}{M_{11}(v,w)}\right] = f_2(v)\left[\frac{F_2(u,w)}{M_{21}(u,w)}\right] = f_3(w)\left[\frac{F_3(u,v)}{M_{31}(u,v)}\right]$$

Only way to satisfy

II. $\frac{F_1(v,w)}{M_{11}(v,w)} = f_2(v) f_3(w) \qquad \frac{F_2(u,w)}{M_{21}(u,w)} = f_1(u) f_3(w) \qquad \frac{F_3(u,v)}{M_{31}(u,v)} = f_1(u) f_2(v)$

II.'

$$\begin{cases} \dfrac{h_2 h_3}{h_1} = f_1 f_2 f_3 M_{11} \\ \dfrac{h_1 h_3}{h_2} = f_1 f_2 f_3 M_{21} \\ \dfrac{h_1 h_2}{h_3} = f_1 f_2 f_3 M_{31} \end{cases}$$

I and II are necessary and sufficient...

From sufficiency proof,

The three separated equations are.

$$\frac{1}{f_i}\frac{d}{du_i}\left(f_i\frac{dU_i}{du_i}\right)+U_i\sum_{j=1}^{3}\alpha_j\,\Phi_{ij}=0$$

here $\alpha_1=0$

For wave equations, only 11 coordinate systems are separable:

The following 11 have the property of wave equation and $\nabla^2 V$ are both separable.

1) Rectangular $u=x,\ v=y,\ w=z$, $x=x,\ y=y,\ z=z$, $h_1=1\quad h_2=1,\ h_3=1$

$-\infty<x<\infty$
$-\infty<y<\infty$
$-\infty<z<\infty$

2) Circular cylindrical $u=r\quad v=\theta\quad w=z$, $0\le r\le\infty$

$x=r\cos\theta,\ y=r\sin\theta\ z=z$ $\quad 0\le\theta<2\pi$

$h_1=1\qquad h_2=r\qquad h_3=1$ $\quad -\infty<z<\infty$

3) Elliptical Cylindrical $u=\eta,\ v=\psi,\ w=z$ $\quad 0\le\eta<\infty$

$x=a\cosh\eta\cos\psi$ $\quad 0\le\psi<2\pi$

$y=a\sinh\eta\sin\psi$ $\quad -\infty<z<+\infty$

$z=z$

$h_1=h_2=a\sqrt{\sinh^2\eta\cos^2\psi+\cosh^2\eta\sin^2\psi}$

$h_3=1$

4) Parabolic Cylindrical $u = \mu,\ v = \nu,\ w = z$

$x = \frac{1}{2}(\mu^2 - \nu^2)$ $\quad 0 \le \mu < \infty$

$y = \mu\nu$ $\quad 0 \le \nu < \infty$

$z = z$ $\quad -\infty < z < \infty$

$h_1 = h_2 = \sqrt{\mu^2 + \nu^2}$

$h_3 = 1$

11-6-67

5) Spherical Coordinates $u = r,\ v = \theta,\ w = \psi$

$x = r \sin\theta \cos\psi$ $\quad 0 \le r < \infty$

$y = r \sin\theta \sin\psi$ $\quad 0 \le \theta \le \pi$

$z = r\cos\theta$ $\quad 0 \le \psi < 2\pi$

$h_1 = 1$

$h_2 = r$

$h_3 = r \sin\theta$

6). Parabolic $u = \mu,\ v = \nu,\ w = \psi$

$x = \mu\nu \cos\psi$ $\quad 0 \le \mu < \infty$

$y = \mu\nu \sin\psi$ $\quad 0 \le \nu < \infty$

$z = \frac{1}{2}(\mu^2 - \nu^2)$ $\quad 0 \le \psi < 2\pi$

$h_1 = h_2 = \sqrt{\mu^2 + \nu^2}$

$h_3 = \mu\nu$

7) Prolate Spheroidal $\mu = \eta,\ v = \theta,\ w = \psi$

$x = a \sinh\eta \cos\psi \sin\theta$ $\quad 0 \le \eta < \infty$

$y = a \sinh\eta \sin\psi \sin\theta$ $\quad 0 \le \theta \le \pi$

$z = a \cosh\eta \cos\theta$ $\quad 0 \le \psi < 2\pi$

$h_1 = h_2 = a\sqrt{\sinh^2\eta + \sin^2\theta}$

$h_3 = a \sinh\eta \sin\theta$

8) OBLATE SPHEROIDAL: $u = \eta$, $v = \theta$, $w = \psi$

$$x = a \cosh\eta \sin\theta \cos\psi \qquad 0 \le \eta \le \infty$$
$$y = a \cosh\eta \sin\theta \sin\psi \qquad 0 \le \theta \le \pi$$
$$z = a \sinh\eta \cos\theta \qquad 0 \le \psi < 2\pi$$
$$h_1 = h_2 = a\sqrt{\cosh^2\eta - \sin^2\theta}$$
$$h_3 = a \cosh\eta \sin\theta$$

TRANSFORMATIONS USE ELLIPTICAL FUNCTIONS

9) ellipsoidal

10) paraboloidal

11) conical

To show $\nabla^2\phi = 0$ is separable in circular cylindrical coordinates:

CONDITIONS FOR SEPARABILITY

$$\frac{f_2 F_2}{f_1 F_1} = \frac{h_1^2}{h_2^2} = \frac{M_{21}}{M_{11}}; \quad \frac{f_3 F_3}{f_1 F_1} = \frac{h_1^2}{h_3^2} = \frac{M_{31}}{M_{11}}; \quad \frac{h_2 h_3}{h_1} = f_1 f_2 f_3 M_{11} \quad \frac{h_1 h_3}{h_2} = f_1 f_2 f_3 M_{21}$$

$$\frac{h_1 h_2}{h_3} = f_1 f_2 f_3 M_{31}$$

Circular Cylindrical $h_1 = 1$, $h_2 = r$ $h_3 = 1$ $\quad u = r, v = \theta, w = z$

$f_1 = f_1(r)$ $f_2 = f_2(\theta)$ $f_3 = f_3(z)$

$$\frac{M_{21}}{M_{11}} = \frac{1}{r^2} \qquad \frac{M_{31}}{M_{11}} = 1 \qquad r = f_1 f_2 f_3 M_{11} \qquad \frac{1}{r} = f_1 f_2 f_3 M_{21}$$

$$r = f_1 f_2 f_3 M_{31}$$

~~$M_{..}(..$~~$w)$

$M_{11} = M_{11}(\theta, z)$ $r = f_1 f_2 f_3 M_{11}$ could be $f_2 = f_3 = M_{11} = 1$, $f_1 = r$

$M_{21} = M_{21}(r, z)$ $\quad \frac{1}{r} = r M_{21}$ $\quad M_{21} = r^2$

$M_{31} = 1$

$M_{11} = 1 \quad M_{21} = r^2 \quad M_{31} = 1$ $\Big\}$ $\therefore$ coordinates are separable

To find Separated Differential Equations:

$$\frac{1}{f_i}\frac{d}{du_i}\left(f_i\frac{dU_i}{du_i}\right) + U_i\sum_{j=1}^{3}\alpha_j\,\Phi_{ij} = 0 \qquad \text{where } \alpha_1 = 0$$

$$M_{21}(r,z) = \begin{vmatrix} -\Phi_{12}(r) & \Phi_{32}(z) \\ -\Phi_{13}(r) & \Phi_{33}(z) \end{vmatrix} = \frac{1}{r^2}$$

11 $$M_{11}(\theta,z) = \begin{vmatrix} \Phi_{22}(\theta) & \Phi_{32}(z) \\ \Phi_{23}(\theta) & \Phi_{33}(z) \end{vmatrix} = 1$$

$$M_{31}(r,\theta) = \begin{vmatrix} \Phi_{22}(\theta) & -\Phi_{12}(r) \\ \Phi_{23}(\theta) & -\Phi_{13}(r) \end{vmatrix} = 1$$

Let $\Phi_{12} = -\frac{1}{r^2}$ $\quad \Phi_{33}(z) = 1$ $\quad$ either $\Phi_{13}(r)$ or $\Phi_{32}(z) = 0$

Then

$\Phi_{22}(\theta) = 1,$ $\qquad$ either $\Phi_{23}(\theta)$ or $\Phi_{32}(z) = 0$

$\Phi_{13}(r) = -1$ $\quad$ so $\Phi_{32}(z) = 0$ $\quad \Phi_{23} = 0$

$$\Phi_{12} = -\frac{1}{r^2} \quad \Phi_{22} = 1 \quad \Phi_{32} = 0$$
$$\Phi_{13} = -1 \quad \Phi_{23} = 0 \quad \Phi_{33} = 1$$

$$\nabla^2 u = \frac{1}{r}\frac{d}{du}\left(r\frac{dR}{du}\right) + R\left(-\frac{\alpha_2}{r^2} + -\alpha_3\right) \overset{=0}{} \; \frac{d^2\Phi}{d\theta^2} + \Phi(\alpha_2) = 0$$

$$+\frac{d^2Z}{dz^2} + Z(\alpha_2 \cdot 0 + \alpha_3) = 0$$

$i=1$

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - R\left(\frac{\alpha_2}{r^2} + \alpha_3\right) = 0 \qquad \text{let } \alpha_2 = q^2$$

$$\frac{d^2\Phi}{d\theta^2} + \alpha_2\Phi = 0 \qquad -\alpha_3 = p^2$$

$$\frac{d^2Z}{dz^2} + \alpha_3 Z = 0$$

Theorem: In addition to the 11 coordinate systems considered, LAPLACE's Equation is separable in any orthogonal cylindrical coordinate system in which ϕ is independent of z.

Case that $\nabla^2\phi$ is not separable, but will be so when z dependence is dropped.

Bicylindrical Coordinates

$u = \eta \quad v = \theta \quad w = z \qquad 0 \le \eta < \infty$

$x = \dfrac{a \sinh\eta}{\cosh\eta - \cos\theta} \qquad 0 \le \theta < 2\pi$

$-\infty < z < +\infty$

$$y = \frac{a \sin\theta}{\cosh\eta - \cos\theta}$$

$$z = z$$

$$h_1 = h_2 = \frac{a}{\cosh\eta - \cos\theta} \qquad h_3 = 1$$

$\dfrac{M_{21}}{M_{11}} = 1 \qquad \dfrac{h_2 h_3}{h_1} = 1 = f_1 f_2 f_3 M_{11}(\theta, z)$

$\dfrac{M_{31}}{M_{11}} = \dfrac{a^2}{(\cosh - \cos\theta)^2} \qquad \dfrac{h_1 h_3}{h_2} = 1 = f_1 f_2 f_3 M_{21}(\eta, z)$

$$\frac{h_1 h_2}{h_3} = \frac{a^2}{(\cosh\eta - \cos\theta)^2} = f_1 f_2 f_3 M_{31}(\eta, \theta)$$

what if $f_1 = f_2 = f_3 = 1 \quad M_{11} = 1,\ M_{21} = 1,\ M_{31} = \dfrac{a^2}{(\cosh\eta - \cos\theta)^2}$

M_{31} is not in permissible form $\quad M_{31} = M_{31}(\eta, \theta) \neq E(\eta)T(\theta)$

11-10-67

Moon & Spencer

In cylindrical coordinate systems, conditions for separability are.

$$h_2^2 = h_1^2 = -[\phi_{13}(u) + \phi_{23}(v)] \qquad h_3 = 1$$

The scale factor must be expressed as a separable sum

If $\phi \neq f(z)$, then $\nabla^2\phi$ is separable in the bicylindrical coordinate system ($\nabla^2\phi$ is separable in any cylindrical coordinate system for $\phi \neq f(z)$)

Bicylindrical.

$$\nabla^2\phi = 0 = \cancel{\left(\frac{\cosh\eta - \cos\theta}{a}\right)^2}\left(\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\theta^2}\right) = 0$$

Separability for Laplace's Equation:

R-separability -

Definition: If the assumption $\phi = \dfrac{U(u)V(v)W(w)}{R(u,v,w)}$ permits the separation of the P.D.E. into 3 ordinary D.E. and if R is not a constant, then the equations are said to be R-separable...

Example: Toroidal (η, θ, ψ)

$$x = \frac{a\sinh\eta\cos\psi}{\cosh\eta - \cos\theta}, \quad y = \frac{a\sinh\eta\sin\psi}{\cosh\eta - \cos\theta} \quad z = \frac{a\sin\theta}{\cosh\eta - \cos\theta}$$

$$h_1 = h_2 = \frac{a}{\cosh\eta - \cos\theta} \qquad h_3 = \frac{a \sinh\eta}{\cosh\eta - \cos\theta}$$

$$\nabla^2 \phi = 0 = \frac{(\cosh\eta - \cos\theta)^3}{a^3 \sinh\eta} \left\{ \left[\frac{\partial}{\partial\eta}\left(\frac{a\sinh\eta}{\cosh\eta - \cos\theta} \frac{\partial\phi}{\partial\eta} \right) + \frac{\partial}{\partial\theta}\left(\frac{a\sinh\eta}{\cosh\eta - \cos\theta} \frac{\partial\phi}{\partial\theta} \right) \right] \right.$$

$$\left. + \frac{\partial}{\partial\psi}\left(\frac{a}{\sinh\eta(\cosh\eta - \cos\theta)} \right) \frac{\partial\phi}{\partial\psi} \right\}$$

Let $\phi = \dfrac{H(\eta)\Theta(\theta)\Psi(\psi)}{1/\sqrt{\cosh\eta - \cos\theta}} = \sqrt{\cosh\eta - \cos\theta}\; H(\eta)\Theta(\theta)\Psi(\psi)$

$$\frac{1}{\sinh\eta}\frac{d}{d\eta}\left(\sinh\eta \frac{dH}{d\eta} \right) + H\left(\frac{1}{4} - \alpha_2 - \frac{\alpha_3}{\sinh^2\eta} \right) = 0$$

$$\frac{d^2\Theta}{d\theta^2} + \alpha_2 \Theta = 0$$

$$\frac{d^2\Psi}{d\psi^2} + \alpha_3 \Psi = 0$$

Call $\alpha_2 = p^2$, $\alpha_3 = q^2$ $\xi = \cosh\eta$

$$(\xi^2 - 1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - \left[\frac{q^2}{\xi^2 - 1} + \left(p^2 - \frac{1}{4}\right) \right] H$$

$$\frac{d^2\Theta}{d\theta^2} + p^2\Theta = 0$$

$$\frac{d^2\psi}{d\psi^2} + q^2\psi = 0$$

$$\phi = \sqrt{\cosh\eta - \cos\theta} \left\{ \begin{matrix} P^q_{p-1/2}(\cosh\eta) \\ Q^q_{p-1/2}(\cosh\eta) \end{matrix} \right\} \begin{matrix} \sin \\ \cos \end{matrix} p\theta \left. \right\} \begin{matrix} \sin \\ \cos \end{matrix} q\psi$$

If axial symmetry,

$$\phi = \left. P_{p-1/2}(\cosh\eta) \right\} \sin p\theta \left. \right\} \sqrt{\cosh\eta - \cos\theta}$$

Moon + Spencer
Chap. 6.

ORTHOGONAL FUNCTIONS:

A set of functions $\{\phi_n(x)\}$ is orthogonal on the interval (a, b) if

$$\int_a^b \phi_n(x)\phi_m(x) = 0 \quad \text{for } m \neq n$$

The n'th norm for the set $\{\phi_n(x)\}$ is

$$N_n = \int_a^b [\phi_n(x)]^2 \, dx$$

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x)$$

where

$$\int_a^b f(x)\, \phi_m(x) dx = \sum_{n=0}^{\infty} A_n \int_a^b \phi_n(x)\phi_m(x) dx$$

$$A_n = \frac{1}{N_n} \int_a^b f(x)\, \phi_n(x) dx$$

Building orthogonal sets:

Let $\{\psi_n'(x)\}$ be a set of linearly independent functions which are non-orthogonal on the interval (a, b). Modify $\{\psi_n(x)\}$ to be orthogonal on (a,b). Call the orthogonal set $\{\phi_n(x)\}$

Let
$$\phi_0(x) = \psi_0(x)$$
$$\phi_1(x) = \psi_1(x) - a_{10}\phi_0(x)$$
$$\phi_2(x) = \psi_2(x) - a_{20}\phi_0(x) - a_{21}\phi_1(x)$$
$$\phi_3(x) = \psi_3(x) - a_{30}\phi_0(x) - a_{31}\phi_1(x) - a_{32}\phi_2(x)$$

Herbert S. Wilf.
Mathematics for Physical Sciences,
Wiley + Co (1962) pg. 78.

$$a_{10} = \frac{\int_a^b \psi_1(x)\, \phi_0(x)\,dx}{N_0}$$

$$a_{20} = \frac{\int_a^b \psi_2(x)\, \phi_0(x)\,dx}{N_0}$$

$$a_{21} = \frac{\int_a^b \psi_2(x)\, \phi_1(x)\,dx}{N_1}$$

$$\int_a^b [\phi_m(x)]^2\,dx = \int_a^b \psi_m(x)\phi_m(x)\,dx$$

$$a_{im} = \frac{\int_a^b \psi_i(x)\, \phi_m(x)\,dx}{N_m} \quad \text{for} \quad m < i$$

11-13-67

For $\{\phi_n(x)\}$ - an orthogonal set
then

$$f(x) = \sum a_n \phi_n(x)$$

if and only if

IF DIRICHLET CONDITIONS SATISFIED

$$f(x) = \sum (b_n \cos nx + c_n \sin nx) \quad \text{converges to } f(x)$$

Let $\psi_0(x) = 1$, $\psi_1(x) = x$, $\psi_2(x) = x^2$, $\psi_3(x) = x^3 \ldots$ $\quad (a,b) = (-1,1)$

$$\phi_0(x) = 1$$
$$\phi_1(x) = x - a_{10}$$
$$\phi_2(x) = x^2 - a_{21}\phi_1(x) - a_{20}$$
$$\phi_3(x) = x^3 - a_{32}\phi_2(x) - a_{31}\phi_1(x) - a_{30}$$

$$\phi_0(x) = 1$$
$$\phi_1(x) = x$$
$$\phi_2(x) = x^2 - \tfrac{1}{3}$$
$$\phi_3(x) = x^3 - \tfrac{3}{5}x$$

$$a_{10} = \frac{1}{N_0}\int_{-1}^{1} x \cdot 1\,dx = \frac{1}{N_0}\frac{x^2}{2}\Big|_{-1}^{1} = 0$$

$$a_{20} = \frac{1}{N_0}\int_{-1}^{1} x^2 \cdot 1\,dx = \frac{1}{N_0}\frac{x^3}{3}\Big]_{-1}^{1} = \frac{2}{3}\frac{1}{N_0} = \frac{1}{3}$$

$$\frac{1}{\ldots}\int_{-1}^{1} x^2 \cdot x\,dx = \frac{1}{\ldots}\frac{x^4}{\ldots}\Big]_{-1}^{1} = 0$$

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

ORTHOGONALITY WITH WEIGHTING FUNCTIONS:

Two functions $\phi_n(x)$ and $\phi_m(x)$ are said to be orthogonal on the interval (a,b) with respect to the weighting function $w(x)$ if

$$\int_a^b \phi_n(x)\phi_m(x)\, w(x)\, dx = 0 \quad \text{for } m \neq n$$

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x)$$

$$\int_a^b f(x)\, w(x)\phi_m(x)\, dx = \sum_{n=0}^{\infty} A_n \int \phi_n(x)\, \phi_m(x)\, w(x)\, dx$$

$$A_m = \frac{\int_a^b f(x)\, w(x)\phi_m(x)\, dx}{\int_a^b w(x)\left[\phi_m(x)\right]^2 dx} = \frac{\int_a^b f(x)\, w(x)\phi_m(x)\, dx}{N_m}$$

$\phi_0(x) = \psi_0(x)$
$\phi_1(x) = \psi_1(x) - a_{10}\phi_0(x)$
$\phi_2(x) = \psi_2(x) - a_{21}\phi_1(x) - a_{20}\phi_0(x)$

where

$$a_{ij} = \frac{1}{N_j}\int_a^b w(x)\, \psi_i(x)\, \phi_j(x)\, dx \qquad N_j = \text{weighted norm}$$

Let $\psi_0(x) = 1$, $\psi_1(x) = x$, $\psi_2(x) = x^2$ $(a,b) = (0,\infty)$

Let $w = e^{-x}$ $\qquad$ $\Gamma(n+1) = n!$

$\phi_0(x) = \psi_0(x) = 1$

$\phi_1(x) = \psi_1(x) - a_{10}\phi_0(x) = x - a_{10} = x - 1$

$$a_{10} = \frac{1}{N_0}\int_0^\infty e^{-x}x\,dx = \frac{1}{N_0}\Gamma(2) = \frac{1}{N_0} = \underline{1}$$

$$N_0 = \int_0^\infty e^{-x} = -e^{-x}\Big|_0^\infty = -(-1) = 1$$

~~Laguerre Functions~~

$\phi_0(x) = 1$

$\phi_1(x) = x - 1$

$\phi_2(x) = x^2 - 4x + 2$

$\phi_3(x) = x^3 - 9x^2 + 18x - 6$

Laguerre Functions.

$L_0(x) = 1$

$L_1(x) = -(x-1)$

$L_2(x) = (x^2 - 4x + 2)$

$L_3(x) = -(x^3 - 9x^2 + 18x - 6)$

Dean Teledyne Filtering Geophysics 10 years ago 1957

$\psi_0(x) = 1$ $\qquad$ $(a,b) = (-\infty, \infty)$

$\psi_1(x) = x$ $\qquad$ $w(x) = e^{-x^2}$

$\psi_2(x) = x^2$

Hermite functions.

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x.$

11-15-67

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \text{ where } n \geq 0$$

For real values of n and finite values of x, one solution is

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

when $n \neq 0$ or integer, a second independent solution is

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$$

For n an integer

$$J_{-n}(x) = (-1)^n J_n(x)$$

$J_n(0)$ is finite for $n \geq 0$

$J_{-n}(0)$ is finite for n integer but infinite for "n" not integer

$J_n(x) \to 0$ as $x \to \infty$ $\quad J_{-n}(x) \to 0$ as $x \to \infty$

$$J_{n-1}(x) + J_{n+1}(x) = (2n/x)\,J_n(x)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

For $n = 0$, integer, non-integer, a second independent solution is. ~~Neumann~~

$$Y_n(x) = \frac{\cos n\pi\, J_n(x) - J_{-n}(x)}{\sin n\pi}$$

for non-integer $\quad Y_{n+\frac{1}{2}}(x) = (-1)^{n+1} J_{-n-1/2}(x)$

for integer series expansion on p.134

IF $n \neq 0$ or integer. General Solution is.

$$y = A J_n(x) + B J_{-n}(x)$$

IF $n = 0$ or integer

$$y = A J_n(x) + B Y_n(x)$$

For very large x for $|x| \gg n$

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{2n+1}{4}\pi\right)$$

$$Y_n(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{2n+1}{4}\pi\right)$$

Modified Bessel's.

$$x^2 y'' + x y' - (x^2 + n^2) y$$

Independent solutions are $J_n(jx)$, $Y_n(jx)$

$$I_n(x) = j^{-n} J_n(jx) = \sum_{r=0}^{\infty} \frac{(x/2)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

Modified Bessel of Second kind.

for non-integer $K_n = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin n\pi}$

n integer $K_n(x) = (-1)^{n+1} \sum_{r=0}^{\infty} \frac{(x/2)^{n+2r}}{r!(n+r)!} \left\{ \ln(x/2) - \frac{1}{2}\phi(r) - \frac{1}{2}\phi(n+r) + \dots \right.$

Equations reducible to Bessel's equation.

$$y'' + \left(\frac{1-2\alpha}{x}\right)y' + \left\{ (\beta\gamma x^{\gamma-1})^2 + \frac{\alpha^2 - n^2\gamma^2}{x^2} \right\} y = 0$$

$$y = x^{\alpha}\left\{ A J_n(\beta x^{\gamma}) + B Y_n(\beta x^{\gamma}) \right\}$$

Equations reducible to Bessel's modified equation.

$$y'' + \left(\frac{1-2\alpha}{x}\right)y' - \left\{ (\beta\gamma x^{\gamma-1})^2 + \frac{n^2\gamma^2 - \alpha^2}{x^2} \right\} y = 0$$

solutions are of type $x^{\alpha} I_n(\beta x^{\gamma})$ $\quad x^{\alpha} K_n(\beta x^{\gamma})$

BESSEL FUNCTIONS OF THIRD KIND (HANKEL FUNCTION)

$$H_n^{(1)}(x) = J_n(x) + j Y_n(x)$$
$$H_n^{(2)}(x) = J_n(x) - j Y_n(x)$$

$H_n(z) \to 0$ as $z \to \infty$
only cylindrical solution doing this.

$\lim_{r\to\infty} H_n^{(1)}(re^{i\theta}) = 0$

multiplication by $\exp(\pm i\omega t)$ gives traveling wave.

$$\lim_{r\to\infty} H_n^{(1)}(re^{i\theta}) = 0 \quad 0 \le \theta \le \pi$$ zero in upper half plane

$$\lim_{r\to\infty} H_n^{(2)}(re^{-i\theta}) = 0 \quad 0 \le \theta \le \pi$$ zero in lower half plane.

$$\frac{d}{dx}\left\{ J_n(x)/x^n \right\} = - J_{n+1}(x)/x^n$$

$$\frac{d}{dx}\left\{ Y_n(x)/x^n \right\} = - Y_n(x)/x^n$$

11-17-67:

Legendre Polynomials:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

General Solution is $y = a_0 u_n(x) + a_1 v_n(x)$

$$u_n(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \dots$$

$$v_n(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5$$

Series converge $|x| < 1$

Special Case n - positive integer.

$n=0$ $\quad y = a_0 + a_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right)$

$n=1$, $\quad y = a_0\left\{1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5}\right\} + a_1 x$

$n=2$ $\quad y = a_0(1-3x^2) + a_1\left\{x - \frac{x^3}{3} - \frac{x^5}{5} - \frac{8x^7}{35} - \dots\right\}$

For n integer.

P 83 $\quad P_n(x) = \frac{u_n(x)}{u_n(1)}$ for $n=0$ or n = even integer (even powers)

$P_n(x) = \frac{v_n(x)}{v_n(1)}$ for n odd integer. (odd powers)

$P_n(x)$ converges for all values of x.

By definition, therefore $P_n(1) = 1$

Also $P_n(-1) = (-1)^n$ or $P_n(-x) = (-1)^n P_n(x)$

Legendre Function of Second Kind

$$Q_n(x) = -v_n(1)\ u_n(x) \quad \text{for } n \text{ odd}$$
$$Q_n(x) = u_n(1)\ v_n(x) \quad \text{for } n \text{ even or zero.}$$

even function
odd function

$$Q_0(x) = \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right| = \tanh^{-1}x$$

$$Q_1(x) = \frac{x}{2} \ln\left|\frac{1+x}{1-x}\right| = P_1(x)\,Q_0(x) - 1$$
$$Q_2(x) = \tfrac{1}{2} P_2(x)\,Q_0(x) - 3x/2$$
$$Q_3(x) = \tfrac{1}{2} P_3(x)\,Q_0(x) - 5x^2/2 + 2/3$$

Rodrigue's Formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

Generating function method:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$
$$f(x,t) = \{1 - (2xt - t^2)\}^{-1/2}$$

Assuming $|2xt - t^2| < 1$ and $|t|$ is small

$$f(x,t) = 1 + \frac{(1/2)}{1!}(2xt - t^2) + \frac{(1/2)(3/2)}{2!}(2xt-t^2)^2 + \dots + \frac{(1/2)(3/2)\cdots\{(2n-1)/2\}}{n!}(2xt-t^2)^n + \dots$$

IF UNIFORMLY CONVERGENT IN x,t can rearrange terms

$$= 1 + xt + \{(3x^2-1)/2\}t^2 + \{(5x^3-3x)/2\}t^3 + \dots + A_n(x)t^n$$

$$\text{where } A_n(x) = \frac{1\cdot3\cdot5\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot4(2n-1)(2n-3)}x^{n-4} - \dots\right]$$

$$f(x,t) = \sum_{n=0}^{\infty} A_n(x)t^n = \sum_{n=0}^{\infty} P_n(x)t^n$$

$$A_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!\,(n-2r)!\,(n-r)!} x^{n-2r} \qquad (11.5)$$

$N = \frac{n}{2}$ for n even and $N = \frac{n-1}{2}$ for n odd.

11-20-67

$A_n(x)$ is a solution of Legendre's equation.

$$f(1,t) = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots \qquad \text{if } |t|<1$$

$$A_1(t) = \sum_{r=0}^{\infty} \frac{(2-2r)!}{2^n r!\,(1-2r)!\,(1-r)!} = 1 \qquad N = \left(\tfrac{n-1}{2}\right) = 0$$

Since $A_n(1) = 1$ and $P_n(1) = 1$ Then $P_n(x)$ and $A_n(x)$ are equivalent.

Recurrence relations:

$$f(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n (P_n(x)$$

differentiate with respect to t.

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nt^{n-1} P_n(x)$$

$$(x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{n=0}^{\infty} nt^{n-1}P_n(x) = (x-t)\sum_{n=0}^{\infty} t^n P_n(x)$$

Equating like coefficients of t,

$$\sum_{n=0}^{\infty} n P_n(x)\left[t^{n-1} - 2xt^n + t^{n+1}\right] = \sum_{0}^{\infty} P_n(x)\left[x t^n - t^{n+1}\right]$$

or $\cancel{\sum_{s=0}^{\infty} P_s(x)\left[(s+1)t^s - 2sxt^s + (s-1)t^s\right]} =$

$$(n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) = xP_n(x) - P_{n-1}(x)$$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_1(x) = x P_0(x)$$
$$P_2(x) = \tfrac{1}{2}(3x^2-1)P_0(x)$$
$$P_3(x) = \tfrac{1}{2}(5x^3-3x)P_0(x)$$
$$P_4(x) = \tfrac{1}{8}(35x^4-30x^2 \cancel{-2x} + 3)P_0$$

$x^3 - x$

$$\int P_3(x)dx = \tfrac{1}{2}\left\{ 5x^4 - \tfrac{3x^2}{2} \right\}$$
$$\int P_4(x)dx = \tfrac{1}{8}\left\{ 7x^5 - 10x^3 + 3x \right\}$$

Differentiate $f(x,t)$ with respect to x.

$$t(1-2xt+t^2)^{-3/2} = \sum_0^\infty t^n P_n'(x)$$

$$\text{but } \frac{t}{(1-2xt+t^2)^{3/2}} = \frac{t}{x-t}\sum n t^{n-1}P_n(x) = \sum_{n=0}^\infty t^n P_n'(x)$$

can equate coefficients since P_n are orthogonal.

$$\sum_0^\infty n t^n P_n(x) = \sum_0^\infty x t^n P_n'(x) - \sum_0^\infty t^n P_{n-1}'(x)$$

$$x P_n'(x) - P_{n-1}'(x) = n P_n(x)$$

differentiate other recursion relation,

$$(n+1)P_{n+1}'(x) - (2n+1)xP_n'(x) - (2n+1)P_n(x) + nP_{n-1}'(x) = 0$$

$\therefore$

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x) \qquad x \geq 1$$

$$\therefore \quad (2n+1)\int P_n(x)dx = P_{n+1}(x) - P_{n-1}(x) \quad + \text{const} \quad n \geq 1$$

$$\int_{-1}^{1} P_n(x) P_m(x) = \begin{cases} 0 & \text{for } m \neq n \\ 2/(2n+1) & \text{for } m = n \end{cases}$$

See p 177 for $Q_n(x)$

$$\int_{-1}^{1} x^m P_n(x)dx = \begin{cases} 0 & m \leq n-1 \\ 2^{n+1}(n!)^2/(2n+1)! & m = n \end{cases}$$

11-22-67

pp. 178-9 in book.

Exercise I. $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$

$$= b_n P_n(x) + b_{n-1} P_{n-1}(x) + \dots + b_0 P_0(x)$$

(11.5) $P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r! (n-2r)!(n-r)!} x^{n-2r}$ $N = \frac{n}{2}$ for n even, $N = \frac{n-1}{2}$ for n odd.

$$= A_r x^{n-2r}$$

term of x^n in $P_n(x)$ for $r=0$

$$(2n)!/(2^n (n!)^2)$$

$f(x) - a_0 \{ 2^n (n!)^2/(2n)! \} P_n(x)$ is a polynomial of degree $n-1$

$$f(x) - a_0 \left[\frac{P_n(x) - A_1 x^{n-1} - A_2 x^{n-2} \dots - A_n}{A_0} \right] = \phi(x)$$

$f(x) = b_n P_n(x) + \phi_1(x)$ $b_n = a_0 \{ 2^n (n!)^2/(2n)! \} = a_0/1$

So $f(x)$ is in declining "powers" of $P_n(x)$

II. Show $\int_{-1}^{1} x^m P_n(x) dx = \begin{cases} 0 & m \le n-1 \\ 2^{n+1}(n!)^2/(2n+1)! & m=n \end{cases}$

$$x^m = b_m P_m(x) + b_{m-1} P_{m-1}(x) + \dots + b_0 P_0(x)$$

$$b_m = 2^m (m!)^2/(2m)!$$

$$\int_{-1}^{1} x^m P_n(x) dx = \int_{-1}^{1} \left[\frac{2^m (m!)^2}{(2m)!} P_m(x) + \dots + P_0(x) \right] P_n(x) dx$$

$$\therefore \int_{-1}^{1} x^m P_n(x) dx = \frac{2^{m+1}(m!)^2}{(2m+1)!}$$ 0 for $m \le n-1$, for $m=n$

Nov 27, 1967

Associated Legendre:

no axial symmetry in spherical coordinates.

$$(1-x^2)v'' - 2xv' + \{n(n+1) - m^2/(1-x^2)\}v = 0$$

$$\begin{cases} P_n^m(x) \\ Q_n^m(x) \end{cases}$$

$$P_n^m(x) = (1-x^2)^{m/2} \frac{d^{(m+n)}}{dx^{(m+n)}} \{ (x^2-1)^n/(2^n n!) \}$$

$$= (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

$$Q_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} Q_n(x)$$

orthogonality.

$$\int_{-1}^{1} P_{n_1}^m(x) P_{n_2}^m(x)\, dx = 0 \quad \text{for } n_1 \neq n_2$$

$$\int_{-1}^{1} (P_n^m(x))^2 dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$$

$Q_m^n(x)$ has infinite singularities at $x = \pm 1$

for $m > n$ $\quad P_n^m(x) = 0$

$$P_n^0(x) = P_n(x)$$

Jahnke + Emde Table of Functions

11-29-67

$$P_n^m(x) = \frac{(1-x^2)^{m/2}}{2^n n!} \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^n$$

$n-m$ zeros of P_n^m of $\underline{x}$

$$\left.\begin{array}{l} r^n \cos m\psi \; P_n^m(\cos\theta) \\ r^n \sin m\psi \; P_n^m(\cos\theta) \end{array}\right\}$$ Solid Spherical harmonics. (Solution of $\nabla^2\varphi$ in spherical coordinates

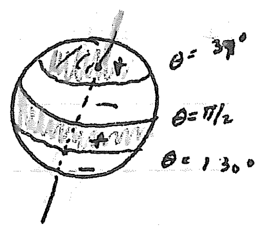
e = even

o = odd.

$$\left.\begin{array}{ll} Y_{mn}^e & \cos m\psi \; P_n^m(\cos\theta) \\ Y_{mn}^o & \sin m\psi \; P_n^m(\cos\theta) \end{array}\right\}$$ Surface spherical harmonics.

$$Y_n^m(\theta,\psi) = (a_{mn}\cos m\psi + b_{mn}\sin m\psi)\, P_n^m(\cos\theta) \qquad \begin{array}{l} 0 \le m \le n \\ 0 \le n \end{array}$$

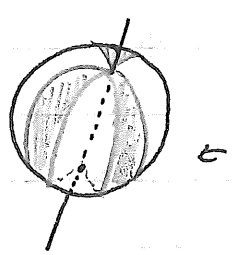
Zonal Harmonics: For $m = 0$



for $n=3$ nodal lines shown...

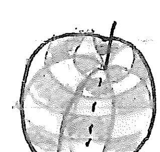
For $m = n$, sectorial harmonics...

$P_n^n(\cos\theta)$ zero only at end points



$\cos 3\psi\, P_3^3(\cos\theta)$

For $m < n$, Tesseral Harmonics (Tessera = square) $m-n$ zeros. of P_{mn}



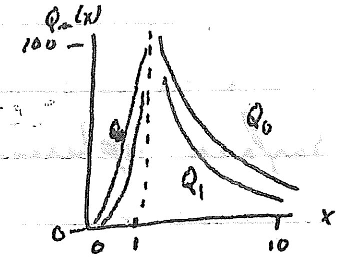
$\cos 3\psi\, P_8^3(\cos\theta)$

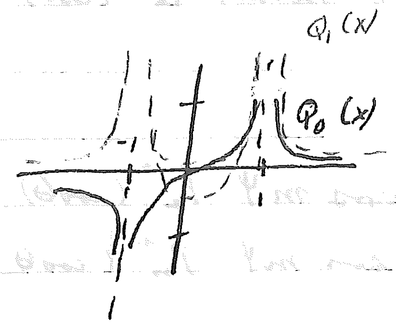
$\lim_{x\to\infty} P_n(x) = \infty$ except for $P_0(x) = 1$ for all x.

$P_n(\cosh x)$ has arguments > 1 so for coordinate systems other than spherical, might have arguments > 1.

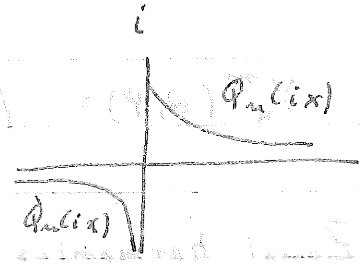
$Q_n(1) = \infty$

$\lim_{x\to\infty} Q_n(x) = 0$





For complex argument,

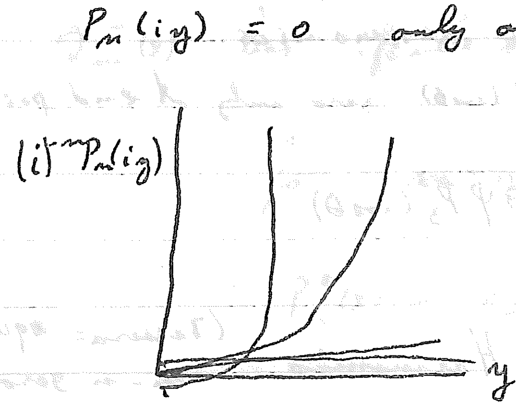


$P_n(z) = P_n(x+iy)$

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n$$

$\lim_{z\to\infty} P_n(z) = \infty$

$P_n(iy) = 0$ only at $y=0$ and only for odd $\underline{n}$



so,

12-1-67

Poisson's Equation:

$$\nabla^2 \phi = -Q(u_1, u_2, u_3)$$

does not apply at a boundary point.

Write $\phi = \Phi + f(u_1, u_2, u_3)$

where $\nabla^2 \Phi = 0$

$$\nabla^2 \phi = \nabla^2 \Phi + \nabla^2 f = \nabla^2 f = -Q$$

Special Case:

(1) Suppose $\nabla^2 \phi$ contains only the term $\frac{\partial^2 \phi}{\partial u_1^2}$ and no other terms in u_1. (e.g. no terms like $\frac{\partial \phi}{\partial u_1}$ or $\frac{1}{u_1}\frac{\partial \phi}{\partial u_2}$

Then

$$\phi = \Phi - Q\frac{u_1^2}{2}$$

(2) Rectangular coordinates.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -Q$$

$$\phi = \Phi - Q\frac{x^2}{2}$$ where $\Phi = {\sin \atop \cos} px \Big\} {\sin \atop \cos} qy \Big\} {\sinh \atop \cosh} \sqrt{p^2+q^2}\, z$

(3) Polar Cylindrical coordinates:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = -Q$$

one solution is $f = -\frac{Qz^2}{2}$

$\phi = \phi(r,\theta)$ only.

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -Q$$

$$r^2\frac{\partial^2\phi}{\partial r^2} + r\frac{\partial\phi}{\partial r} + Qr^2 + \frac{\partial^2\phi}{\partial\theta^2} = 0$$

Assume $\phi = \phi(r)$:

$$r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + Qr^2 = 0$$

$$\phi = \Phi - Q\frac{r^2}{4}$$

Assume $\phi = \phi(r,z)$ only.

One solution is $\phi = \Phi - \frac{Qz^2}{2}$

Another solution is $\phi = \Phi - \frac{Q}{4}r^2$

$$\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + Q + \frac{\partial^2\phi}{\partial z^2}\right) = 0$$

Polar spherical (r, θ, ψ)

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{1}{r^2\sin\theta}\frac{\partial^2\phi}{\partial\psi^2} + Q = 0$$

Multiply by r^2,

$$\boxed{r^2\frac{\partial^2\phi}{\partial r^2} + 2r\frac{\partial\phi}{\partial r} + Qr^2} + \frac{\partial^2\phi}{\partial\theta^2} + \cot\theta\frac{\partial\phi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2} = 0$$

$$\phi = \Phi - \frac{Qr^2}{6}$$

If $\phi = \phi(r,\theta)$ only,

$$\phi = \begin{Bmatrix} r^n \\ r^{-(n+1)} \end{Bmatrix}\begin{Bmatrix} P_n(\cos\theta) \\ \ \end{Bmatrix} - Q\frac{r^2}{6}$$

If $\phi = \phi(r)$ only,

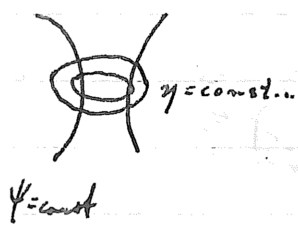
$$\phi = \left.\begin{matrix} r^{n} \\ r^{-(n+1)} \end{matrix}\right\} - Q\frac{r^2}{6}$$

If $\phi = \phi(\theta, \psi)$ only or $\phi = \phi(\theta)$ only or $\phi = \phi(\psi)$ only...
cannot find a solution... if Q = constant...

MOON & SPENCER
ONE EXAMPLE
WORKED
INCORRECTLY...

Elliptic Cylindrical:

$$\nabla^2 \phi = \frac{1}{a^2(\cosh^2\eta - \cos^2\psi)}\left\{\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\psi^2}\right\} + \frac{\partial^2\phi}{\partial z^2} = -Q \text{ (constant)}$$



If $\phi = \phi(\eta, \psi, z)$ or if $\phi = \phi(\eta, z)$ or $\phi = \phi(\psi, z)$

$$\phi = \Phi - \frac{Qz^2}{2}$$

If $\phi = \phi(\eta, \psi)$ only,

$$\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\psi^2} = -Qa^2(\cosh^2\eta - \cos^2\psi) = -\frac{Qa^2}{2}(\cosh 2\eta - \cos 2$$

$$\frac{\partial^2\phi}{\partial\eta^2} + \frac{Qa^2}{2}\cosh 2\eta = -\left[\frac{\partial^2\phi}{\partial\psi^2} - \frac{Qa^2}{2}\cos 2\psi\right]$$

Let $\phi = \Phi + f_1(\eta) + f_2(\psi)$

$$\left[\frac{\partial^2 f_1(\eta)}{\partial\eta^2} + \frac{Qa^2}{2}\cosh 2\eta\right] = -\left[\frac{\partial^2 f_2(\psi)}{\partial\psi^2} - \frac{Qa^2}{2}\cos 2\psi\right] \equiv 0$$

$$f_1(\eta) = -\frac{Qa^2}{8}\cosh 2\eta \qquad f_2(\psi) = -\frac{Qa^2}{8}\cos 2\psi$$

$$\phi = \Phi - f_1(\eta) + f_2(\psi)$$

$$\phi = \left.\begin{matrix} e^{p\eta} \\ e^{-p\eta} \end{matrix}\right\}\left.\begin{matrix} \sin p\psi \\ \cos p\psi \end{matrix}\right\} - \frac{Qa^2}{8}(\cosh 2\eta + \cos 2\psi)$$

12-4-67

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = -j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

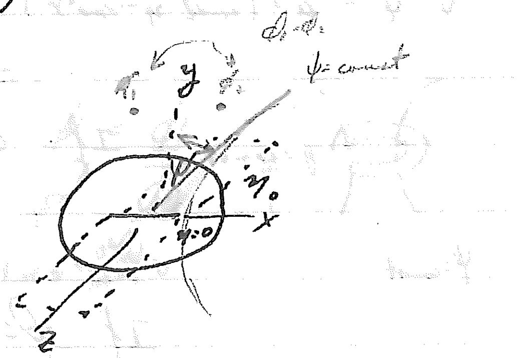
$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Solution of Poisson's Equation in Elliptical Cylindrical (Moon + Spencer)

$$\frac{1}{a^2(\cosh^2\eta - \cos^2\psi)}\left(\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\psi^2}\right) = -Q \quad \text{where } Q \text{ is constant}$$

$$\phi(\eta_0, \psi) = \phi_0$$

$$\phi(0, \psi) = 0$$



$$\phi = \left.\begin{matrix} e^{p\eta} \\ e^{-p\eta}\end{matrix}\right\} \left.\begin{matrix}\sin \\ \cos\end{matrix} \; p\psi\right\} - \frac{Qa^2}{8}(\cosh 2\eta + \cos 2\psi)$$

ϕ is even function of ψ

$$\phi = (A\cosh p\eta + B\sinh p\eta)\cos p\psi - \frac{Qa^2}{8}(\cosh 2\eta + \cos 2\psi) + C$$

$\phi(0, \psi)$,

$$A\cos p\psi - \frac{Qa^2}{8}(1 + \cos 2\psi) + C = 0$$

$$C = \frac{Qa^2}{8} \quad p = 2 \quad A = \frac{Qa^2}{8}$$

$$\phi = \left(\frac{Qa^2}{8}\cosh 2\eta + B\sinh 2\eta\right)\cos 2\psi - \frac{Qa^2}{8}(\cosh 2\eta + \cos 2\psi) + \frac{\Phi a}{8}$$

$$\phi(\eta_0,\psi) = \phi_0 = \frac{Qa^2}{8}\left[1 - \cosh 2\eta_0\right] + \frac{Qa^2}{8}\left[\cosh 2\eta_0 + \frac{\sum_{p=1}^{\infty} 8 B_p}{\Phi a^2}\sinh p\eta_0 - 1\right]\cos 2\psi$$

$$\int_0^{2\pi} \phi_0 \, d\psi = \int_0^{2\pi} \frac{Qa^2}{8}\left[1 - \cosh 2\eta_0\right] d\psi.$$

$$\int_0^{2\pi} \phi_0 \cos 2\psi \, d\psi = \frac{Qa^2}{8}\pi\left[\cosh 2\eta_0 + \frac{8B}{\Phi a^2}\sinh 2\eta_0 - 1\right]$$

if $\phi_0 =$ constant, $\qquad B_1 = 0 \quad B_m = 0$ for $m \geq 3$.

$$\frac{Qa_0^2}{8}\left\{\frac{1 - \cosh 2\eta_0}{\sinh 2\eta_0}\right\} = B_2$$

Assume. $\phi = \sum_{p=1}^{\infty}(A_p \cosh p\eta + B_p \sinh p\eta)\cos p\psi - \frac{Qa^2}{8}(\cosh 2\eta + \cos 2\psi)$

B.C. (2) $\quad C = A_2 = \frac{Qa^2}{8} \quad A_1 = A_3 = A_n = 0$ for $n \geq 4$

B.C (1)

$$\phi_0 = \sum_{p=1}^{\infty}(A_p \cosh p\eta_0 + B_p \sinh p\eta_0)\cos p\psi - \frac{Qa^2}{8}\cosh 2\eta_0 + C - \frac{Qa^2}{8}\cos 2\psi$$

$$Q = \frac{8\phi_0}{a^2(1-\cosh 2\eta_0)}$$

$$\underline{\underline{B_2 = \frac{Qa^2}{8}\,\frac{1-\cosh 2\eta_0}{\sinh 2\eta_0}}} \qquad B_1 = B_3 = B_4 = \cdots = 0.$$

$$\therefore \quad \phi(\eta, \psi) = \frac{8\phi_0}{(1-\cosh 2\eta_0)} \left[\left\{ \cosh 2\eta + \frac{(1-\cosh 2\eta_0)}{\sinh 2\eta_0} \sinh 2\eta \right\} \cos 2\psi \right.$$

$$\left. + \; 1 - (\cosh 2\eta + \cos 2\psi) \right]$$

Complex potential function,

$$\Omega = \Omega(z)$$

$$\Omega = \phi + i\psi \qquad \text{where } \phi \text{ and } \psi \text{ are real functions of } (x, y)$$

Results apply only to ~~two~~ dimensional problems — cylindrical problems.

Cauchy - Rieman conditions since $\nabla^2 \Omega = 0$ and thus Ω is analytic,

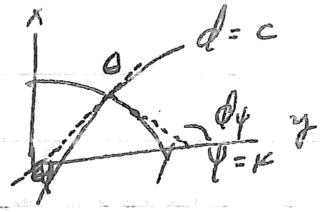
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \; ; \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

ϕ and ψ are harmonic functions.

$$\nabla^2 \phi = 0, \quad \nabla^2 \psi = 0.$$

ϕ + ψ are also conjugate harmonic functions.
The curve of ϕ intersects curve of ψ at right angles.



Prove that $\theta = \pi/2$.

$$\tan\theta = \tan(\theta_\psi - \theta_\phi) = \frac{\tan\theta_\psi - \tan\theta_\phi}{1 + \tan\theta_\psi \tan\theta_\phi}$$

$$\tan\theta_\phi = \left(\frac{dy}{dx}\right)_{\phi=c} \qquad \tan\theta_\psi = \left(\frac{dy}{dx}\right)_{\psi=K}.$$

$$\tan\theta = \frac{\left(\frac{dy}{dx}\right)_{\psi=K} - \left(\frac{dy}{dx}\right)_{\phi=c}}{1 + \left(\frac{dy}{dx}\right)_{\psi=K}\left(\frac{dy}{dx}\right)_{\phi=c}}$$

On $\phi = \text{const}$ $\quad d\phi = 0 = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$

$$\left(\frac{dy}{dx}\right)_\phi = -\partial\phi/\partial x / \partial\phi/\partial y \Big|_{\phi=c}$$

$$\psi = K \quad dK = 0\ ; \quad \left(\frac{dy}{dx}\right)_{\psi=K} = -\left(\frac{\partial\psi/\partial x}{\partial\psi/\partial y}\right)_{\psi=K}.$$

$$\tan\theta = \frac{\left(\frac{-\frac{\partial\psi}{\partial x}}{\frac{\partial\psi}{\partial y}}\right)_{\psi=K} - \left(\frac{-\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}}\right)_{\phi=c}}{1 + (\quad)(\quad)}$$

$$\frac{\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial x}}{\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial y}} = \frac{\frac{\partial\psi}{\partial x}}{\frac{\partial\psi}{\partial y}}\,\frac{\frac{\partial\psi}{\partial y}}{-\frac{\partial\psi}{\partial x}} = -1$$

$\tan\theta = \infty \quad , \theta = \pi/2 \qquad$ Q.E.D.

12-6-67 $\Omega = \phi + i\psi$

Usually want uniform field at large values of z. or $\frac{d\Omega}{dz}$ = finite...

$$\Omega = Az + B \ln z + \sum_{m=1}^{\infty} \left({a_m}/{z^m} \right)$$

$$v_x = \frac{-\partial\phi}{\partial x}, \quad v_y = \frac{-\partial\phi}{\partial y} \qquad \frac{d\Omega}{dz} = \frac{\partial\phi}{\partial x} + j\frac{\partial\psi}{\partial x} = v_x - j v_y = \text{const}$$

5.1 Uniform stream. inclined at angle α to x axis.

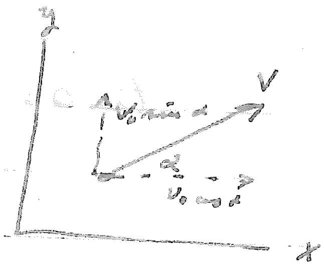
$$\Omega = V_0 e^{-i\alpha} z \qquad V_0, \alpha \text{ are real constants.}$$

$$= V_0(\cos\alpha - i \sin\alpha)(x + iy)$$

$$= \overbrace{V_0(x\cos\alpha + y\sin\alpha)}^{\phi} + \overbrace{i(y\cos\alpha - x\sin\alpha)}^{i\psi} \ldots$$

$$V_x = \frac{\partial\phi}{\partial x} = V_0 \cos\alpha$$

$$V_y = \frac{\partial\phi}{\partial y} = V_0 \sin\alpha$$



$-\pi \leq \alpha < \pi$ to keep everything single valued...

5.2. Source, Sink, Vortex.

ψ = stream lines – no flow across

$$\Omega = m \ln z \qquad \phi = m \ln|z| = m \ln r \qquad \psi = m\,\theta$$

$$\text{Flow} = \psi_c = \int_c d\psi = \int_0^{2\pi} m\, d\theta = 2\pi m.$$

if m is positive; fluid emanates from origin.
point source.
if m is negative, fluid is being destroyed at origin at rate $2\pi m$ and is point sink. m is strength.

Circulation $= \int_C d\phi = \int_C m \, d(\ln r) = 0$

IF. $\Omega = -jk \ln z$ $\qquad \phi = k\theta \qquad \psi = -k \ln r.$

$$\psi_C = -\int k \, d(\ln r) = 0$$

$$\int_C d\phi = 2\pi k \quad = \text{circulation.}$$

Point vortex of strength k at origin.

$$\phi = k \tan^{-1} y/x$$

$$V_x = \frac{-k \sin\theta}{r} \qquad V_y = k \frac{\cos\theta}{r}$$

radial velocity.

$$V_r = V_x \cos\theta + V_y \sin\theta = 0$$

tangential velocity $= \frac{1}{r} \frac{\partial\phi}{\partial\theta} = \frac{k}{r}$

IF source or sink is located at $z = a$,

$$\Omega = m \ln(z-a)$$

n, Vortex $\qquad \Omega = -jk \ln(z-a)$

8-12-67 Conformal Transformation:

$w = f(z)$ assume $f(z)$ is an analytic function.

$$f'(z) = \frac{dw}{dz} = a e^{i\alpha}$$

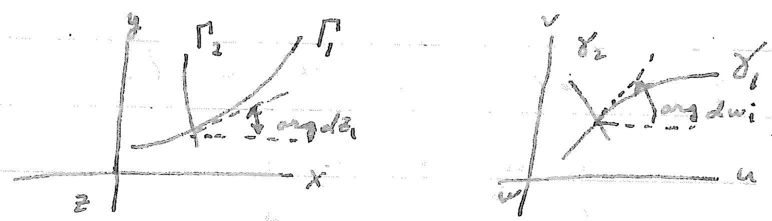
$a = |f'(z)| \neq 0$

$\alpha = \arg |f'(z)|$

when $f'(z) = 0$ change in angle upon transformation... and mapping not conformal.

(1) $|dw| = a\,|dz|$

(2) $\arg(dw) = \alpha + \arg(dz)$



(1) if take infinitesimal ~~argument~~ element on γ_1, it is "a" ~~the~~ times the corresponding infinitesimal element on Γ_1.

(2) curve γ_1 is also rotated through angle α.
If sense of angles are preserved, then mapping conformal.

$w = f(z)$ has a single valued inverse iff

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$ — does not vanish or become infinite

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2$$

Dictionary of Conformal Representations - Dover, Kober \$2.00

if $w = u + jv = f(z) = f(x+iy)$ is a conformal transformation, and ~~then~~ $\phi(x,y)$ for which $\nabla^2\phi = 0$, then

$\phi(u,v)$ also $\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2} = 0.$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = |f'(z)|^2 \left\{ \frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2} \right\}$$

12-12-67

Bilinear Transformation.

$$w = \frac{az+b}{cz+d}$$

a, b, c, d are complex constants... and $c \neq 0$.

if $c = 0$, just a linear transformation

$\equiv \quad Awz + Bz + Cw = D$ linear in both w and z

$$w = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}$$

two fixed points are roots z_1, z_2 of $\quad$ points remain unchanged under transformation...

$$c z^2 + z(d-a) - b = 0$$

$$z = \frac{-(d-a) \pm \sqrt{(d-a)^2 + 4bc}}{2c}$$

since $z_1 = \dfrac{az_1+b}{cz_1+d}$, $\quad z_2 = \dfrac{az_2+b}{cz_2+d}$

$$\frac{w - z_1}{w - z_2} = K \, \frac{z - z_1}{z - z_2}, \text{ where } K = \frac{cz_2+d}{cz_1+d}$$

for z_1, z_2 any points,

$$\frac{w - w_1}{w - w_2} = \frac{\dfrac{az+b}{cz+d} - \dfrac{az_1+b}{cz_1+d}}{\dfrac{az+b}{cz+d} - \dfrac{az_2+b}{cz_2+d}} = \left(\frac{cz_2+d}{cz_1+d}\right)\left(\frac{z - z_1}{z - z_2}\right)$$

Equation of a circle in z plane is:

$$\left| \frac{z - z_1}{z - z_2} \right| = \lambda$$

by transformation,

$$\left| \frac{w - w_1}{w - w_2} \right| = \lambda \, |K|$$

if $\lambda |K| = 1$, circle transforms into line

$\lambda |K| \neq 1$, circle transforms into another circle.

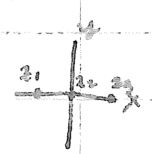
given 3 points in each plane, transformation can be made up. to accomplish this.

$z_1 = 1$, $z_2 = 0$, $z_3 = -1$ $\quad w_1 = i$, $w_2 = 1$, $w_3 = \infty$.

one constant is arbitrary - let $c = 1$

$$i = \frac{\cancel{z}a + b}{\cancel{z}c + d} \qquad \cancel{0 = a+b} \qquad 1 = \frac{b}{d} \qquad \infty = \frac{-a+b}{-c+d}.$$

$$i = \frac{a+b}{1+d} \qquad 1 = \frac{b}{d} \qquad \infty = \frac{-a+b}{-1+d}$$

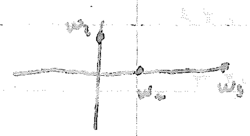


$\therefore \; d = +1$

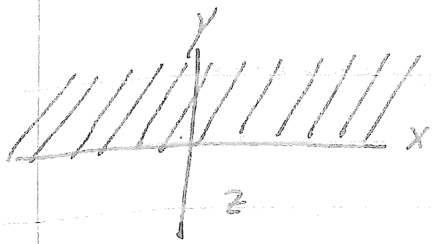
$b = d = +1$

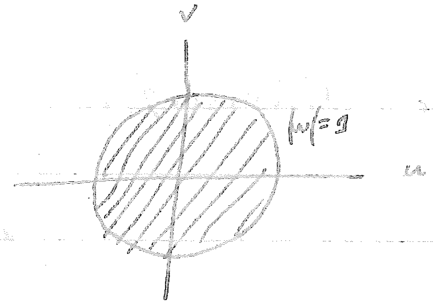
$a + 1 = 2i$

$a = 2i - 1$



$$w = \frac{(2i-1)z + 1}{z + 1}$$





$$w = \frac{az+b}{cz+d}$$

Boundary is $y=0$, so transform boundaries.

$$|w| = 1 = \left|\frac{ax+b}{cx+d}\right| = \left|\frac{a}{c}\right| \left|\frac{x+b/a}{x+d/c}\right|$$

with $x \Rightarrow \infty$ $\left|\frac{a}{c}\right| = 1$ or $\frac{a}{c} = e^{i\sigma}$

also $|x+b/a| = |x+d/c|$

at $x=0$ $|b/a| = |d/c|$

if $\frac{b}{a} = \rho e^{i\beta}$ $\frac{d}{c} = \rho e^{i(\beta+\gamma)}$

$$(x+\rho\cos\beta)^2 + \rho^2\sin^2\beta = \{x+\rho\cos(\gamma+\beta)\}^2 + \rho^2\sin^2(\beta+\gamma)$$

if $x \neq 0$, $\rho \neq 0$.

$$\cos\beta = \cos(\beta+\gamma)$$

$\beta = \beta+\gamma$, $\gamma = 0$.

$\beta+\gamma = 2\pi - \beta \Rightarrow \overline{\left(\frac{b}{a}\right)} = \left(\frac{d}{c}\right)$

$$w = e^{i\sigma}\frac{z-\alpha}{z-\bar{\alpha}} \quad \text{where } \alpha = -b/a$$

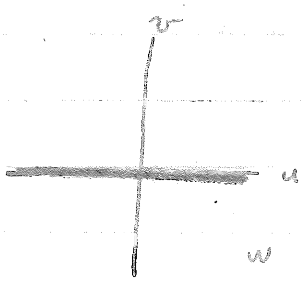
if $\gamma=0$, trivial solution $w=a/c$ if $\rho=0$ $|b/a| = |d/c| = 0$, $w=a/c$

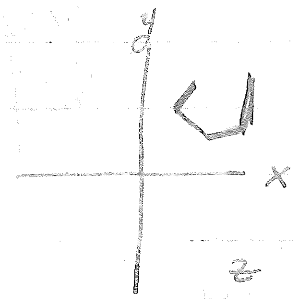
if $z=\alpha$ where α is any point in the upper plane,

$w=0$, inside circle...

$\alpha = \alpha_1 + i\alpha_2$ where $\alpha_2 > 0$

Schwarz - Christoffel Transformation:



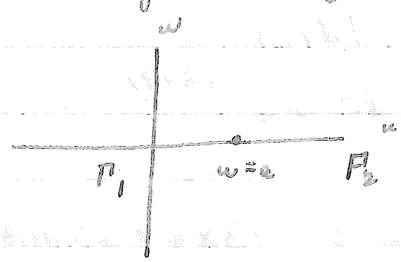


from real axis to polygon.

Let $\frac{dz}{dw} = K(w-a)^t$ $\quad K = \text{complex}, \; a, t \text{ real}$

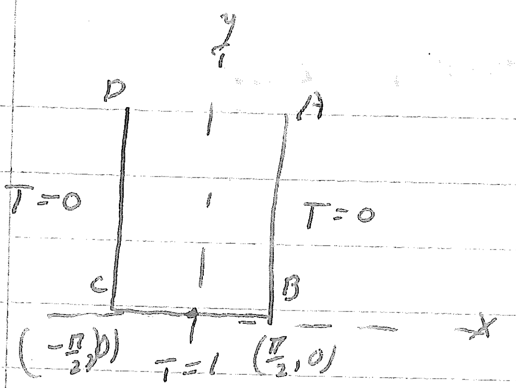
$$\left|\frac{dz}{dw}\right| = |K|\,|w-a|^t$$

$$\arg \frac{dz}{dw} = \arg K + t \arg(w-a)$$



on Γ_1: $u-a<0 \Rightarrow \arg(w-a)=\pi$

Γ_2 $u-a>0 \Rightarrow \arg(w-a)=0$



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$-\frac{\pi}{2} < x < \pi/2$$

$$y > 0$$

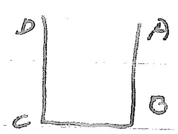
angle of $C = \frac{\pi}{2}$, at $B = \frac{\pi}{2}$

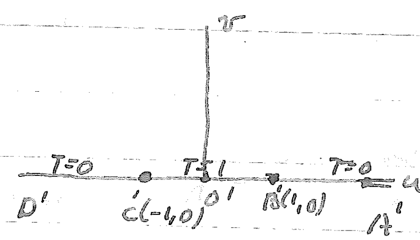
$T(-\frac{\pi}{2}, y) = T(\frac{\pi}{2}, y) = 0$ for $y > 0$

$T(x, 0) = 1$ for $-\frac{\pi}{2} < x < \pi/2$

$|T(x,y)| < M$ where M is some positive number.

Use $w = \sin z$



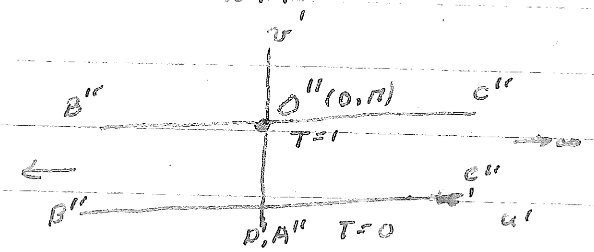


$$\frac{\partial^2 T}{\partial v^2} + \frac{\partial^2 T}{\partial u^2} = 0$$

$u = \sin x \cosh y$

$v = \cos x \sinh y$

$$w' = \ln \frac{w-1}{w+1} \qquad w' = \ln\left|\frac{w-1}{w+1}\right| + i \arg\left(\frac{w-1}{w+1}\right)$$



$$\therefore \phi = \frac{v'}{\pi}$$

transformation -

$$v' = \arg\left(\frac{u-1+iv}{u+1+iv}\right) = \tan^{-1}\left(\frac{2v}{u^2+v^2-1}\right)$$

MKP

O': $u' = \ln\left|\frac{0-1}{0+1}\right| = 0$; $v' = \tan^{-1}\left(\frac{0}{-1}\right) = \pi$

A': $u' = \lim_{a'\to\infty} \ln\left(\frac{a'-1}{a'+1}\right) = 0$, $v' = \tan^{-1}\left(\frac{0}{\infty}\right) = 0$

D': $u' = \lim_{d'\to\infty} \ln\left(\frac{d'-1}{d'+1}\right) = 0$, $v' = \tan^{-1}\left(\frac{0}{\infty}\right) = 0$

B': $u' = \ln\left|\frac{1-1}{1+1}\right| = -\infty$; $v' = \tan^{-1}\left(\frac{0}{1-1}\right)$ indeterminant.

let $u_+ = 1+\epsilon$, $u_- = 1-\epsilon$,
$v_+' = \tan^{-1}\left(\frac{0}{(1+\epsilon)^2-1}\right) = 0$ $\quad v_-' = \tan^{-1}\left(\frac{0}{(1-\epsilon)^2-1}\right) = \pi$

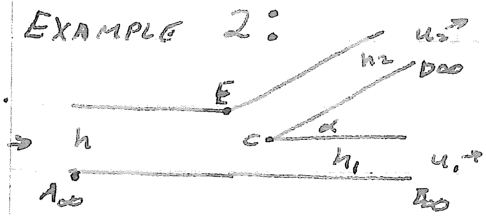
$$\phi = \frac{v'}{\pi} = \frac{1}{\pi}\tan^{-1}\left(\frac{w-1}{w+1}\right) = \frac{1}{\pi}\tan^{-1}\left(\frac{\sin z-1}{\sin z+1}\right)$$

$$= \frac{1}{\pi}\tan^{-1}\left(\frac{2v}{u^2+v^2-1}\right) = \frac{1}{\pi}\tan^{-1}\left(\frac{2\cos x\sinh y}{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y-1}\right)$$

$$\therefore \phi = \frac{2}{\pi}\tan^{-1}\left(\frac{\cos x}{\sinh y}\right)$$

EXAMPLE 2:

p468-470.

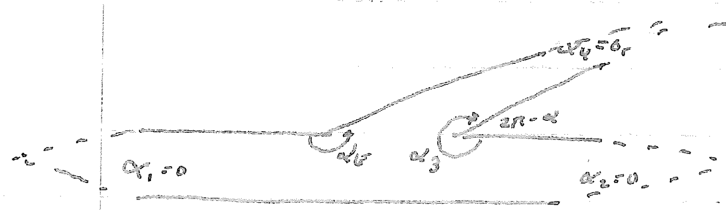


at $-\infty$. $v = u$
at $+\infty$ $v_1 = u_1$, $v_2 = u_2$

Source at A_∞, sinks at B_∞, D_∞.

Rate of creation $= Uh$, destroyed U_1h_1 , U_2h_2

polygon in limit.



don't know α_E, so put E at infinity and point drops from transformation.

In transformation, source transformed into source of same strength $b,$

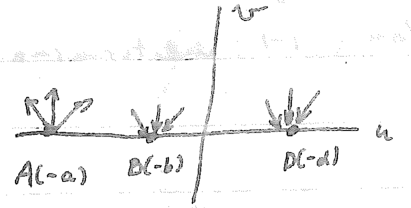
z plane	w plane
E	∞.
C	origin
A_∞	$w=-a$
B_∞	$w=-b$
D_∞	$w=+d$

$\left.\begin{matrix} w=-b \\ w=+d \end{matrix}\right\}$ b, d unknown.

$$\alpha_g = 2\pi-\alpha$$

$$\text{exponent} = \frac{2\pi-\alpha}{\pi}-1 = 1-\frac{\alpha}{\pi}$$

$$\frac{dz}{dw} = K(w+a)^{-1}(w+b)^{-1}(w-0)^{1-\frac{\alpha}{\pi}}(w-d)^{-1}$$



$$\Omega = \frac{Uh}{\pi}\ln(w+a) - \frac{U_1h_1}{\pi}\ln(w+b) - \frac{U_2h_2}{\pi}\ln(w-d)$$

V = complex fluid velocity = $V_u \mp i V_v$

$$\Omega = \phi + i\psi$$

$$\frac{d\Omega}{dw} = \frac{\partial\phi}{\partial u} + \frac{i\partial\psi}{\partial u}$$

$$\frac{\partial\phi}{\partial u} = V_u \quad , \quad \frac{\partial\phi}{\partial v} = -iV_v$$

by Cauchy Riemann conditions.

$$\frac{d\Omega}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = V_x - iV_y$$

$$\frac{d\Omega}{dz} = \frac{d\Omega}{dw}\cdot\frac{dw}{dz}$$

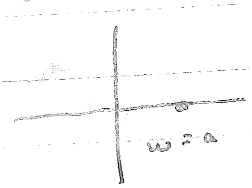
$$\frac{d\Omega}{dw} = \frac{Uh}{\pi(w+a)} - \frac{U_1h_1}{\pi(w+b)} - \frac{U_2h_2}{\pi(w-d)}$$

$\frac{dz}{dw}$ is found from transformation.

b, d, K, u_1, ju_2 are to be found.
make use of known ... $\Delta = |1 - \frac{\alpha}{\pi}|$

12-13-67 Schwarz - Christoffel Transformation

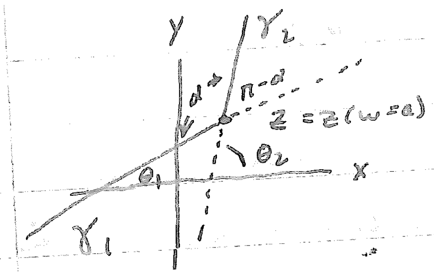
$$\frac{dz}{dw} = K(w-a)^t$$



$$\left(\arg \frac{dz}{dw}\right)_{\Gamma_1} = \arg K + t\pi = \theta_1$$

$$\left(\arg \frac{dz}{dw}\right)_{\Gamma_2} = \arg K + 0 = \theta_2$$

$$\theta_2 - \theta_1 = -\pi t$$



On γ_1 arg of $dz = \tan^{-1}\left[\frac{Im(dz)}{Re(dz)}\right]$

$$= \tan^{-1}\frac{dy}{dx}$$

$$\arg\frac{dz}{dw} = \arg\frac{dx+idy}{dw} = \tan^{-1}\frac{dy}{dx}$$

since dw = real ...

$$\therefore \left(\frac{dy}{dx}\right)_{\gamma_1} = \tan^{-1}\theta_1 \qquad \left(\frac{dy}{dx}\right)_{\gamma_2} = \tan^{-1}\theta_2$$

$$\theta_2 - \theta_1 = \pi - \alpha = -\pi t$$

$$t = \left(\frac{\alpha}{\pi} - 1\right)$$

$$\frac{dz}{dw} = K(w-a)^{\left(\frac{\alpha}{\pi}-1\right)}$$

if (P) w moves along real axis $v = 0$ from $-\infty$ to ∞, the (Q) z moves along the two arms of an angle whose magnitude is α.

$$z = \left(\frac{K\pi}{\alpha}\right)(w-a)^{\alpha/\pi} + c \qquad \alpha \neq 0$$

$$= K\ln(w-a) + c \qquad \alpha = 0.$$

b, d, k, U_1, U_2 are to be found.
make use of known velocities $\beta = \left(1 - \frac{\alpha}{\pi}\right)$.
at A_∞, C, E

at $w = -a = ae^{i\pi}$ corresponding to A_∞ $V_x = U$, $V_y = 0$

and.
$$U = \frac{Uh}{\pi k} \frac{(a-b)(a+d)}{a^{\beta} e^{i\pi\beta}}$$

at $w = -b = be^{i\pi}$ corresponding to B_∞ $V_x = U_1$, $V_y = 0$.

$$U_1 = \frac{U_1 h_1}{\pi K} \frac{(a-b)(b+d)}{b^{\beta} e^{i\pi\beta}}$$

at $w = d = de^{i0}$ corresponding to D_∞, $V_x = U_2 \cos\alpha$, $V_y = U_2 \sin\alpha$

$$U_{2 \cdot 0} = \frac{U_2 h_2}{\pi K} \frac{(a+d)(b+d)}{a^{\beta} e^{i\pi\beta}}$$

hence,
$$\pi k e^{i\pi\beta} = \frac{h(a-b)(a+d)}{a^{\beta}} = h_1 \frac{(a-b)(b+d)}{b^{\beta}} = h_2 \frac{(a+d)(b+d)}{d^{\beta}}$$

$z = 0$ corresponds to $w = 0$, $\frac{d\Omega}{dz}$ since α is stagnation point.

coefficient of $w^{-\beta}$ = zero...

$$-Uhbd + U_1 h_1 ad - U_2 h_2 ab = 0$$
$$\frac{Uh}{a} = \frac{U_1 h_1}{b} - \frac{U_2 h_2}{d}.$$

equation of continuity. : $Uh = U_1 h_1 + U_2 h_2$

by manipulation, and $\lambda = h_1/h$ $\mu = h_2/h$, $\nu = \frac{U_1 h_1}{Uh}$

$$\frac{U_2 h_2}{Uh} = 1 - \nu$$

$$1 = \lambda\left(\frac{\nu}{\lambda}\right)^{1-\pi/\alpha} - \mu\left(\frac{1-\nu}{\mu}\right)^{1-\pi/\alpha}$$

which is a transcendental equation for ν and hence U_1.
from U_1, U_2 can be found.

~~Trans~~

1-3-68 Transform Methods --

Laplace Transform

$$g(p) = \int_0^{\infty} e^{-px} f(x)\,dx \qquad p \geq 0$$

Transform of $\delta(x)$ - Dirac delta function.

Ordinary Differential Equation with constant coefficients.

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) x = F(t) \quad \text{for } t \geq 0$$

$D^r = \frac{d^r}{dt^r}$

given $x, \frac{dx}{dx}, \dots, \frac{d^{n-1}(x)}{dt^{n-1}}$ at $t=0$

$$(x)_{t=0} = x_0 \ , \ \left(\frac{dx}{dt}\right)_{t=0} = x_1 \ , \ \left(\frac{d^{n-1}}{dx^{n-1}}\right)_{t=0} = x_{n-1}$$

$$\int_0^{\infty} e^{-pt} [D^n x + a_1 D^{n-1} x + \dots + a_{n-1} Dx + a_n x]\,dt = \int_0^{\infty} e^{-pt} F(t)\,dt$$

Assume $x \to$ finite as $t \to \infty$

$$\int_0^{\infty} (e^{-pt} Dx)\,dt = [x e^{-pt}]_0^{\infty} + p\int_0^{\infty} e^{-pt} x\,dt$$

$$= -x_0 + p\int_0^{\infty} e^{-pt} x\,dt$$

$$\int_0^{\infty} e^{-pt} D^2 x\,dt = [e^{-pt} Dx]_0^{\infty} + p\int_0^{\infty} e^{-pt} Dx\,dt$$

$$-x_1 - p x_0 + p^2 \int_0^{\infty} e^{-pt} x\,dt$$

$$\int_0^{\infty} e^{-pt} D^r x\,dt = -(p^{r-1} x_0 + p^{r-2} x_1 + \dots + p^{r-(r-1)} x_{r-2}$$

$$+ p^{r-r} x_{r-1}) + p^r \int_0^{\infty} e^{-pt} x\,dt$$

$\bar{x} = \int_0^\infty e^{-pt} x\,dt$

$$= -(p^{r-1}x_0 + p^{r-2}x_0 + \dots + p x_{r-2} + x_{r-1}) + p^r \bar{X}(p)$$

$$\int_0^\infty e^{-pt}\left[D^n x + a_1 D^{n-1} x + \dots + a_n x\right]dt = \int_0^\infty e^{-pt} F(t)\,dt$$

$$\begin{aligned}
= \; & -(p^{n-1}x_0 + p^{n-2}x_1 + \dots + p x_{n-2} + x_{n-1}) \\
& + p^n \bar{x}(p) \\
& + a_1 \left\{ -(p^{n-2}x_0 + p^{n-3}x_1 + \dots + p x_{n-3} + x_{n-2}) \right. \\
& \qquad \left. + p^{n-1}\bar{x}(p) \right\} \\
& + \dots + \\
& a_{n-2} \left\{ -(p x_0 + x_1) + p^2 \bar{x}(p) \right\} \\
& + a_{n-1} \left\{ -x_0 + p\bar{x}(p) \right. \\
& + a_n \left. \bar{x}(p) \right\}.
\end{aligned}$$

Define $\phi(p) \equiv \left\{ p^n + a_1 p^{n-1} + \dots + a_{n-1} p^1 + a_n \right\}$

Add terms together,

Subsidiary equation.

$$\begin{aligned}
\bar{x}(p)\,\phi(p) = \; & (p^{n-1}x_0 + p^{n-2}x_1 + \dots + p x_{n-2} + x_{n-1}) \\
& + a_1 (p^{n-2}x_0 + p^{n-3}x_1 + \dots + p x_{n-3} + x_{n-2}) \\
& + \dots \\
& + a_{n-2}(p x_0 + x_1) \\
& + a_{n-1} x_0 \\
& + \int_0^\infty e^{-pt} F(t)\,dt
\end{aligned}$$

Next step, take inverse transform...

$x(t)$ = response of system to excitation function $F(t)$

$$(D+1)x = 1 \qquad \text{for } t \geq 0$$
$$x_0 = 0$$

Subsidiary equation.

$$\bar{x}(p)\,[\,p+1\,] = p^{n-1}x_0 + \ldots + \int_0^\infty e^{-pt}F(t)\,dt$$

$n=1, \quad x_0=0, F(t)=0$

$$(p+1)\bar{x}(p) = \int_0^\infty e^{-pt}\,dt = \frac{1}{p}$$

$$\bar{x}(p) = \frac{1}{p(p+1)} = \frac{A}{p} + \frac{B}{p+1}$$

$$Ap + A + Bp = 1$$
$$A = 1, \quad B = -1$$

$$\bar{x}(p) = \frac{1}{p} - \frac{1}{p+1}$$

Applying ~~inverse~~

$$x(t) = 1 - e^{t}$$

Example 2. $(D^2 - 5D + 4)x = 12 + 9e^t + 5\sin 2t \quad$ for $t \geq 0$.

$x_0 = 1, \quad x_1 = -2, \quad n=2$

$$(p^2 - 5p + 4)\,\bar{x}(p) = p \cdot 1 - p^0 2$$
$$-5(\cdot 1) + \int_0^\infty e^{-pt}(12 + 9e^t + 5\sin 2t)\,dt$$

Solution of Simultaneous System:

$$(D^2 - 3D + 2)x + (D-1)y = 0$$
$$(D-1)x - (D^2 - 5D + 4)y = 0$$

$x = 0,\ y = 1,\ Dx = 0,\ Dy = 0$ at $t = 0$

$$\left.\begin{aligned} (p^2 - 3p + 2)\bar{x} + (p-1)\bar{y} &= p x_0 + x_1 - 3x_0 + y_0 \\ (p-1)\bar{x} - (p^2 - 5p + 4)\bar{y} &= x_0 - p y_0 - y_1 + 5y_0 \end{aligned}\right\} \text{subsidiary Equations}$$

After including B.C.

$$(p^2 - 3p + 2)\bar{x} + (p-1)\bar{y} = 1$$
$$(p-1)\bar{x} - (p^2 - 5p + 4)\bar{y} = -p + 5$$

$$\bar{x} = \frac{1}{(p-1)(p-3)^2} = \frac{1}{4}\left[\frac{1}{p-1} - \frac{1}{p-3} + \frac{2}{(p-3)^2}\right]$$

$$\bar{y} = \frac{p^2 - 7p + 11}{(p-1)(p-3)^2} = \frac{1}{4}\left[\frac{5}{p-1} - \frac{1}{p-3} - \frac{2}{(p-3)^2}\right]$$

$$x = \mathcal{L}^{-1}(\bar{x}) = \frac{1}{4}\left[e^{t} - e^{-3t}(1-2t)\right]$$

$$y = \mathcal{L}^{-1}(\bar{y}) = \frac{1}{4}\left[5e^{t} - e^{-3t}(1+2t)\right]$$

1-8-68 ... Fourier Transform:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

Now on integration by parts.

$$\int_{-\infty}^{\infty} e^{-i\omega t} \frac{d^2y}{dt^2} = \left\{ \left[e^{-i\omega t} \frac{dy}{dt} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-i\omega) e^{-i\omega t} \frac{dy}{dt} \right\}$$

set $\frac{dy}{dt} = 0$ at $t = \pm\infty$...
$y = 0$

$$= \frac{i\omega}{\sqrt{2\pi}} \left\{ \left[e^{-i\omega t} y \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} y (-i\omega) e^{-i\omega t} dt \right.$$

$$= (i\omega)^2 \bar{y}(\omega)$$

Sneddon's "Elements of P.D.E." McGraw-Hill p. 126 ff

Integral Transforms...

(1) $$a(x_1) \frac{\partial^2 u}{\partial x_1} + b(x_1) \frac{\partial u}{\partial x_1} + c(x_1) u + Lu = f(x_1, x_2, \dots, x_n) \quad (1)$$

for $\alpha \le x_1 \le \beta$

where L is linear differential operator in variables $x_2, \dots, x_n$

(2) Define transform of u, $\bar{u}(\xi, x_2, x_3, \dots, x_n) \equiv \int_{\alpha}^{\beta} u(x_1, x_2, \dots, x_n) K(\xi, x_1) dx_1$

$\bar{u}$ integral transform of u corresponding to kernel K

$$\int_{\alpha}^{\beta} \left\{ a(x_1) \frac{\partial^2 u}{\partial x_1} + b(x_1) \frac{\partial u}{\partial x_1} + c(x_1) u \right\} K(\xi, x_1) dx_1$$

$$= \left[a \frac{\partial u}{\partial x_1} K(\xi x_1) + b u K(\xi, x_1) \right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \left\{ \frac{\partial u}{\partial x_1} \frac{\partial (aK)}{\partial x_1} + u \frac{\partial (bK)}{\partial x_1} \right.$$

$$\left. - ucK \right\} dx_1$$

$$\equiv g(\xi, x_2, x_3, \ldots, x_n)$$

$$\int_\alpha^\beta \{\quad\} K(\xi, x_1)\, dx_1 = \left[a \frac{\partial u}{\partial x_1} K(\xi, x_1) + b u K(\xi, x_1) - u \frac{\partial (aK)}{\partial x} \right]_\alpha^\beta$$

$$- \int_\alpha^\beta \left\{ -u \frac{\partial^2 (aK)}{\partial x_1} + u \frac{\partial (bK)}{\partial x_1} - ucK \right\} dx_1$$

$$\int_\alpha^\beta \left\{ a(x_1) \frac{\partial^2 u}{\partial x_1^2} + b(x_1) \frac{\partial u}{\partial x_1} + c(x_1) u \right\} K(\xi, x_1)\, dx_1 = g(\xi, x_2 \ldots x_n)$$

$$\int_\alpha^\beta \frac{\partial u}{\partial x_1} \frac{\partial (aK)}{\partial x_1} dx = u \frac{\partial (aK)}{\partial x}\Big|_\alpha^\beta - \int_\alpha^\beta u \frac{\partial^2 aK}{\partial x_1}$$

$$+ \int_\alpha^\beta u \left\{ \frac{\partial^2 (aK)}{\partial x_1^2} - \frac{\partial (bK)}{\partial x_1} + cK \right\} dx_1$$

(3) pick $\underline{K}$ such that $\left\{ \frac{\partial^2 (aK)}{\partial x_1^2} - \frac{\partial (bK)}{\partial x_1} + cK \right\} = \lambda K$ $\boxed{\text{where } \lambda = \text{constant}}$ (3)

Going back to 1, multiply by K $\int_\alpha^\beta dx_1$

$$\int_\alpha^\beta u \lambda K\, dx_1 + g(\xi, x_2, \ldots, x_n) + \int_\alpha^\beta Lu \cdot K\, dx_1 = \int_\alpha^\beta f(x_1, \ldots x_n) K\, dx_1$$

by eq. 2.

$$\lambda \bar{u} + g(\xi, x_n) + L \bar{u} = \int_\alpha^\beta f K\, dx_1 = \bar{f}(\xi, x_2, x_3, \ldots, x_n)$$

(4) $(\lambda + L)\bar{u} = \bar{f} - g$ (P.D.E. in $x_2, \ldots, x_n$)

keep operating to get O.D.E., solve, then go back inverting all the way

~~assume~~ $\cancel{u \frac{\partial (aK)}{\partial x}\Big|_\alpha^\beta = 0}$

Hankel $\alpha = 0$ $\beta = \infty$
$K = r J_0(\xi r)$

$$g(\xi, z) = \left[1 \cdot \frac{\partial V}{\partial r} K + \frac{V}{r} K - V \frac{\partial K}{\partial r} \right]_\alpha^\beta$$

$$= \left[r J_0(\xi r) \frac{\partial V}{\partial r} + V J_0(\xi r) - V J_0(\xi r) + \xi V r J_1(\xi r) \right]_{\alpha=0}^{\beta=\infty}$$

choose J_0 since no azimuthal dependence...

$\left|\frac{\partial V}{\partial r}\right| = O\left(\frac{1}{r^2}\right)$ for large r $\quad \lim_{r \to \infty} r \frac{\partial V}{\partial r} = 0$

$\lim_{r \to \infty} rV = \text{Constant}$ $\quad \lim_{r \to \infty} J_1(\xi r) = 0$

osp

$\therefore$ ~~$g(\xi, r) = 0$~~

(1) assume $\lim_{r \to 0} r \frac{\partial V}{\partial r} = \text{finite}$ (exclude a point source)
$\lim_{r \to 0} rV = \text{constant}$.

$\therefore g(\xi, z) = 0$

e.q. 4 becomes...

$$\left(\frac{d^2}{dz^2} + \lambda \right) \bar{V}(\xi, z) = 0$$

$$\frac{\partial^2 (aK)}{\partial x_1^2} - \frac{\partial (bK)}{\partial x_1} + cK = \lambda K.$$

$$\frac{\partial^2}{\partial x_1^2} (r J_0(\xi r)) - \frac{\partial}{\partial x_1} (J_0(\xi r)) = \lambda K = \lambda\, r J_0(\xi r)$$

$$\frac{\partial}{\partial r} \left[J_0(\xi r) - r \xi J_1(\xi r) - r \xi J_1(\xi r) \right] = \frac{\partial}{\partial r} (-r \xi J_1(\xi r))$$

$$- \cancel{\xi J_1(\xi r) - \xi J_1(\xi r)} = - \xi r \frac{\partial}{\partial r} J_1(\xi r) - \xi J_1(\xi r) = \lambda r J_0(\xi$$

by $n J_n(x) + x J_n'(x) = x J_{n-1}(x)$

$$= - \xi^2 r J_0(\xi r) = \lambda r J_0(\xi r)$$

$$\therefore \quad \lambda = -\xi^2$$

Hence. $\bar{V}(\xi z) = A \sinh \xi z + B \cosh \xi z = A' e^{-\xi z} + B e^{+\xi z}$

now $V \to 0$ as $z \to \infty$

$\boxed{\bar{V}(\ ) = \bar{F}(\ ) e^{-\xi z}}$ $\quad \bar{V} \to 0 \quad B = 0$

$$\int_0^a x^{m+1} J_m(\xi x)dx = \frac{a^{m+1}}{\xi} J_{m+1}(\xi a)$$

$$(\bar{V})_{z\to\infty} = \lim_{z\to\infty} \int_0^\infty V\, r J_0(\xi r)dr = \lim_{\epsilon\to 0} \epsilon \int_0^\infty r J_0(\xi r)dr$$

$$= \lim_{\substack{\epsilon\to 0\\ a\to\infty}} \epsilon \int_0^a r J_0(\xi r)dr = \lim_{\substack{\epsilon\to 0\\ a\to\infty}} \epsilon \frac{a}{\xi} J_1(\xi a)$$

$$\lim_{a\to\infty} a J_1(\xi a) = \text{const}$$

$$(\bar{V})_{z=\infty} = 0$$

$$\bar{V}(\xi,z) = \bar{F}(\xi) e^{-\xi z}$$

$$V(\xi) = \int_0^\infty \xi \bar{F}(\xi) e^{-\xi z} J_0(\xi r) d\xi$$

1-12-68

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \quad \text{for } t \geq 0, \quad -\infty < \begin{matrix}x\\y\end{matrix} < \infty$$

1). $z = F(x,y)$ at $t=0$

2) $dz/dt = 0$ at $t=0$

3) z and $\frac{\partial z}{\partial x} = 0$ at $|x| = \infty$

4) z and $\frac{\partial z}{\partial y} = 0$ at $|y| = \infty$

$$a_1 = 1;\ b_1 = 0;\ c_1 = 0;\quad L_1 = \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\ ;\ F_1 = 0$$

$$f(\xi, y, t) = 0$$

$$g(\xi, y, t) = \left[\frac{\partial z}{\partial x} K(\xi,x) - z\frac{\partial K(\xi,x)}{\partial x}\right]_{-\infty}^{\infty} = 0$$

$$\frac{\partial^2 K(\xi,x)}{\partial x^2} = \lambda_1 K(\xi,x) \qquad K = \frac{1}{\sqrt{2\pi}} e^{i\xi x}$$

$$-\xi^2 K = \lambda_1 K$$

$$(L_1 + \lambda_1)\bar{Z}(\xi, y, t) = 0 \qquad \lambda_1 = -\xi^2$$

$$\frac{\partial^2 \bar{Z}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 \bar{Z}}{\partial t^2} + \lambda_1 \bar{Z} = 0$$

$$(L_1 - \xi^2)\, Z(\xi, y, t) = 0.$$

$$\frac{\partial^2 Z}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 Z}{\partial t^2} - \xi^2 Z = 0$$

$$a_2 = 1, \quad b_2 = 0 \quad c_2 = -\xi^2 \qquad L = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad f = 0.$$

$$g(\xi, \eta, t) = \left[1 \frac{\partial Z}{\partial y} K - Z \frac{\partial K}{\partial x} \right]_{-\infty}^{\infty} = 0$$

$$\frac{\partial^2 K}{\partial y^2} - \xi^2 K = \lambda K \qquad K = \frac{1}{\sqrt{2\pi}} e^{i\eta y}$$

$$-\eta^2 - \xi^2 = \lambda$$

$$(L_1 \; - \eta^2 - \xi^2)\, \bar{Z}(\xi, \eta, t) = 0$$

$$+\frac{1}{c^2}\frac{\partial^2 \bar{Z}(\xi,\eta,t)}{\partial t^2} + (\eta^2 + \xi^2)\, \bar{Z}(\xi, \eta, t) = 0.$$

$$a_3 = +\frac{1}{c^2} \quad b_3 = 0, \quad c_3 = +(\eta^2 + \xi^2) \quad f_3 = 0 \quad L_3 = 0$$

$$\bar{Z}(\xi, \eta, t) = A e^{ic\sqrt{\eta^2+\xi^2}\, t} + B e^{-ic\sqrt{\eta^2+\xi^2}\, t}.$$

$$\cancel{Z(\xi, y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\eta y}\, \bar{Z}(\xi, \eta, t)\, d\eta}$$

$$\cancel{= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A e^{i\left[\sqrt{\frac{\eta^2+\xi^2}{c^2}}\, t - \eta y\right]} + B e^{-i\left[\sqrt{\frac{\eta^2+\xi^2}{c^2}}\, t + \eta y\right]}\, d\eta}$$

At $t = 0$

$$z = F(x, y, 0) \qquad \frac{\partial z}{\partial t} = 0$$

$$Z(\xi, y, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} F(x, y, 0)\, dx$$

$$\bar{Z}(\xi, \eta, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\eta y} F(\xi, y, 0)\, dy = A + B.$$

$$\bar{Z}(\xi,\eta,0) = A + B = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\, e^{i(\xi x+\eta y)}\, dx\,dy$$

$$\frac{\partial z}{\partial t}(\xi,\eta,0) = A - B = 0$$

$$A = \tfrac{1}{2}(\xi,\eta,0)$$
$$B = t\tfrac{1}{2}(\xi,\eta,0)$$

$$\bar{Z}(\xi,\eta,t) = \tfrac{1}{2}(\xi,\eta,0)\left[e^{ic\sqrt{\eta^2+\xi^2}\,t} + e^{-ic\sqrt{\eta^2+\xi^2}\,t}\right]$$

$$\bar{z}(\xi,y,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\eta y}\, \bar{Z}(\xi,\eta,t)\,d\eta$$

$$z(x,y,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi x}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\eta y}\, \bar{Z}(\xi,\eta,t)\,dy\right]dx$$

Ans. $$z(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi x+\eta y)}\, \bar{Z}(\xi,\eta,t)\,dy\,dx$$

Generally.

$$\frac{\partial z}{\partial t}(\xi,\eta,0) = ic\sqrt{\xi^2+\eta^2}\,[A - B] = \left[\frac{1}{2\pi}\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial z(x,y,t)}{\partial t}\, e^{i(\xi x+\eta y)}\,dx\,dy$$

$$z(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\xi x+\eta y)}\left[\tfrac{1}{2}\bar{F}(\xi,\eta)\left\{e^{ic\sqrt{\eta^2+\xi^2}\,t} - e^{-ic\sqrt{\eta^2+\xi^2}\,t}\right\}\right]dx\,dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\xi x+\eta y)}\, \bar{F}(\xi,\eta)\cos\left[c\sqrt{\eta^2+\xi^2}\,t\right]dx\,dy$$

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{for } -\infty < x < \infty, \; -\infty < y < \infty.$$

1). z and its partial derivatives $\to 0$ as $x \to \pm\infty$.

2) $z = f(x)$ at $y = 0$

3) $dz/dy = 0$ at $y = 0$

$a = 1$, $b = c = d = e = 0$ $\quad L = \frac{\partial^2}{\partial y^2}$, $f \equiv 0$, $\bar{f}(0)$.

$a\frac{\partial^4}{\partial x^4} + b\frac{\partial^3}{\partial x^3} + c\frac{\partial^2}{\partial x^2}\; d\frac{\partial}{\partial x}\; e$

$$g(\xi, y) = \left[aK\frac{\partial^3 z}{\partial x^3} - \frac{\partial(aK)}{\partial x}\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2(aK)}{\partial x^2}\frac{\partial z}{\partial x} - \frac{\partial^3(aK)}{\partial x^3} z \right]_{-\infty}^{\infty}$$

$= 0$

$\lambda K = \frac{\partial^4(aK)}{\partial x^4} = +\xi^4 K. \quad K = \frac{1}{\sqrt{2\pi}} e^{i\xi x}$

$$\frac{d^2 \bar{z}(\xi, y)}{\partial y^2} + \xi^4 \bar{z}(\xi, y) = 0$$

$$\bar{z} = C_1 e^{i\xi^2 y} + C_2 e^{-i\xi^2 y}$$

$z = f(x)$ at $y = 0 \qquad \bar{z} = C_1 + C_2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x} f(x)\,dx$

$\frac{\partial \bar{z}}{\partial y} = i\xi^2[C_1 - C_2] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x} z(x,y)\,dx$

$$\therefore \quad \bar{z} = \bar{f}(\xi)\cos(\xi^2 y)$$

$$z = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi x} \bar{f}(\xi)\cos(\xi^2 y)\, d\xi$$

1-15-68 $\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial y^2}$ Use Laplace transform to solve

(1) $z = 0$ at $y = \infty$

(2) $z = f(x)$ at $y = 0$, $x > 0$

(3) $z = 0$ at $y > 0$, $x = 0$

$$\bar{Z}(\xi, y) = \int_0^\infty e^{-\xi x} z(x,y)\,dx$$

$$Z(\xi, 0) = \int_0^\infty e^{-\xi x} z(x,0)\,dx$$

Consider as a special case $f(x) = k$

$$\int_0^\infty e^{-\xi x}\frac{\partial z}{\partial x}\,dx = \left(e^{-\xi x} z\right)\Big|_0^\infty + \xi \bar{Z}$$

$$= -z(0,y) + \xi \bar{Z}.$$

$$\frac{d^2\bar{Z}}{dy^2} - \xi \bar{Z} = -z(0,y)$$

$$\frac{d^2\bar{Z}}{dy^2} - \xi\bar{Z} = 0, \qquad \bar{Z} = A e^{-\sqrt{\xi}\,y}$$

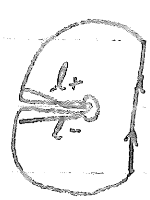
at

$$y = 0, \quad z(x,0) = K, \quad \bar{Z}(\xi, 0) = \frac{K}{\xi}$$

$$\bar{Z} = \frac{K}{\xi} e^{-\sqrt{\xi}\,y}.$$

$$z(x,y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\xi x}\,\frac{K}{\xi} e^{-\sqrt{\xi}\,y}\,d\xi.$$

Branch point at $\xi = 0$. and residue at $\xi = 0$



$$\frac{1}{2\pi i}\int_C = \sum \mathrm{Res} - \int_{l_+} - \int_{l_-}$$

on l_+

$\xi = \eta e^{i\pi}$ $\quad \sqrt{\xi} = i\sqrt{\eta}$ $\quad 0 \to \infty$

on l_-

$\xi = \eta e^{-i\pi}$ $\quad \sqrt{\xi} = -i\sqrt{\eta}$ $\quad \infty \to 0$

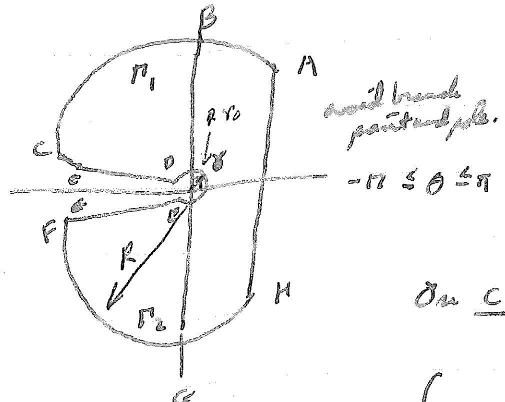
$$z(x,y) = K - \frac{K}{2\pi i}\int_\infty^0 e^{-\eta x} e^{-i\sqrt{\eta}\, y}\frac{d\eta}{\eta} - \frac{1}{2\pi i}\int_0^\infty e^{-\eta} e^{+i\sqrt{\eta}\, y}\frac{d\eta}{\eta}$$

$$= K\left[1 + \frac{1}{2\pi i}\int_0^\infty e^{-\eta x}\left[-e^{+i\sqrt{\eta}\, y} + e^{-i\sqrt{\eta}\, y}\right]\frac{d\eta}{\eta}\right]$$

$$= K\left[1 + \frac{-1}{\pi}\int_0^\infty e^{-\eta x} \sin\sqrt{\eta}\, y \frac{d\eta}{\eta}\right]$$

$$z(x,y) = K\left[1 - \frac{1}{\pi}\int_0^\infty e^{-\eta x} \sin(\sqrt{\eta}\, y)\frac{d\eta}{\eta}\right]$$

Note other line integrals about contour $= 0$ since $|z(\xi,y)| < 1$ as $\xi \geq 1$



$$\int_C \frac{1}{\xi} e^{x\xi - y\sqrt{\xi}}\, d\xi = \int_A + \int_{\Gamma_1} + \int_{CD} + \int_\gamma + \int_{EF} + \int_{\Gamma_2} = 0$$

On $\underline{CD}$: as $\epsilon \to 0$ $\quad \xi = re^{i\theta} = -r \quad \sqrt{\xi} = i\sqrt{r}$

$$\int_{CD} = \int_R^{r_0} \frac{1}{r} e^{-xr - iy\sqrt{r}}\, dr$$

$$\int_{EF} = \int_{r_0}^R \frac{1}{r} e^{-xr + iy\sqrt{r}}\, dr$$

$$\int_{\substack{CD\\EF}} = \int_{r_0}^R \frac{e^{-xr}}{r}\, 2i \sin\sqrt{r}\, y\, dr$$

$$\int_\gamma = \int_\pi^{-\pi} \frac{1}{r_0 e^{i\theta}} e^{x r_0 e^{i\theta} - y\sqrt{r_0}\, e^{i\theta/2}}\, \cancel{i r_0 e^{i\theta}}\, d\theta$$

$$= i\int^{-\pi} e^{(x r_0 e^{i\theta} - y\sqrt{r_0}\, e^{i\theta/2})}\, d\theta \qquad i\int^{-\pi} d\theta = -2i\pi$$

π

let $s = y\sqrt{r}$ $\quad r = \frac{s^2}{y^2}$

$$\int_{CD} + \int_{EF} = 2i\int_0^\infty \frac{e^{-s^2/y^2}}{s^2/y^2}\frac{2s}{y^2}\sin s\, ds$$

$$= 4i\int_0^\infty e^{-xs^2/y^2}\sin s\frac{ds}{s}$$

I-M p602 - theorem the $\int_{\Gamma_1} \int_{\Gamma_2} = 0$ if $|f(z)| \leq KR^{-k}$ $\quad \xi = Re^{i\theta} \quad -\pi \leq \theta \leq \pi$

then limit $R \to \infty \int_{\Gamma} e^{\xi x} f(\xi) d\xi = 0$

Thus $\int_{\Gamma_1} \int_{\Gamma_2}$ as $R = \infty$ $= 0$

$$\left| f(\xi) \right| = \left| \frac{e^{-\sqrt{\xi}\, y}}{\xi} \right| = \left| \frac{1}{e^{y\sqrt{\xi}} Re^{i\theta}} \right| \leq \frac{1}{e^{y\sqrt{\xi}} R} < R^{-1}$$

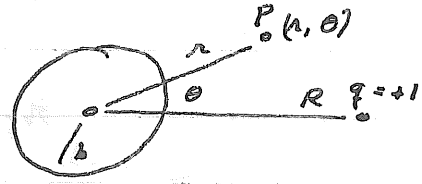
$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\xi} e^{\xi x - y\sqrt{\xi}} d\xi + \frac{2}{\pi} \int_0^{\infty} e^{-\frac{x}{y} s^2} \frac{\sin s}{s} ds - 1 = 0.$$

$$\Psi = L^{-1}(F) = k\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{x}} \right) \right]$$

8. A conducting sphere of radius b is brought into the neighborhood of a unit positive point charge. And then the sphere is grounded. Show that the potential owing to the induced charge on the conductor at any point outside the conductor is.

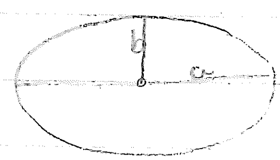
$$\phi = \frac{-b}{R r}\left[P_0(\cos\theta) + \frac{b^2}{R r} P_1(\cos\theta) + \frac{b^4}{(R r)^2} P_2(\cos\theta) + \ldots\right.$$

$\phi = 0$ = total potential due to charge induced + point charge.



9. Find the gravitational potential of a homogeneous oblate spheroid both at points outside and within the spheroid.

(Poisson's equation)



$$\phi = -\pi G \sigma (a^2 - b^2) \sin^2\theta \cosh^2\eta$$

$$\sin^2\theta = \frac{2}{3}\left[P_0(\cos\theta) - P_2(\cos\theta)\right]$$

$2\cosh\eta\ldots$

$2\cosh^3\eta + 4\cosh\eta\sinh^2\eta$

$$\frac{1}{\cosh\eta}\frac{d}{d\eta}\left(\cosh\eta \frac{d\phi}{d\eta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\eta}\right) + Q(\cosh^2\eta - \sin^2\theta)$$

$$\sin^2\theta\left[2\cosh^2\eta + 4\sinh^2\eta\right] + \cosh^2\eta\left[4\cos^2\theta - 2\sin^2\theta\right] + Q(\cosh^2\eta - \sin^2\theta)$$

$$\sin^2\theta\left[6\cosh^2\eta - 4\right] + \cosh^2\eta\left[4 - 6\sin^2\theta\right] + Q(\cosh^2\eta - \sin^2\theta)$$

$$\frac{\phi}{4}$$

$2\sin^2\cos$

$4\sin\cos^2$

$-2\sin^3$

$\cosh^2\eta - 1$

$4\cosh^2\theta \qquad -4\sin^2\theta$