Due
Nov 3, 1967

Potential Theory (Gph 204)
Problems

Assigned: October 20, 1967 O. Nuttli

1. Solve the following Neumann interior boundary value problems:

   a) $\nabla^2\phi = 0$ in the interior of a circle of radius a, $\left(\frac{\partial\phi}{\partial n}\right)_{r=a} = K(x^2 - y^2)$

   b) $\nabla^2\phi = 0$ in the interior of a circle of radius a, $\left(\frac{\partial\phi}{\partial n}\right)_{r=a} = K.$

2. Given: $\nabla^2\phi = 0$ in the spherical region $a<r<b$, and $(\phi)_{r=a} = C_1$, $(\phi)_{r=b} = C_2$.

   Show that, for $a\leq r\leq b$,

   no dependence on θ, φ.

   $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 0$

$$\phi = \frac{(C_1 - C_2)a\,b}{(b-a)r} + \frac{C_2 b - C_1 a}{b - a} .$$

   $\phi = \frac{A}{r} + B$ $aC_1 = \frac{A}{a} + aB$

   $bC_2 = \frac{A}{b} + bB$

   $C_1 - C_2 = A\left(\frac{b-a}{ab}\right)$

   $B = \frac{aC_1 - bC_2}{a - b}$

3. Determine the potential ϕ, given that $\nabla^2\phi = 0$ inside the rectangle $0<x<b$, $0<y<a$, and $\phi(0,y) = 0$, $\phi(b,y) = B$, $(\partial\phi/\partial y)_{y=0} = 0$, $(\partial\phi/\partial y)_{y=a} = 0$.

4. Given: $\nabla^2\phi = 0$ in the cylindrical region $0<r<a$, $0<z<h$ and $\phi(a,z,\theta) = 0$ for $0<z<h$

   $\phi(r,0,\theta) = f_1(r)$ for $r<a$

   $\phi(r,h,\theta) = f_2(r)$ for $r<a$.

   Show that inside the cylindrical region

$$\phi = \sum_{n=1}^{\infty} A_n \sinh\left\{p_n(h - z)\right\} J_0(p_n r) + \sum_{n=1}^{\infty} B_n \sinh(p_n z) J_0(p_n r)$$

   where

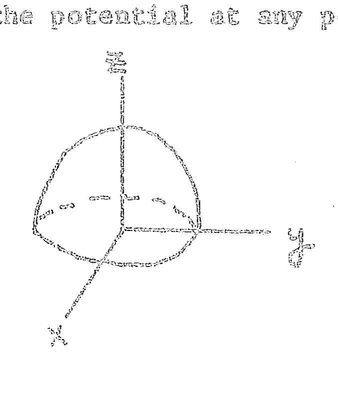
$$A_n = \frac{2\int_0^a r f_1(r) J_0(p_n r)\,dr}{a^2\left[J_1(p_n a)\right]^2 \sinh(p_n h)}$$

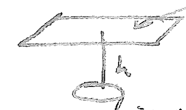
$$B_n = \frac{2\int_0^a r f_2(r) J_0(p_n r)\,dr}{a^2\left[J_1(p_n a)\right]^2 \sinh(p_n h)}$$

   and the p_n are the roots of $J_0(pa) = 0$.

5. Find the potential at any point in the space $z>0$ outside a homogeneous spherical shell of infinitesimal thickness and of radius b.

   or hemispherical



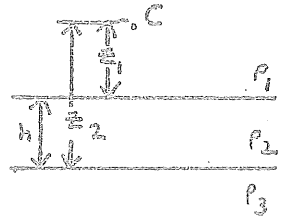


find g on this plane

expand in series (end of potential theory notes - properties of harmonic functio

6. Derive an expression for the gravity anomaly, valid anywhere in the plane $z = 0$, for the anomalous field produced by a horizontal circular disc of radius b and depth h.

7. Consider a layer of electrical resistivity ρ_2 and thickness h sandwiched between two half-spaces of resistivity ρ_1 and ρ_3, respectively. Show that the electric potential produced by a point current source in the upper half-space is



$$U_1 = \frac{\rho_1 I}{4\pi}\left[\frac{1}{R} + K_{21}\int_0^\infty \frac{e^{-\lambda(2z_1 - z)}J_o(\lambda r)d\lambda}{1 + K_{21}K_{32}e^{-2\lambda h}} + \int_0^\infty \frac{K_{32}e^{-\lambda(2z_2 - z)}J_o(\lambda r)d\lambda}{1 + K_{21}K_{32}e^{-2\lambda h}}\right]$$

$$U_2 = \frac{\rho_1 I}{4\pi}(1 + K_{21})\left[K_{32}\int_0^\infty \frac{e^{-\lambda(2z_2 - z)}J_o(\lambda r)d\lambda}{1 + K_{21}K_{32}e^{-2\lambda h}} + \int_0^\infty \frac{e^{-\lambda z}J_o(\lambda r)d\lambda}{1 + K_{21}K_{32}e^{-2\lambda h}}\right]$$

$$U_3 = \frac{\rho_1 I}{4\pi}(1 + K_{21})(1 + K_{32})\int_0^\infty \frac{e^{-\lambda z}J_o(\lambda r)d\lambda}{1 + K_{21}K_{32}e^{-2\lambda h}}$$

where $R = \sqrt{r^2 + z^2}$ is the distance from C to the point at which the potential is being determined, and

$$K_{21} = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}, \quad K_{32} = \frac{\rho_3 - \rho_2}{\rho_3 + \rho_2} \quad .$$

Robert B. Herrmann
Gph-204.
Nov 3, 1967

1. Solve the following Neuman interior boundary value problems:

a) $\nabla^2\phi = 0$ in the interior of a circle of radius a,

$\left(\frac{\partial\phi}{\partial n}\right)_{r=a} = K(x^2-y^2)$

$$\nabla^2\phi = \frac{1}{r}\left(\frac{\partial}{\partial r} r \frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

The general solution is

$$\phi = \left.\begin{matrix}\ln r \\ 1\end{matrix}\right\} + \left.\begin{matrix} r^n \\ r^{-n}\end{matrix}\right\} \left.\begin{matrix}\cos n\theta \\ \sin n\theta\end{matrix}\right\}$$

Since there will be a θ dependence, the general solution is of the form. and ϕ exists for $r=0$,

$$\phi = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \sum_{n=0}^{\infty} n a^{n-1}(A_n \cos n\theta + B_n \sin n\theta) = K(x^2-y^2)$$

now at $r=a$ $\quad x = a\cos\theta \quad y = a\sin\theta$

$$x^2 = a^2\cos^2\theta \quad y^2 = a^2\sin^2\theta$$

and $\quad x^2-y^2 = a^2(\cos^2\theta - \sin^2\theta) = a^2\cos 2\theta$

It is seen that $A_1 = B_1 = B_2 = A_j = B_j = 0$ for $j>2$

and that

$$2a A_2 \cos 2\theta = Ka^2 \cos 2\theta$$

and

$$A_2 = \frac{Ka}{2}$$

The solution for ϕ is

$$\underline{\underline{\text{Ans 1(a)} \quad \phi = \frac{Ka}{2} r^2 \cos 2\theta + A_0}}$$

(another condition is required to determine A_0)

(b) $\nabla^2\phi = 0$ in the interior of a circle of radius a, $\left(\frac{\partial\phi}{\partial n}\right)_{r=a} = K$. This solution implies no θ dependence, so

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = 0$$

$$\phi = A \ln r + B$$

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \frac{A}{a} = K$$

$$A = Ka.$$

The solution is $\phi = Ka \ln r + B$.

Not valid solution

$\iint \frac{\partial\phi}{\partial n}\, dS \neq 0$

Since the solution is not valid at the origin let another boundary condition be $\phi = \phi(b)$ at $r = b$

Then

$$\phi(b) = Ka \ln b + B$$

$$\therefore \quad \phi = Ka \ln r - Ka \ln b + \phi(b)$$

Robert B. Herrmann
Gph-204
Nov. 3, 1967

No solution possible

Ans 1(b) $\phi = \cancel{K...}\ K a \ln r + B$.

(another condition defining ϕ is required to determine B)...

2. Given: $\nabla^2\phi = 0$ in the spherical region $a < r < b$, and $(\phi)_{r=a} = c_1$, $(\phi)_{r=b} = c_2$. Show that for $a \le r \le b$

$$\phi = \frac{(c_1 - c_2)ab}{(b-a)r} + \frac{c_2 b - c_1 a}{b-a}$$

Since there is no θ dependence,

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r}\right) = 0$$

The solution is of the form

$$\phi = \frac{A}{r} + B$$

Using the boundary conditions,

$$c_1 = \frac{A}{a} + B \quad (1)$$

$$c_2 = \frac{A}{b} + B \quad (2)$$

Solving simultaneously for A and B:

Subtracting (2) from (1) and rearrang.

$$A = \frac{c_1 - c_2}{b-a}\, ab$$

Multiply (1) by "a", (2) by "b" and subtract the products

$$B = \frac{a c_1 - b c_2}{a-b}$$

Ans 5.2 $\phi = \dfrac{(c_1 - c_2)ab}{(b-a)r} + \dfrac{c_1 a - c_2 b}{a-b}$

3. Determine the potential ϕ, given that $\nabla^2\phi = 0$ inside the rectangle $0<x<b$, $0<y<a$, and $\phi(0,y)=0$, $\phi(b,y)=B$, $(\partial\phi/\partial y)_{y=0}=0$, $(\partial\phi/\partial y)_{y=a}=0$.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Let

$$\phi(x,y) = X(x)\,Y(y)$$

Separating the variables.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \beta^2$$

The solutions are of the form.

$$\phi = \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} y \\ 1 \end{matrix}\right\} + \left.\begin{matrix} \sinh\beta x \\ \cosh\beta x \end{matrix}\right\} \left.\begin{matrix} \sin\beta y \\ \cos\beta y \end{matrix}\right\}.$$

Not the solution $\phi = kx$

The conditions $\phi(0,y)=0$, $(\partial\phi/\partial y)_{y=0}=0$, eliminate the first group of solutions and the coefficients of $\cosh\beta y$ and $\sin\beta y$.

The solution of the condition

$$\left(\partial\phi/\partial y\right)_{y=a} = 0 \quad = \quad \sin\beta a = 0$$

$\beta = \frac{n\pi}{a}$

Leave as

$$\therefore \quad \beta = \frac{2n\pi}{a} \qquad n = 1, 2$$

choice of $\frac{2n\pi}{a}$ explored later.

Robert B. Herrmann
Gph 204
November 3, 1967

The solutions are of the form.

$$\phi = \sum_{n=\phi}^{\infty} A_n \sinh\frac{2n\pi}{a}x \cos\frac{2n\pi}{a}y$$

The condition $\phi(b,y) = B$ gives

$$B = \sum_{n=\phi}^{\infty} A_n \sinh\frac{2n\pi b}{a} \cos\frac{2n\pi}{a}y$$

The coefficients A_n are found by considering the series.

$$B = \sum_{n=\phi}^{\infty} C_n \cos\frac{2n\pi}{a}y \quad \text{, where}$$

$$C_n = A_n \sinh\frac{2n\pi b}{a}$$

The coefficients C_n are found from

$$C_0 = \frac{2}{a}\int_0^{a/2} B \, dy$$

Let the $C_m = 0$. Then $\phi(x,y) = K^x$, $B = K^b$, where $\phi(b,y)$ ... $\phi = B/b \, x$

$$C_m = \frac{4}{a}\int_0^{a/2} B \cos\frac{n4\pi y}{2a} \, dy$$

But the C_0 term is not used,

For B = constant,

$$C_0 = B \quad C_1 = \frac{4B}{\pi}, \; C_3 = \frac{-4B}{3\pi}, \; C_5 = \frac{4B}{5\pi}$$

So,

$$A_1 = \frac{4B}{\pi \sinh 2\pi b/a}$$

$$A_3 = \frac{-4B}{3\pi} \frac{1}{\sinh \frac{6\pi b}{a}}$$

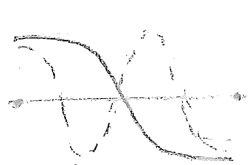
$$A_5 = \frac{4B}{5\pi} \frac{1}{\sinh \frac{10\pi b}{a}}$$

$$\phi = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4B}{n\pi} \frac{\sinh \frac{2\pi n x}{a}}{\sinh \frac{2n\pi b}{a}} \cos \frac{2n\pi y}{a}$$

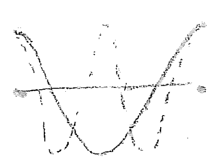
The series can also be written as

$$\phi = \sum_{n=0}^{\infty} \frac{4B}{(2n+1)\pi} \frac{\sinh \frac{(2n+1)\pi x}{a}}{\sinh \frac{(2n+1)\pi b}{a}} \cos (2n+1) \frac{y}{a}$$

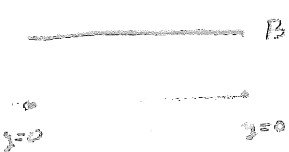
The value for β was chosen to be $\beta = \frac{2n\pi}{a}$ $n=1,2,\ldots$ instead of $\beta = \frac{K\pi}{a}$ $K=1,2,3.$ So that a solution could be found.



SOLUTION FOR $\beta = \frac{K\pi}{a}$



$\beta = \frac{2n\pi}{a}$



BOUNDARY CONDITION

These graphs show that a summation of ~~solutions~~ curves of type $\beta = \frac{K\pi}{a}$ could never give $\phi = B$ at (b, y).

Robert B. Herrmann
6 ph - 204
November 9, 1967

4. Given: $\nabla^2\phi = 0$ in the cylindrical region $0 < r < a$, $0 < z < h$

and $\phi(a, z, \theta) = 0$ for $0 < z < h$.

$\phi(r, 0, \theta) = f_1(r)$ for $r < a$

$\phi(r, h, \theta) = f_2(r)$ for $r < a$

Show that inside the cylindrical region

$$\phi = \sum_{n=1}^{\infty} A_n \sinh\{p_n(h-z)\} J_0(p_n r)$$

$$+ \sum_{n=1}^{\infty} B_n \sinh\{p_n z\} J_0(p_n r)$$

where

$$A_n = \frac{2\int_0^a r f_1(r) J_0(p_n r)\,dr}{a^2 [J_1(p_n a)]^2 \sinh(p_n h)}$$

$$B_n = \frac{2\int_0^a r f_2(r) J_0(p_n r)\,dr}{a^2 [J_1(p_n a)]^2 \sinh(p_n h)}$$

and the p_n are the roots of $J_0(pa) = 0$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

From the ~~boundary~~ conditions, it is seen that ϕ is independent of Θ;

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2} = 0$$

For a solution, let $\phi = R(r)\,Z(z)$

Then, $\nabla^2\phi = \frac{Z(z)}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + R(r)\frac{d^2Z}{dz^2} = 0$

or. $\frac{1}{R(r)\,r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -p^2$

The solutions become

$$Z(z) = A_1 \cosh pz + A_2 \sinh pz$$

$$R(r) = A_3 J_0(\cancel{a}pr) + A_4 Y_0(pr)$$

Since the solution is valid for $r=0$, $A_4 = 0$

For $\phi(a, z, \theta) = 0$

$$R(r) = A_3 J_0(pa) = 0$$

If α_n are the zeroes, nth zero of $J_0(pa) = 0 = J_0(\alpha_n)$

$$pa = \alpha_n$$
$$p = \frac{\alpha_n}{a} = p_n$$

So the solution is of the form.

$$\phi = \sum_1^\infty \left(A_n' \cosh p_n z + B_n' \sinh p_n z\right) J_0(p_n r)$$

(i) for $z=0$, $\phi = f_1(r)$

or $f_1(r) = \sum_1^\infty A_n' \, J_0(p_n r)$

and where

$$A_n' = \frac{\frac{2}{a^2}\int_0^a r f_1(r)\, J_0(p_n r)}{\{J_1(p_n r)\}^2}$$

(ii) for $z = h$,

$$f_2(r) = \sum_1^\infty \left(A_n' \cosh(p_n h) + B_n' \sinh(p_n h)\right) J_0(p_n r) = \sum_1^\infty C_n J_0(p_n r)$$

where

$$C_n = \frac{\frac{2}{a^2}\int_0^a r f_2(r)\, J_0(p_n r)}{\{J_1(p_n r)\}^2}$$

The coefficients B_n' are therefore found to be

$$B_n' = \frac{C_n - A_n' \cosh(p_n h)}{\sinh(p_n h)}$$

The solution for ϕ becomes

$$\phi = \sum_1^\infty \left\{ A_n \cosh(p_n z) + \frac{C_n - A_n \cosh(p_n h)}{\sinh(p_n h)} \sinh(p_n z) \right\} J_0(p_n r)$$

$$= \sum_1^\infty \frac{J_0(p_n r)}{\sinh(p_n h)} \left\{ A_n \cosh(p_n z)\sinh(p_n h) + C_n \sinh(p_n z) - A_n \cosh(p_n h)\sinh(p_n z) \right\}$$

Since $\sinh(a-b) = \sinh a \cosh y - \sinh y \cosh x$

we have,

$$\phi = \sum_{n=1}^{\infty} \frac{J_0(p_n r)}{\sinh(p_n h)} \left\{ A_n' \sinh\{p_n(h-z)\} + C_n \sinh(p_n z) \right\}$$

where

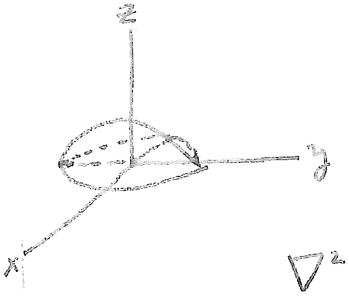
$$A_n' = \frac{\frac{2}{a^2}\int_0^a r\, J_0(p_n r)\, f_1(r)\, dr}{\{J_1(p_n a)\}^2}$$

$$C_n = \frac{\frac{2}{a^2}\int_0^a r\, J_0(p_n r)\, f_2(r)\, dr}{\{J_1(p_n a)\}^2}$$

Ans 4

---

5. Find the potential at any point in space out side a homogeneous hemispherical shell of infinitesimal thickness and of radius b.



In this coordinate system there is symmetry about the z axis and the Laplacian has the form.

$$\nabla^2 \phi = \frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right\} + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0$$

Let $\phi = R(r)\,\Theta(\theta)$

then on separation of variables,

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = -\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = n(n+1)$$

R. Herrmann
Gph-204
Nov. 9, 1969

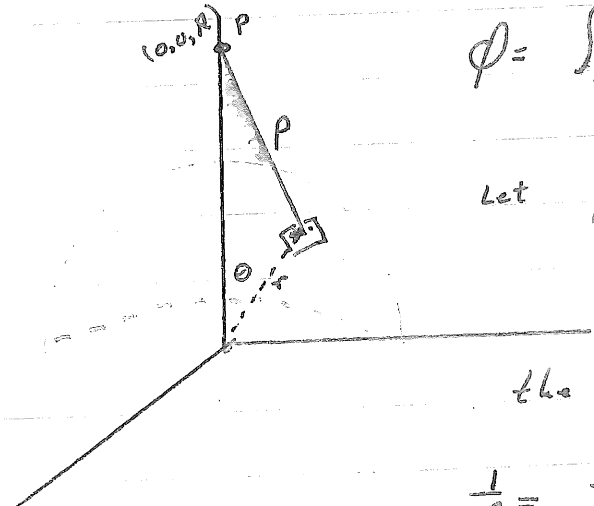
The general solution is

$$\phi = \Theta R = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} \left.\begin{matrix} P_n(\cos\theta) \\ Q_n(\cos\theta) \end{matrix}\right\}$$

Consider the region $r > b$, the solution must be of the form.

$$\phi = \sum_{n=0}^{\infty} A_n r^{-(n+1)} P_n(\cos\theta)$$

To determine the coefficients A_n consider the potential at a point on the z axis.



$$\phi = \int_S \frac{\sigma\, dS}{\rho}$$

Let $\rho^2 = R^2 - 2rR\cos\theta + r^2$

Upon the expansion under the condition that $r/R < 1$.

$$\frac{1}{\rho} = \sum_{n=0}^{\infty} \frac{r^n}{R^{n+1}} P_n(\cos\theta)$$

The potential at P due to all points on the surface of the hemisphere is

$$\phi = \int_{\theta=0}^{\theta=\pi/2} \int_{\phi=0}^{\phi=2\pi} \sigma a^2 \sum \frac{a^n}{R^{n+1}} P_n(\cos\theta) \sin\theta \, d\theta \, d\phi$$

since $r=a$ on the surface of the sphere.

$$\phi = 2\pi\sigma a^2 \sum_{n=0}^{\infty} \frac{a^n}{R^{n+1}} \int_{\theta=0}^{\theta=\pi/2} P_n(\cos\theta)\sin\theta\, d\theta$$

Letting $x=\cos\theta$, the potential on the axis becomes

$$\phi = 2\pi\sigma a^2 \sum_{n=0}^{\infty} \frac{a^n}{R^{n+1}} \int_0^1 P_n(x)dx$$

Now it is known that

$$C_n = \int_0^1 P_n(x)dx = \frac{\pi^{1/2}\, 2^{-1}}{\Gamma(1-\frac{n}{2})\Gamma(\frac{n}{2}+\frac{3}{2})}$$

Some values of the integral are as follow:

n	$\int_0^1 P_n(x)dx$
0	1
1	1/4
2	0
3	-1/8
4	0
5	1/16
6	0
7	-5/128
8	0
9	7/256
10	0
11	3/2048

R. Herrmann
Gph-204
Nov. 9, 1967

ON THE AXIS:

$$\phi_A = 2\pi\sigma a \sum_{n=0}^{\infty} \frac{a^{n+1}}{R^{n+1}} C_n$$

Going back to the original solution,

$$\phi = \sum_{n=0}^{\infty} A_n r^{-(n+1)} P_n(\cos\theta).$$

On the axis for $r=R$, $\theta=0$,

$$\phi(R,0,\phi) = \sum_{n=0}^{\infty} A_n R^{-(n+1)}$$

From the previous derivation, the coefficients A_n are found to be.

$$A_n = 2\pi\sigma a \cdot a^{n+1} C_n$$

The complete solutions for the potential at points away from the hemisphere, for $R>a$,

$$\phi(R,\theta) = 2\pi\sigma a \sum_{n=0}^{\infty} C_n \frac{a^{n+1}}{R^{n+1}} P_n(\cos\theta)$$

where $C_n = \int_0^1 P_n(x)dx = \dfrac{\pi^{1/2}\, 2^{-1}}{\Gamma(1-\frac{n}{2})\Gamma(\frac{n}{2}+\frac{3}{2})}$

For regions for $r < a$,

$$V = \int_S \frac{\sigma\, dS}{\rho}$$

where, again, $\rho^2 = R^2 - 2rR\cos\theta + r^2$

and $\dfrac{1}{\rho} = \dfrac{1}{a}\left(1 - \dfrac{2r}{a}\cos\theta + \dfrac{r^2}{a^2}\right)^{-1/2}$

The expansion is

$$\frac{1}{\rho} = \sum \frac{r^n}{a^{n+1}} P_n(\cos\theta)$$

Consider the potential on the axis of symmetry at $\overset{x}{(}0, \overset{y}{0}, \overset{z}{R})$

$$V = \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} \sigma a^2 \sin\theta\, d\theta\, d\phi \sum \frac{R^n}{a^{n+1}} P_n(\cos\theta)$$

$$\therefore \quad V(\overset{r}{R}, \overset{\theta}{0}, \overset{\phi}{0}) = 2\pi\sigma a \sum_{n=0}^{\infty} \frac{R^n}{a^n} C_n$$

where C_n has the same meaning as before.

The general solution of $\nabla^2 V = 0$ in the region $r < a$ is.

$$\phi(r, \theta) = \sum_{n=0}^{\infty} B_n r^n P_n(\cos\theta)$$

Now the B_n can be determined by using the above value of $V(R,0)$

$$V(R,0) = 2\pi\sigma a \sum_{n=0}^{\infty} \frac{R^n}{a^n} C_n = \sum_{n=0}^{\infty} B_n R^n$$

$$\therefore \quad B_n = \frac{2\pi\sigma a\, C_n}{a^n}$$

R. Herrmann
Gph-204
Nov. 9, 1967

The solution for the interior region is

for $r < a$ $\quad \phi_0(r,\theta) = 2\pi\sigma a \sum_{n=0}^{\infty} c_n \frac{r^n}{a^n} P_n(\cos\theta)$

$$\text{where } C_n = \int_0^1 P_n(x)\,dx = \frac{\pi^{1/2}\,2^{-1}}{\Gamma(1-\frac{n}{2})\,\Gamma(\frac{n}{2}+\frac{3}{2})}$$

for $r > a$

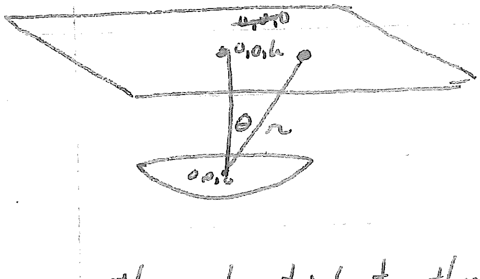
$$\phi_{1n}(r,\theta) = 2\pi\sigma a \sum_{n=0}^{\infty} c_n \frac{a^{n+1}}{r^{n+1}} P_n(\cos\theta)$$

It is seen that $\phi_0(r,\theta) = \phi_{1n}(r,\theta)$ for $r=a$, $\pi \leq \theta < \pi/2$!

However these solutions are not valid in the region $r=0$, $\frac{\pi}{2} \leq \theta \leq 0$

Herrmann, Robert B.
Gph-204
Nov-3, 1967

6. Derive the expression for the gravity anomaly valid anywhere in the plane $z=0$, for the anomalous field produced by a horizontal circular disc of radius "a" and a depth h below the plane for which $h > a$



The potential of the disc at a point a distance h above the disc on its axis is

$$\phi_{axis} = \iint \frac{\sigma\, dS}{\sqrt{r^2+z^2}} = 2\pi\sigma \left\{ \sqrt{a^2+z^2} - z \right\}$$

The gravity anomaly A on the axis is

$$A_{AXIS} = \left.\frac{d\phi}{dz}\right|_{z=h} = 2\pi\sigma \left\{ \frac{1}{\sqrt{1+a^2/h^2}} - 1 \right\}$$

For $h > a$,

$$A_{AXIS} = 2\pi\sigma \left\{ -\frac{a^2}{2h^2} + \frac{3}{8}\frac{a^4}{h^4} - \frac{5}{16}\frac{a^6}{h^6} + \frac{35}{128}\frac{a^8}{h^8} - \frac{63}{256}\frac{a^{10}}{h^{10}} + \frac{231}{1024}\frac{a^{12}}{h^{12}} \cdots \right\}$$

IF ϕ IS A SOLUTION OF $\nabla^2\phi = 0$, THEN $\nabla^2 A = 0$ ALSO.

The general solution is

$$A = 2\pi\sigma \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta).$$

Now $A(h,0) = \sum_{n=0}^{\infty} B_n h^{-(n+1)} = A_{AXIS}$.

The B_n are found to be

$$B_0 = 0 \qquad B_4 = 0 \qquad B_8 = 0$$
$$B_1 = -a^2 \qquad B_5 = \frac{-5}{16}a^6 \qquad B_9 = \frac{-63}{256}a^{10}$$
$$B_2 = 0 \qquad B_6 = 0 \qquad B_{10} = 0$$
$$B_3 = \frac{3}{8}a^4 \qquad B_7 = \frac{35}{128}a^8 \qquad B_{11} = \frac{231}{1024}a^{12}$$

The solution for the anomaly on the plane $z=h$, is found by choosing r, and θ such that $h = r\cos\theta$ is.

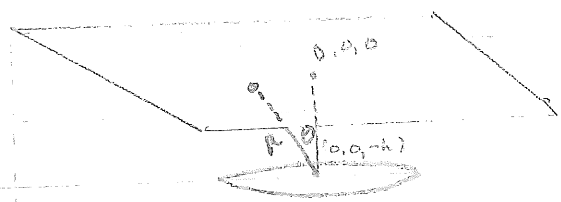
$$A = 2\pi\sigma \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n\left(\frac{h}{r}\right)$$

$$= 2\pi\sigma \left\{ -\frac{a^2}{r^2} P_1\left(\frac{h}{r}\right) + \frac{3}{8}\frac{a^4}{r^4} P_3\left(\frac{h}{r}\right) - \frac{5}{16}\frac{a^6}{r^6} P_5\left(\frac{h}{r}\right) + \frac{35}{128}\frac{a^8}{r^8} P_7\left(\frac{h}{r}\right) \right.$$

$$\left. - \frac{63}{256}\frac{a^{10}}{r^{10}} P_9\left(\frac{h}{r}\right) + \frac{231}{1024}\frac{a^{12}}{r^{12}} P_{11}\left(\frac{h}{r}\right) + \dots \right\}$$

Robert B. Herrmann
Gph-204
November 9, 1967

6. Derive the expression for the gravity anomaly valid anywhere in the plane $z=0$, for the anomalous field produced by a horizontal circular disc of radius b and at a depth h ~~j~~ below the plane.



IT is seen from the problem that there is axial symmetry.

The potential of the disk at the origin on the plane, is

$$\phi_{axis} = \int \frac{\sigma\, dS}{\rho} = \int_{r=0}^{r=b}\int_{\phi=0}^{\phi=2\pi} \frac{\sigma r\, dr\, d\phi}{\sqrt{r^2+h^2}} = 2\pi\sigma\left\{\sqrt{b^2+\cancel{k}^2}-h\right\}$$

$$= 2\pi\sigma\left\{\sqrt{b^2+h^2}-h\right\}$$

Expressing Laplaces equation is spherical coordinates, with axial symmetry, the solution is

$$\phi(R,\theta) = \sum_{n=0}^{\infty} A_n R^{-(n+1)} P_n(\cos\theta)$$

where R is the distance from the center of the ~~d~~disc ~~t~~ to the plane and θ is the angle between the z axis and R.

This solution was taken instead of R^n so that $\lim_{n\to\infty} \phi = 0$; This solution is taken for $R > b$

IF $h > b$ For $R = h, \theta = 0$

$$\phi(R=h,\theta=0) = \phi(h,0) = \sum_{n=0}^{\infty} A_n R^{-(n+1)}_{R=h} = 2\pi\sigma\left\{\sqrt{b^2+h^2} - h\right\}$$

$$\therefore \sum_{n=0}^{\infty} A_n h^{-(n+1)} = 2\pi\sigma\left\{\sqrt{b^2+h^2} - h\right\} = 2\pi\sigma h\left\{\sqrt{\frac{b^2}{h^2}+1} - 1\right\}$$

$$\sqrt{1+\frac{b^2}{h^2}} = 1 + \frac{1}{2}\frac{b^2}{h^2} - \frac{1}{8}\frac{b^4}{h^4} + \frac{1}{16}\frac{b^6}{h^6} - \frac{5}{128}\frac{b^8}{h^8} + \frac{7}{256}\frac{b^{10}}{h^{10}} - \frac{21}{1024}\frac{b^{12}}{h^{12}}$$

$$\sum_{n=0}^{\infty} A_n h^{-(n+1)} = 2\pi\sigma h\left(\frac{1}{2}\frac{b^2}{h^2} - \frac{1}{8}\frac{b^4}{h^4} + \frac{1}{16}\frac{b^6}{h^6} - \frac{5}{128}\frac{b^8}{h^8} \text{ turn}\right)$$

$$\therefore \quad A_0 = 2\pi\sigma\left\{\tfrac{1}{2}b^2\right\} \qquad A_1 = 0$$

$$A_2 = 2\pi\sigma\left\{-\tfrac{1}{8}b^4\right\} \qquad A_3 = 0$$

$$A_4 = 2\pi\sigma\left\{\tfrac{1}{16}b^6\right\} \qquad A_5 = 0 \quad \text{etc.}$$

For $h > b$

$$\phi(R,\theta) = 2\pi\sigma\left\{\frac{1}{2}\frac{b^2}{R}P_0(\cos\theta) - \frac{1}{8}\frac{b^4}{R^3}P_2(\cos\theta) + \frac{1}{16}\frac{b^6}{R^5}P_4(\cos\theta) - \frac{5}{128}\frac{b^8}{R^7}P_6(\cos\theta) + \frac{7}{256}\frac{b^{10}}{R^9}P_8(\cos\theta)\ \text{t}\ldots\right\}$$

For $h > b$ $h = R\cos\theta$

Let us now consider the solution for $h < b$.

In this case,

$$\phi_{AXIS} = \left\{\sqrt{b^2+h^2} - h\right\}2\pi\sigma = 2\pi\sigma\left\{b - h + \frac{1}{2}\frac{h^2}{b} - \frac{1}{8}\frac{h^4}{b^3} + \frac{1}{16}\frac{h^6}{b^5} - \frac{5}{128}\frac{h^8}{b^7} \text{ turn}\right\}$$

Robert B. Herrmann.
Gph-204
Nov. 8, 1967

The solution to Laplace's Equation takes on two values, according to whether $R > b$ or $R < b$.

For $R > b$.

$$\phi(R,\theta) = 2\pi\sigma \sum_{n=0}^{\infty} A_n R^{-(n+1)} P_n(\cos\theta)$$

For $R < b$

$$\phi(R,\theta) = 2\pi\sigma \sum_{n=0}^{\infty} B_n R^{n} P_n(\cos\theta)$$

at $R = h, \theta = 0,$

$$\phi(h,0) = 2\pi\sigma \sum_{n=0}^{\infty} B_n h^n = 2\pi\sigma \left\{ b - h + \frac{1}{2}\frac{h^2}{b} - \frac{1}{8}\frac{h^4}{b^3} + \frac{1}{16}\frac{h^6}{b^5} - \frac{5}{128}\frac{h^8}{b^7} + \dots \right.$$

So that

$B_0 = b$ $\quad B_3 = 0$ $\quad B_6 = \frac{1}{16}\frac{1}{b^5}$ $\quad B_9 = 0$

$B_1 = -1$ $\quad B_4 = \frac{-1}{8b^3}$ $\quad B_7 = 0$ $\quad B_{10} = \frac{7}{256}\frac{1}{b^9}$

$B_2 = \frac{1}{2b}$ $\quad B_5 = 0$ $\quad B_8 = -\frac{5}{128}\frac{1}{b^7}$

Another condition is that $\phi_{R>b} = \phi_{R<b}$ at $R = b$ for $\theta \neq \pm\pi/2$

Then,

$$\left(2\pi\sigma \sum_{n=0}^{\infty} A_n b^{-(n+1)} P_n(1) = 2\pi\sigma \sum B_n b^n (P_n(1)) \right.$$

Then $A_n = B_n b^{2n+1}$

$A_0 = b^2$ $\quad A_2 = b^4/2$ $\quad A_4 = -\frac{1}{8}b^6$ $\quad A_6 = \frac{1}{16}b^8$ $\quad A_8 = \frac{-5}{128}b^{10}$

$A_1 = -b^3$ $\quad A_3 = 0$ $\quad A_5 = 0$ $\quad A_7 = 0$ $\quad A_9 = 0$

The general solution in the case $h < b$ is

for $R > b$

here $R \equiv r$

$$\phi(r,\theta) = 2\pi\sigma \left\{ \frac{b^2}{r} P_0(\cos\theta) - \frac{b^3}{r^2} P_1(\cos\theta) + \frac{b^4}{2r^3} P_2(\cos\theta) - \frac{1}{8}\frac{b^6}{r^5} P_4(\cos\theta) + \frac{1}{16}\frac{b^8}{r^7} P_6(\cos\theta) - \frac{5}{128}\frac{b^{10}}{r^9} P_8(\cos\theta) + \dots \right\}$$

for $R < b$

Solution for $h < b$

$$\phi(r,\theta) = 2\pi\sigma \left\{ b\, P_0(\cos\theta) - r\, P_1(\cos\theta) + \frac{r^2}{2b} P_2(\cos\theta) - \frac{r^4}{8b^3} P_4(\cos\theta) + \frac{r^6}{16 b^5} P_6(\cos\theta) - \frac{5 r^8}{128 b^7} P_8(\cos\theta) + \frac{7}{256}\frac{r^{10}}{b^9} P_{10}(\cos\theta) - \dots + \right\}$$

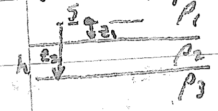
To obtain the potential on a horizontal plane a distance h above the disc, choose the radius r is such a way that

$$h = r\cos\theta$$

So that, $\phi = \phi(\theta) = \phi(r)$

Herrmann, R.B.

7. Consider a layer of electrical resistivity ρ_2 and thickness h sandwiched between two halfspaces of resistivity ρ_1 & ρ_3, respectively. ~~Show~~ Find the field for a source in the upper half space.



Absolute value signs because z can be negative in this half space.

$$\phi_1 = \frac{\rho_1 I}{4\pi}\int_0^\infty \left\{\left[e^{-\lambda|z|} + f_1(\lambda)e^{-\lambda z}\right] + g_1(\lambda)e^{\lambda z}\right\} J_0(\lambda r)\,d\lambda$$

$$\phi_2 = \frac{\rho_1 I}{4\pi}\int_0^\infty \left[e^{-\lambda z} + f_2(\lambda)e^{-\lambda z} + g_2(\lambda)e^{\lambda z}\right\} J_0(\lambda r)\,d\lambda$$

$$\phi_3 = \frac{\rho_1 I}{4\pi}\int_0^\infty \left[e^{-\lambda z} + f_3(\lambda)e^{-\lambda z} + g_3(\lambda)e^{\lambda z}\right] J_0(\lambda r)\,d\lambda$$

BOUNDARY CONDITIONS:

i) $\lim\limits_{z\to-\infty}\phi_1 = 0$

ii) $\lim\limits_{z\to\infty}\phi_3 = 0$

iii) $\frac{1}{\rho_1}\frac{\partial\phi_1}{\partial z} = \frac{1}{\rho_2}\frac{\partial\phi_2}{\partial z}$ at $z = z_1$

iv) $\phi_1 = \phi_2$ at $z = z_1$

v) $\phi_2 = \phi_3$ at $z = z_2$

vi) $\frac{1}{\rho_2}\frac{\partial\phi_2}{\partial z} = \frac{1}{\rho_3}\frac{\partial\phi_3}{\partial z}$ at $z = z_2$

vii) ϕ is finite everywhere except at the source ($r=0, z=0$)

Condition (i), requires $[illegible] + f_1(\lambda) = 0$

condition (ii), requires $g_3(\lambda) = 0$

(iii) $$\frac{1}{\rho_1}\left[-\lambda e^{-\lambda z_1} + \lambda g_1(\lambda) e^{\lambda z_1}\right] = \frac{1}{\rho_2}\left[-\lambda f_2(\lambda) e^{-\lambda z_1} + \lambda g_2 e^{\lambda z_1} - \lambda e^{-\lambda z_1}\right]$$

(iv) $$\left[e^{-\lambda z_1} + g_1(\lambda) e^{\lambda z_1}\right] = \left[f_2(\lambda) e^{-\lambda z_1} + g_2(\lambda) e^{\lambda z_1} + e^{-\lambda z_1}\right]$$

(v) $$\left[f_2(\lambda) e^{-\lambda z_2} + g_2(\lambda) e^{\lambda z_2} + e^{-\lambda z_2}\right] = \left[e^{-\lambda z_2} + f_3(\lambda) e^{-\lambda z_2}\right]$$

(vi) $$\frac{1}{\rho_2}\left[-\lambda f_2(\lambda) e^{-\lambda z_2} + \lambda g_2(\lambda) e^{\lambda z_2} - \lambda e^{-\lambda z_2}\right] = \frac{1}{\rho_3}\left[-\lambda f_3(\lambda) e^{-\lambda z_2} - \lambda e^{-\lambda z_2}\right]$$

On rearranging,

$$\begin{pmatrix} e^{\lambda z_1} & +\frac{\rho_1}{\rho_2} e^{-\lambda z_1} & -\frac{\rho_1}{\rho_2} e^{\lambda z_1} & 0 \\ e^{\lambda z_1} & -e^{-\lambda z_1} & -e^{\lambda z_1} & 0 \\ 0 & e^{-\lambda z_2} & e^{\lambda z_2} & -e^{-\lambda z_2} \\ 0 & -e^{-\lambda z_2} & +e^{\lambda z_2} & +\frac{\rho_2}{\rho_1} e^{-\lambda z_2} \end{pmatrix} \begin{pmatrix} g_1(\lambda) \\ f_2(\lambda) \\ g_2(\lambda) \\ f_3(\lambda) \end{pmatrix} = \begin{pmatrix} \left(\frac{\rho_2-\rho_1}{\rho_2}\right) e^{-\lambda z_1} \\ 0 \\ 0 \\ \left(\frac{\rho_3-\rho_2}{\rho_3}\right) e^{-\lambda z_2} \end{pmatrix}$$

Herrmann, R.B.

After some manipulation, this 4x4 system of equations takes the form.

$$
\begin{array}{c}
(1)\\ (2)\\ (3)\\ (4)
\end{array}
\begin{bmatrix}
1 & \frac{\rho_1}{\rho_2} e^{-2\lambda z_1} & -\frac{\rho_1}{\rho_2} & 0 \\
0 & 1 & K_{21} e^{2\lambda z_1} & 0 \\
0 & 0 & K_{21} e^{2\lambda z_1} - e^{2\lambda z_2} & +1 \\
0 & 0 & \frac{1+K_{32}}{K_{32}} e^{2\lambda z_2} & -1
\end{bmatrix}
\begin{bmatrix}
g_1(\lambda) \\ f_2(\lambda) \\ g_2(\lambda) \\ f_3(\lambda)
\end{bmatrix}
=
\begin{bmatrix}
\left(\frac{\rho_2 \cdot \rho_1}{\rho_2}\right) e^{-2\lambda z_1} \\
K_{21} \\
K_{21} \\
\left(\frac{\rho_3 - \rho_2}{\rho_3}\right) e^{-\lambda z_2}
\end{bmatrix}
$$

By adding (3) and (4), it is seen that.

$$g_2(\lambda) = \frac{K_{32}(1+K_{21}) e^{-2\lambda z_2}}{(K_{32}K_{21} e^{-2\lambda h} + 1)}$$

Using the value for $g_2(\lambda)$ in (4)

$$f_3(\lambda) = \frac{(1+K_{32})(1+K_{21})}{(K_{32}K_{21} e^{-2\lambda h} + 1)} - 1$$

Using the value for $g_2(\lambda)$ in (2)

$$f_2(\lambda) = \frac{K_{21}\left[1 - K_{32} e^{-2\lambda h}\right]}{\left[K_{32} K_{31} e^{-2\lambda h} + 1\right]}$$

Using the values for $f_2(\lambda)$ and $g_2(\lambda)$ in (iv)

$$f_2(\lambda) e^{-2\lambda z_1} + g_2(\lambda) = g_1(\lambda)$$

$$= \left[K_{21} e^{-2\lambda z_1} + K_{32} e^{-2\lambda z_2}\right] / \left[K_{32}K_{21} e^{-2\lambda h} + 1\right]$$

The potentials satisfying the boundary conditions are

$$\phi_1 = \frac{\rho_1 I}{4\pi}\left[\int_0^\infty\left\{e^{-\lambda z} + \frac{K_{21}e^{-2\lambda z_1} + K_{32}e^{-2\lambda z_2}}{\left[1+K_{21}K_{32}e^{-2\lambda h}\right]}e^{\lambda z}\right\}J_0(\lambda r)\,d\lambda\right]$$

$$\phi_2 = \frac{\rho_1 I}{4\pi}\int_0^\infty\left\{e^{-\lambda z} + \frac{K_{21}\left[1-K_{32}e^{-2\lambda h}\right]}{\left[1+K_{21}K_{32}e^{-2\lambda h}\right]}e^{-\lambda z} + \frac{K_{32}(1+K_{21})e^{-2\lambda z_2}}{\left[1+K_{21}K_{32}e^{-2\lambda h}\right]}e^{\lambda z}\right\}J_0(\lambda r)\,d\lambda$$

$$\phi_3 = \frac{\rho I}{4\pi}\int_0^\infty\left\{e^{-\lambda z} + \left\langle\frac{(1+K_{32})(1+K_{21})}{1+K_{21}K_{32}e^{-2\lambda h}}\right\rangle e^{-\lambda z} - e^{-\lambda z}\right\}J_0(\lambda r)\,d\lambda .$$

NOTING THAT $\frac{1}{R} = \frac{1}{\sqrt{r^2+z^2}} = \int_0^\infty e^{-\lambda z}J_0(\lambda r)\,d\lambda$

The solutions, upon simplification become

$$\phi_1 = \frac{\rho_1 I}{4\pi}\left[\frac{1}{R} + K_{21}\int_0^\infty \frac{e^{-\lambda(2z_1-z)}J_0(\lambda r)\,d\lambda}{1+K_{21}K_{32}e^{-2\lambda h}} + K_{32}\int_0^\infty \frac{e^{-\lambda(2z_2-z)}J_0(\lambda r)\,d\lambda}{1+K_{21}K_{32}e^{-2\lambda h}}\right]$$

$$\phi_2 = \frac{\rho_1 I}{4\pi}(1+K_{21})\left[\int_0^\infty \frac{K_{32}e^{-\lambda(2z_2-z)}J_0(\lambda r)\,d\lambda}{1+K_{21}K_{32}e^{-2\lambda h}} + \int_0^\infty \frac{e^{-\lambda z}J_0(\lambda r)\,d\lambda}{1+K_{21}K_{32}e^{-2\lambda h}}\right]$$

$$\phi_2 = \frac{\rho_1 I}{4\pi}(1+K_{32})(1+K_{21})\int_0^\infty \frac{e^{-\lambda z}J_0(\lambda r)\,d\lambda}{1+K_{21}K_{32}e^{-2\lambda h}}$$

where $K_{21} = \frac{\rho_2-\rho_1}{\rho_2+\rho_1}$ $K_{32} = \frac{\rho_3-\rho_2}{\rho_3+\rho_2}$

$$\phi_1 = \frac{\rho_1 I}{2\pi}\int_0^\infty \left[e^{-\lambda z} + f_1(\lambda) e^{-\lambda z} + g_1(\lambda) e^{\lambda z}\right] J_0(\lambda r)\, d\lambda$$

$$\phi_2 = \frac{\rho_1 I}{2\pi}\int_0^\infty \left[e^{-\lambda z} + f_2(\lambda) e^{-\lambda z} + g_2(\lambda) e^{+\lambda z}\right] J_0(\lambda r)\, d\lambda$$

$$\phi_3 = \frac{\rho_1 I}{2\pi}\int_0^\infty \left[e^{-\lambda z} + f_3(\lambda) e^{-\lambda z} + g_3(\lambda) e^{+\lambda z}\right] J_0(\lambda r)\, d\lambda$$

Boundary conditions.

i) $\frac{\partial \phi}{\partial z} = 0$ at $z=0$ (except $r=0$, $z=0$)

ii) $\phi_1 = \phi_2$ at $z = H = z_1$

iii) $\frac{1}{\rho_1}\frac{\partial \phi_1}{\partial z} = \frac{1}{\rho_2}\frac{\partial \phi_2}{\partial z}$ at $z = z_1$

iv) $\phi_3 = \phi_2$ at $z = z_2$

v) $\frac{1}{\rho_2}\frac{\partial \phi_2}{\partial z} = \frac{1}{\rho_3}\frac{\partial \phi_3}{\partial z}$ at $z = z_2$

vi) $\phi_1 = \phi_2 = \phi_3 = 0$ at $r = \infty$

vii) $\phi_3 = 0$ at $z = \infty$

viii) ϕ is finite everywhere except at the source ($r=0$, $z=0$)

Condition (vii) is satisfied if $g_3(\lambda) = 0$

Condition (i) is satisfied by $f_1(\lambda) = g_1(\lambda)$

Condition (vi) is satisfied by the values of ϕ_1, ϕ_2, ϕ_3

The equations resulting from application of the boundary conditions are as follows:

ii) $f_1(\lambda)\{e^{-\lambda z_1} + e^{\lambda z_1}\} = f_2(\lambda)e^{-\lambda z_1} + g_2(\lambda)e^{\lambda z_1}$

iii) $\frac{1}{\rho_1}\left[-\lambda f_1(\lambda)e^{-\lambda z_1} + \lambda f_1(\lambda)e^{\lambda z_1} - \lambda e^{-\lambda z_1}\right] = \frac{1}{\rho_2}\left[-\lambda f_2 e^{-\lambda z_1} + \lambda g_2 e^{\lambda z_1} - \lambda e^{-\lambda z_1}\right]$

iv) $f_2(\lambda)e^{-\lambda z_2} + g_2(\lambda)e^{\lambda z_2} = f_3(\lambda)e^{-\lambda z_2}$

v) $\frac{1}{\rho_2}\left[-\lambda f_2(\lambda)e^{-\lambda z_2} + \lambda g_2(\lambda)e^{\lambda z_2} - \lambda e^{-\lambda z_2}\right] = \frac{1}{\rho_3}\left[-\lambda f_3(\lambda)e^{-\lambda z_2} - \lambda e^{-\lambda z_2}\right]$

Upon rearranging,

$$\begin{pmatrix} (e^{-\lambda z_1} + e^{\lambda z_1}) & -e^{-\lambda z_1} & -e^{\lambda z_1} & 0 \\ (e^{-\lambda z_1} - e^{\lambda z_1}) & -\frac{\rho_1}{\rho_2}e^{-\lambda z_1} & +\frac{\rho_1}{\rho_2}e^{\lambda z_1} & 0 \\ 0 & e^{-\lambda z_2} & e^{\lambda z_2} & -e^{-\lambda z_2} \\ 0 & e^{-\lambda z_2} & -e^{\lambda z_2} & -\frac{\rho_2}{\rho_3}e^{-\lambda z_2} \end{pmatrix} \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \\ g_2(\lambda) \\ f_3(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\rho_1 - \rho_2}{\rho_2}e^{-\lambda z_1} \\ 0 \\ \frac{\rho_2 - \rho_3}{\rho_3}e^{-\lambda z_2} \end{pmatrix}$$

$$K_{21} = \frac{\rho_2-\rho_1}{\rho_2+\rho_1} \qquad K_{32} = \frac{\rho_3-\rho_2}{\rho_3+\rho_2}$$

$$\begin{array}{c|cccc|c}
1 & -\frac{(1+\frac{\rho_1}{\rho_2})}{2} & +\frac{(\frac{\rho_1}{\rho_2}-1)e^{2\lambda z_1}}{2} & 0 & \frac{\rho_1-\rho_2}{2\rho_2} \\
0 & [1-K_{21}e^{-2\lambda z_1}] & [K_{21}e^{2\lambda z_1}-1] & 0 & K_{21}[1+e^{-2\lambda z_1}] \\
0 & e^{-\lambda z_2} & e^{\lambda z_2} & -e^{-\lambda z_2} & 0 \\
0 & 2e^{-\lambda z_2} & 0 & -(\frac{\rho_2}{\rho_3}+1)e^{-\lambda z_2} & \frac{\rho_2-\rho_3}{\rho_3}e^{-\lambda z_2}
\end{array}$$

$$\begin{array}{rccccc}
 & 1 & \frac{-(1+\rho_1/\rho_2)}{2} & +\frac{(\frac{\rho_1}{\rho_2}-1)}{2}e^{2\lambda z_1} & 0 & \frac{\rho_1-\rho_2}{2\rho_2} \\
 & 0 & 1 & \frac{[K_{21}e^{2\lambda z_1}-1]}{[1-K_{21}e^{-2\lambda z_1}]} & 0 & \frac{K_{21}[1+e^{-2\lambda z_1}]}{[1-K_{21}e^{-2\lambda z_1}]} \\
3 \rightarrow & 0 & 1 & e^{2\lambda z_2} & -1 & 0 \\
4 \rightarrow & 0 & 1 & 0 & \frac{-(\frac{\rho_2}{\rho_3}+1)}{2} & \frac{(\frac{\rho_2}{\rho_3}-1)}{2}
\end{array}$$

$$\begin{array}{rccccc}
3' \rightarrow & 0 & 0 & \frac{[K_{21}-1][e^{2\lambda z_1}+1]}{[1-K_{21}e^{-2\lambda z_1}]} & +1 & \frac{K_{21}[1+e^{-2\lambda z_1}]}{[1-K_{21}e^{-2\lambda z_1}]} \\
4' \rightarrow & 0 & 0 & e^{2\lambda z_2} & (\frac{\rho_2}{\rho_3}-1)/2 & -\frac{(\frac{\rho_2}{\rho_3}-1)}{2} \\
4' \rightarrow & 0 & 0 & -\frac{2e^{2\lambda z_2}}{(\frac{\rho_2}{\rho_3}-1)} & -1 & +1 \\
 & 0 & 0 & \frac{(1+K_{32})}{K_{32}}e^{2\lambda z_2} & -1 & +1
\end{array}$$

$$2e^{-\lambda z_1} \qquad -\left(1+\frac{\rho_1}{\rho_2}\right)e^{-\lambda z_1} \qquad +\left(\frac{\rho_1}{\rho_2}-1\right)e^{\lambda z_1} \qquad 0 \qquad \frac{\rho_1-\rho_2}{\rho_2}\,e^{-\lambda z}$$

$$\boxed{1 \qquad -\frac{(1+\rho_1/\rho_2)}{2} \qquad +\frac{\left(\frac{\rho_1}{\rho_2}-1\right)}{2}e^{2\lambda z_1} \qquad 0 \qquad \frac{\rho_1-\rho_2}{2\rho_2}}$$

$$2e^{\lambda z_1} \qquad \left(\frac{\rho_1}{\rho_2}-1\right)e^{-\lambda z_1} \qquad -\left(\frac{\rho_1}{\rho_2}+1\right)e^{\lambda z} \qquad 0 \qquad -\frac{(\rho_1-\rho_2)}{2\rho_2}\,e^{-\lambda z_1}$$

$$\boxed{1 \qquad \frac{(\rho_1/\rho_2-1)}{2}e^{-2\lambda z_1} \qquad -\frac{\left(\frac{\rho_1}{\rho_2}+1\right)}{2} \qquad 0 \qquad -\frac{(\rho_1-\rho_2)}{2\rho_2}e^{-\lambda z_2}}$$

$$0 \qquad \frac{1}{2}\left[\left(1+\frac{\rho_1}{\rho_2}\right)+\left(\frac{\rho_1}{\rho_2}-1\right)e^{-2\lambda z_1}\right] \qquad -\frac{1}{2}\left[\left(\frac{\rho_1}{\rho_2}+1\right)+\left(\frac{\rho_1}{\rho_2}-1\right)e^{2\lambda z_1}\right] \qquad 0 \qquad \frac{-1}{2}\left[\left(\frac{\rho_1}{\rho_2}-1\right)+\left(\frac{\rho_1}{\rho_2}-1\right)e^{-2\lambda z_1}\right]$$

$$0 \qquad \left[1+\frac{(\rho_1-\rho_2)}{(\rho_1+\rho_2)}e^{-2\lambda z_1}\right] \qquad -\left[1+\frac{\rho_1-\rho_2}{\rho_1+\rho_2}e^{2\lambda z_1}\right] \qquad 0 \qquad 2e^{-2\lambda z_1}$$

$$0 \qquad \left[1-K_{21}e^{-2\lambda z_1}\right] \qquad \left[K_{21}e^{2\lambda z_1}-1\right] \qquad 0 \qquad \left(1+\cancel{z}\,e^{-2\lambda z_1}\right)K$$

$$
\left[\begin{array}{cccc|c}
(1+K_{21}) & -1 & -K_{21}\,e^{2\lambda z_1} & 0 & -K_{21} \\
0 & 1 & +\dfrac{K_{21}\,e^{2\lambda z_1}-1}{1-K_{21}\,e^{-2\lambda z_1}} & 0 & \dfrac{K_{21}\left[1+e^{-2\lambda z_1}\right]}{\left[1-K_{21}\,e^{-2\lambda z_1}\right]} \\
0 & 0 & \left[\dfrac{K_{21}\,e^{2\lambda z_1}-1}{1-K_{21}\,e^{-2\lambda z_1}}\right] - e^{2\lambda z_2} & +1 & \dfrac{K_{21}\left[1+e^{-2\lambda z_1}\right]}{\left[1-K_{21}\,e^{-2\lambda z_1}\right]} \\
0 & 0 & \dfrac{(1+K_{32})}{K_{32}}\,e^{2\lambda z_2} & -1 & +1
\end{array}\right]
$$

Dec 4. Problem Set #3.

1. Given $\nabla^2\phi = 0$ in the interior of a cylindrical region, $0 < z \le h$, $0 \le r < b$ and $\phi(r,0) = \phi(r,h) = 0$, $\phi(b,z) = Az(1 - z/h)$

Show that $\phi = \frac{8Ah}{\pi^3} \sum_0^\infty \frac{1}{(2m+1)^3} \frac{I_0\left(\frac{2m+1}{h}\pi r\right)}{I_0\left(\frac{2m+1}{h}\pi b\right)} \sin\frac{(2m+1)\pi z}{h}$

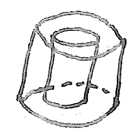
1, 3, 5, 7

$\frac{\pi z}{h}$

General solution is $\left.\begin{matrix} I_0(kr) \\ K_0(kr)\end{matrix}\right\} \begin{matrix}\sin \\ \cos\end{matrix}\, kz$

or $\left.\begin{matrix} J_0(kr) \\ Y_0(kr)\end{matrix}\right\} \left.\begin{matrix}\sinh \\ \cosh\end{matrix}\, kz\right\}$

2. Given $\nabla^2\phi = 0$ in the cylindrical region $a < r < b$, $0 < z < h$



$\phi(r,0) = \phi(r,h) = \phi(a,z) = 0$

$\phi(b,z) = C$

Show that $\phi = \frac{4C}{\pi} \sum_{m=0}^\infty \frac{\sin\left(\frac{2m+1}{h}\pi z\right)}{(2m+1)} \frac{\left[I_0\left(\frac{2m+1}{h}\pi r\right)K_0\left(\frac{2m+1}{h}\pi a\right) - I_0\left(\frac{2m+1}{h}\pi a\right)K_0\left(\frac{2m+1}{h}\pi r\right)\right]}{I_0\left(\frac{2m+1}{h}\pi b\right)K_0\left(\frac{2m+1}{h}\pi a\right) - I_0\left(\frac{2m+1}{h}\pi a\right)K_0\left(\frac{2m+1}{h}\pi b\right)}$

Irving & Mullineaux

12, 13, 15, 16. Chapter 3.

7. Find the potential in the source free spherical region $r < b$, given that $\phi = 0$ on $r = b$ except for the part cut out by a cone of angle α, on which $\phi = c$

$\phi(r,\theta) = \sum \left[1 - \cos\alpha\right] = \sum_0^\infty \left\{P_{n+1}(\cos\alpha) - P_{n-1}(\cos\alpha)\right\}\left(\frac{r}{b}\right)^n P_n$

Herrmann, Robert B
6ph 204

1. Given $\nabla^2\phi = 0$ in the interior of a cylindrical region, $0 < z < h$, $0 \le r < b$ and $\phi(r,0) = \phi(r,h) = 0$,

$$\phi(b,z) = Az(1 - z/h)$$

Show that

$$\phi = \frac{8Ah}{\pi^3} \sum_0^\infty \frac{1}{(2n+1)^3} \frac{I_0\left(\frac{2n+1}{h}\pi r\right)}{I_0\left(\frac{2n+1}{h}\pi b\right)} \sin\left(\frac{2n+1}{h}\pi z\right)$$

The solutions to Laplace's equation is cylindrical coordinates with no θ dependence are

$$\phi = \left.\begin{matrix} I_0(kr) \\ K_0(kr) \end{matrix}\right\} \left.\begin{matrix} \sin \\ \cos \end{matrix}\, kz\right\}$$

or

$$\phi = \left.\begin{matrix} J_0(kr) \\ Y_0(kr) \end{matrix}\right\} \left.\begin{matrix} \sinh \\ \cosh \end{matrix}\, kz\right\}$$

The boundary condition $\phi(r,0) = \phi(r,h) = 0$ eliminates the second solution. The existence of the solution at $r = 0$, causes the $K_0(kr)$ term to be dropped. as also the $\cos kz$ term.

$$\therefore \quad \phi = A\, I_0(kr) \sin kz$$

$$\phi(r,h) = A I_0(kr) \sin kh = 0$$

$$\text{or} \quad k = \frac{n\pi}{h}$$

$$\phi(r,z) = \sum A_n I_0\left(\frac{n\pi r}{h}\right) \sin\left(\frac{n\pi z}{h}\right).$$

$$\phi(b,z) = Az\left(1 - \frac{z}{h}\right) = \sum A_n I_0\left(\frac{n\pi b}{h}\right) \sin \frac{n\pi z}{h}$$

Because of the orthogonality of $\sin\left(\frac{n\pi z}{h}\right)$,

$$\int_{z=0}^{z=h} Az\left(1-\frac{z}{h}\right)\sin\left(\frac{n\pi z}{h}\right)dz = A_n I_0\left(\frac{n\pi b}{h}\right)\int_0^h \sin^2\left(\frac{n\pi z}{h}\right)dz$$

Upon integration, it is found that

$$A_n = \frac{4Ah}{n^3\pi^3 I_0\left(\frac{n\pi b}{h}\right)}\left[1-\cos n\pi\right]$$

For n even A_n is zero

For n odd,

$$A_{2n+1} = \frac{8Ah}{(2n+1)^3\pi^3}\,\frac{}{I_0\left(\frac{n\pi b}{h}\right)}$$

$$\therefore\quad \phi = \frac{8Ah}{\pi^3}\sum_0^{\infty}\frac{1}{(2n+1)^3}\,\frac{I_0\left(\frac{2n+1}{h}\pi r\right)}{I_0\left(\frac{2n+1}{h}\pi b\right)}\sin\left(\frac{2n+1}{h}\pi z\right)$$

2. Given $\nabla^2\phi=0$ in the cylindrical region $a<r<b$, $0<z<h$, $\phi(r,0)=\phi(r,h)=\phi(a,z)=0$
$\phi(b,z)=C$.

Find ϕ.

The solution is.

$$\phi = \{A I_0(Kr) + B K_0(Kr)\}\sin Kz$$

From $\phi(r,h)=0$, $K_n = \frac{n\pi}{h}$

$$\phi(a,z) = \Sigma\{A_n I_0(K_n a) + B_n K_0(K_n a)\}\sin Kz = 0$$

or $B_n = -\dfrac{A_n I_0(K_n a)}{K_0(K_n a)}$

RBH

So, $\phi = \sum_{n=0}^{\infty} \frac{A_n}{K_0(k_n a)} \left\{ K_0(k_n a) I_0(k_n r) - I_0(k_n a) K_0(k_n r) \right\} \sin k_n z$

$\phi(b, z) = c$

$$c = \sum \frac{A_n}{K_0(k_n a)} \left\{ K_0(k_n a) I_0(k_n b) - I_0(k_n a) K_0(k_n b) \right\} \sin k_n z$$

By orthogonality condition,

$$c \int_0^h \sin \frac{n\pi z}{h} dz = \sum_0^{\infty} \frac{A_n}{K_0(k_n a)} \left\{ \qquad \right\} \int_0^h \sin^2 \frac{n\pi z}{h} dz$$

$$A_n = \frac{-2c}{n\pi} \frac{K_0(k_n a)}{(I_0(k_n b) K_0(k_n a) - I_0(k_n a) K_0(k_n b))} [\cos n\pi - \cos 0]$$

$A_{2n} = 0$ since $[\cos 2n\pi - \cos 0] = 0$

$A_{2n+1} = \frac{4c}{(2n+1)\pi} \left\{ \qquad \right\}.$

Ans 2. $\phi = \frac{4c}{\pi} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{2n+1}{h}\pi z\right)}{(2n+1)} \left[\frac{I_0\left(\frac{2n+1}{h}\pi r\right) K_0\left(\frac{2n+1}{h}\pi a\right) - I_0\left(\frac{2n+1}{h}\pi a\right) K_0\left(\frac{2n+1}{h}\pi r\right)}{I_0(k_{2n+1} b) K_0(k_{2n+1} a) - I_0(k_{2n+1} a) K_0(k_{2n+1} b)} \right]$

RBH

[illegible] Chapter 8.

12(a) Expand the function $f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}$ as a series of Legendre polynomials and evaluate the first five coefficients, using the explicit form of each Legendre polynomial.

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

$$\int_{-1}^{+1} f(x) P_m(x)dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x) P_m(x)dx.$$

$$= \begin{cases} 0 & \text{for } n \ne m \\ 2a_m/(2m+1) & \text{for } n = m. \end{cases}$$

$$a_m = \left(\frac{2m+1}{2}\right) \int_{-1}^{1} f(x) P_m(x)dx.$$

$$a_0 = \frac{1}{2}\int_{-1}^{0} 0 \cdot P_0(x)dx + \frac{1}{2}\int_0^1 x\, P_0(x)dx = 1/4$$

$$a_1 = \frac{3}{2}\int_{-1}^{0} 0 \cdot P_1(x)dx + \frac{3}{2}\int_0^1 x^2 dx = 1/2$$

$$a_2 = \frac{5}{2}\int_{-1}^{0} 0 \cdot P_2(x)dx + \frac{5}{2}\int_0^1 x \cdot \frac{1}{2}(3x^2-1)dx = \frac{5}{16}$$

$$a_3 = \frac{7}{2}\int_{-1}^{0} 0 \cdot P_3(x)dx + \frac{7}{2}\int_0^1 x \cdot \frac{1}{2}(5x^3-3x)dx = 0$$

$$a_4 = \frac{9}{2}\int_{-1}^{0} 0 \cdot P_4(x)dx + \frac{9}{2}\int_0^1 x \cdot \frac{1}{8}(35x^4 - 9x^2 - 3)dx = -3/32$$

$$a_5 = \frac{11}{2}\int_{-1}^{0} 0 \cdot P_5(x)dx + \frac{11}{2}\int_0^1 x \cdot P_5(x)\, dx = 0$$

b) Deduce the coefficient of $P_n(x)$ in the above expansion using Rodrigue's formula.

Since $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \qquad n \geq 1$

$$(2n+1)\int_0^1 x P_n(x)dx = \int_0^1 \{x P'_{n+1}(x) - xP'_{n-1}(x)\} dx$$

$$= \left[x P_{n+1}(x) - x P_{n-1}(x) - \int P_{n+1}(x)dx + \int P_{n-1}(x)dx \right]_0^1$$

$$= \int_0^1 P_{n-1}(x)dx - \int_0^1 P_{n+1}(x)dx$$

$$= \frac{1}{2n-1}\left[P_n(x) - P_{n-2}(x)\right]_0^1 - \frac{1}{2n+3}\left[P_{n+2}(x) - P_n(x)\right]_0^1$$

$$= -\left\{\frac{1}{2n-1}\left[P_n(0) - P_{n-2}(0)\right] - \frac{1}{2n+3}\left[P_{n+2}(0) - P_n(0)\right]\right\}$$

For n odd, $P_n(0) = 0$.

See below. for proof. $\Rightarrow$ For n even, $P_n(0) = \frac{1}{2^n}(-1)^n \frac{n!}{(n/2)!\,(n/2)!}$

Change of notation : Let $2m = n \qquad m = 1, 2, \ldots$

$$-(4m+1)\int_0^1 x P_{2m}(x)dx = \frac{1}{4m-1}\left[P_{2m}(0) - P_{2m-2}(0)\right] - \frac{1}{4m+3}\left[P_{2m+2}(0) - P_{2m}(0)\right]$$

R B4

Now.

$$P_{2m}(0) = \frac{1}{2^{2m}}(-1)^m \frac{2m!}{(m!)^2}$$

$$P_{2m+2}(0) = +P_{2m}(0) \cdot \frac{1}{4}(-1)^1 \frac{(2m+2)(2m+1)}{(m+1)(m+1)}$$

$$P_{2m-2}(0) = P_{2m}(0) \cdot 4(-1)^{-1} \frac{m^2}{2m(2m-1)}$$

$$P_{2m}(0) - P_{2m-2}(0) = P_{2m}(0)\left[1 + \frac{4m^2}{2m(2m-1)}\right] = P_{2m}(0)\left[\frac{4m-1}{2m-1}\right]$$

$$P_{2m+2}(0) - P_{2m}(0) = -P_{2m}(0)\left[\frac{1}{4}\frac{(2m+2)(2m+1)}{(m+1)(m+1)} + 1\right] = -P_{2m}(0)\frac{4m+3}{2(m+1)}$$

$\therefore$

$$(4m+1)\int_0^1 x P_{2m}(x)dx = (-1)^1 P_{2m}(0)\left[\frac{1}{2m-1} + \frac{1}{2(m+1)}\right] = (-1)^1 P_{2m}(0)\left[\frac{4m+1}{2(m-1)(2(m+1)}\right]$$

$$= \frac{1}{2^{2m+1}}(-1)^{m+1}\frac{(2m)!}{(m!)^2}\frac{4m+1}{(2m-1)(m+1)}$$

$$= \frac{1}{2^{2m+1}}(-1)^{m+1}\frac{2\cancel{m}\cancel{(2m-1)}(2m-2)!\,(4m+1)}{\cancel{m}(m-1)!(m+1)m!\,\cancel{(2m-1)}}$$

$$= \frac{1}{2^{2m}}(-1)^{m+1}\frac{(2m-2)!\,(4m+1)}{(m-1)!\,(m+1)!}$$

The coefficients of $P_{2m}(x)$ in the expansion are

$$a_{2m} = \frac{4m+1}{2}\int_0^1 x P_{2m}(x)dx = \frac{1}{2^{2m+1}}(-1)^{m+1}\frac{(2m-2)!\,(4m+1)}{(m-1)!\,(m+1)!}$$

Ans #12 Coefficient of $P_{2m+1} = 0$
Coefficient of $P_{2n} =$

$$a_n = \frac{(-1)^{n+1}\,(2n-2)!\,(4n+1)}{2^{2n+1}\,(n+1)!\,(n-1)!}, \quad n \geq 1.$$

Proof that $P_n(0) = \frac{1}{2^n}(-1)^{n/2}\frac{n!}{(n/2)!(n/2)!}$:

By Rodrigues formula,

$$P_n(x) = \frac{1}{2^n\,n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

$(x^2-1)^n$ can be expanded using the binomial expansion,

$$(x^2-1)^n = \sum_{r} \frac{n!}{(n-r)!\,r!}\,x^{2(n-r)}(-1)^r$$

For n even $P_n(x)$ contains the terms

$$a_n x^{2n} + a_{n-1}x^{2n-2} + \dots + a_0 x^0$$

and $P_n(0) = a_0$

So it is just necessary to find the coefficient of x^0

To get $a_0 x^0$, the term $b_0 x^n$ in Rodrigues formula is differentiated n times.

or $(x^2-1)^n = x^{2n} + \dots + b_0 x^n + \dots \; c x^0$

$P_n(0) = n!\,b_0$ where $b_0 = \frac{n!}{(n/2)!(n/2)!}$

R BH

Therefore.

$$\underline{\underline{P_n(0) = \frac{1}{2^n}(-1)^{n/2}\frac{n!}{(n/2)!(n/2)!} \quad \text{for } n \text{ even.}}}$$

13. Show that.

(1) $\int_{-1}^{1} x\, P_n(x) P_{n-1}(x)\, dx = 2n/(4n^2-1)$

By equation 11.1.4.

$$(2n+1)\, x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$

$$(2n+1)\int_{-1}^{1} x\, P_n(x) P_{n-1}(x)\, dx = \int_{-1}^{+1} (n+1) P_{n+1}(x) P_{n-1}(x)\, dx + n\int_{-1}^{1} P_{n-1}^2(x)\, dx$$

$$= \frac{2n}{2n-1}$$

$$\underline{\underline{\int_{-1}^{1} x\, P_n(x) P_{n-1}(x)\, dx = \frac{2n}{(2n-1)(2n+1)} = \frac{2n}{(4n^2-1)}}}$$

(2) $\int_{-1}^{1} \{\cosh 2u - x\}^{-1/2} P_n(x)\, dx = 2\sqrt{2}\, \exp\frac{\{-(2n+1)u\}}{2n+1}$ for $u > 0$.

By some rearrangement, it can be shown that

$$\{\cosh 2u - x\}^{-1/2} = \sqrt{2}\, e^{-u}\left[1 - 2x e^{-2u} + e^{-4u}\right]^{-1/2}$$

$$= \sqrt{2}\, e^{-u} \sum_{n=0}^{\infty} e^{-2nu} P_n(x)$$

Integrating and using the orthogonality of $P_n(x)$,

$$\int_{-1}^{1}\{\cosh 2u - x\}^{-1/2} P_n(x)dx = \sqrt{2}\, e^{-u}\sum_{m=0}^{\infty}\int_{-1}^{1} e^{-2mu} P_m(x) P_n(x)dx$$

$$= \frac{2\sqrt{2}}{2n+1} e^{-(2n+1)u}$$

Ans 13(2) $\displaystyle\int_{-1}^{1}\{\cosh 2u - x\}^{-1/2} P_n(x)dx = \frac{2\sqrt{2}}{2n+1} e^{-(2n+1)u}$

13.(3) $\displaystyle\int_0^1 x^2 P_{n+1}(x)P_{n-1}(x)dx = n(n+1)/\{(4n^2-1)(2n+3)\}$

By formula (11.14),

$$(2n+3)\,x P_{n+1}(x) = (n+2)P_{n+2}(x) + (n+1)P_n(x)$$
$$(2n-1)\,x P_{n-1}(x) = n P_n(x) + (n-1)P_{n-2}(x)$$

$$(2n-1)(2n+3)x^2 P_{n+1}(x)P_{n-1}(x) = n(n+1)P_n(x)P_{n+2}(x) + n(n+1)P_n^2(x) + (n+2)(n-1)P_{n+2}(x)P_{n-2}(x) + (n+1)(n-1)P_n(x)P_{n-2}(x)$$

Now note that

$P_n(x)P_{n+2}(x)$, $P_n^2(x)$, $P_{n+2}(x)P_{n-2}(x)$, $P_n(x)P_{n-2}(x)$ are all even functions.

Since $P_n(-x) = (-1)^n P_n(x)$

RBW

Denote one of these even functions by $E(x)$
then
$$\int E(x)\,dx = K(x) \text{ an odd function.}$$
or. $K(-x) = -K(x)$

So,
$$\int_{-x}^{x} E(x)\,dx = K(x) - K(-x) = 2K(x)$$

Since $K(x)$ will be an odd function,
$$K(0) = 0$$

So
$$\int_{-x}^{x} E(x)\,dx = 2K(x) = 2\int_0^x E(x)\,dx$$
$$\therefore \quad \int_0^1 E(x)\,dx = \tfrac{1}{2}\int_{-1}^{1} E(x)\,dx.$$

$$(2n-1)(2n+3)\int_0^1 x^2 P_{n+1}(x)P_{n-1}(x)\,dx = \tfrac{1}{2}n(n+1)\int_{-1}^{1} P_n(x)P_{n+2}(x)\,dx$$
$$+\ \tfrac{1}{2}n(n+1)\int_{-1}^{1} P_n^2(x)\,dx + \tfrac{1}{2}(n+2)(n-1)\int_{-1}^{1} P_{n+2}(x)P_{n-2}(x)\,dx$$
$$+\ \tfrac{1}{2}(n+1)(n-1)\int_{-1}^{1} P_n(x)\cdot P_{n-2}(x)\,dx$$
$$= \tfrac{1}{2}(n)(n+1)\int_{-1}^{1} P_n^2(x)\,dx = \frac{n(n+1)}{(2n+1)}$$

$$\therefore \quad \int_0^1 x^2 P_{n+1}(x)P_{n-1}(x)\,dx = \frac{n(n+1)}{(4n^2-1)(2n+3)}$$

15. A conducting spherical shell of radius a is placed in a uniform electric field E_0 in free space and maintained at zero potential (cf. Example 9 of text). The potential ϕ at any point in space then satisfies the equation

$$\nabla^2 \phi = 0$$

with the boundary conditions that ϕ is continuous and finite at all points, and $\phi = 0$ on the sphere; also $\phi \to -E_0 z$ at infinity, the z axis being taken through the centre of the sphere in the direction of the electric field. Show that

$$\phi = 0, \quad r \leq a.$$

and $$\phi = -E_0 r \cos\theta + (E_0 a^3/r^2)\cos\theta \quad r \geq a$$

Deduce the potential if the sphere is given a total charge q, and deduce that the charge density σ, given by $-\frac{1}{4\pi} \partial\phi/\partial r|_{r=a}$ on the sphere is everywhere positive if $q > 3a^2 E_0$.

A solution valid for $0 \leq r \leq a$ is.

$$\phi = \sum B_n r^n P_n(\cos\theta)$$

The solution for $a \leq r$ is.

$$\phi = \sum A_n r^{-(n+1)} P_n(\cos\theta). + C_n r^n P_n(\cos\theta)$$

R BH

Applying the B.C. for large r, $\phi \to -E_0 z = -E_0 r\cos\theta$

$\therefore \quad C_0 = C_2 = C_3 = \cdots C_n = 0$

$C_1 = -E_0.$

Applying B.C. $\phi(r=a) = 0$. using orthogonality feature of $P_n(\theta)$, we get $A_0 = A_2 = A_3 = \cdots = A_n = 0$.

$$\frac{A_n}{a^2} - E_0 a = 0$$

$$A_n = +E_0 a^3$$

$\therefore$ for $r \geq a \quad \phi = +E_0\left(\frac{a^3}{r^2} - r\right)\cos\theta$

For the interior solution, it is required that $\phi(a) = 0$ for all θ. This can only occur for $B_n = 0$.

$$\phi = 0 \quad r \leq a$$

$$\phi = -E_0\left(r - a^3/r^2\right)\cos\theta \qquad r \geq a$$

By Maxwells equations,

$$\nabla\cdot\vec{D} = +\rho_0$$

or $\iint \vec{D}\cdot\hat{n}\, d\vec{s} = q_{enclosed}$. around the surface of spherical conductor

, or $\hat{n}\cdot\vec{D}_n = \sigma$

$\sigma = \vec{D}_n\cdot\hat{n} = \epsilon_0 \frac{\partial\phi}{\partial r}\Big|_{r=a}$

$\therefore \quad \sigma = -\epsilon_0 E_0 \left(1 - \frac{2a^3}{r^3}\right)\cos\theta$

$$q = \iint \sigma\, ds = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sigma \; a^2 \sin\theta\, d\phi\, d\theta$$

$$= \epsilon_0 E_0 a^2 2\pi \int_0^{\pi} \cos\theta \sin\theta\, d\theta = 0$$

The total charge induced on one half the surface is

$$q = \epsilon_0 E_0 2\pi a^2 \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta = \epsilon_0 E_0 \pi a^2$$

If the sphere is given a total charge q, the potential is.

$$\phi = \frac{q}{4\pi\epsilon_0 r} - E_0 r\cos\theta + E_0 (a^3/r^2)\cos\theta \quad \text{for } r \geq a.$$

The ~~induced~~ charge density $= \sigma = |\vec{D}\cdot\vec{n}| = |\epsilon_0 \vec{E}\cdot n|$

$$\sigma = \epsilon_0 E_r = -\epsilon_0 \frac{\partial\phi}{\partial r}\Big|_{r=a}$$

$$= \epsilon_0 \left\{ \frac{q}{4\pi\epsilon_0} \frac{1}{a^2} + E_0 \cos\theta + 2E_0 \left(\frac{a^3}{a^3}\right) \cos\theta \right\}$$

$$= \epsilon_0 \left\{ \frac{+q}{4\pi\epsilon_0} \frac{1}{a^2} + 3E_0 \cos\theta \right\}$$

since $-1 \leq \cos\theta \leq +1$

σ is everywhere positive

for $\frac{q}{4\pi\epsilon_0} \frac{1}{a^2} > 3E_0$

or $q > 3E_0 a^2 4\pi\epsilon_0$

RBA

Ans 15. $\phi = 0 \quad r \le a$

$\phi = -E_0 r \cos\theta + E_0 a^3/r^2 \cos\theta \quad , r \ge a$

giving a charge q on sphere, surface charge density is everywhere positive for.

$q > 3 E_0 a^2 \cdot 4\pi\epsilon_0$ in MKS. units.

16. If the conducting sphere of the previous problem is replaced by a non-conducting sphere of dielectric constant K, the boundary conditions with ϕ_1 and ϕ_2 denoting the potentials inside and outside the sphere, respectively, are, with $q = 0$;

(a) ϕ_1 and ϕ_2 are finite and continuous at all points.

(b) $K \frac{\partial \phi_1}{\partial r} = \partial \phi_2 / \partial r$ for $r = a$

(c) $\phi_2 \to -E_0 z$ at infinity.

Show that $\phi_1 = -3 E_0 z / (K+2)$

$\phi_2 = -E_0 z \left\{ 1 - \frac{a^3}{r^3} \frac{(K-1)}{(K+2)} \right\}$

The interior and exterior solutions to $\nabla^2 \phi = 0$ are.

$\phi_1 = \sum_0^\infty A_n r^n P_n(\cos\theta)$

$\phi_2 = \sum_0^\infty \{ B_n r^n + C_n r^{-(n+1)} \} P_n \cos\theta.$

From B.C. (c) it is seen that $B_0 = B_2 = B_3 = \dots = B_n = 0, \; B_1 = -E_0$

So, $\phi_1 = \sum_0^\infty A_n r^n P_n(\cos\theta)$

$\phi_2 = \sum_0^\infty C_n r^{-(n+1)} P_n(\cos\theta) - E_0 r\cos\theta$

B.C.

(a). $\phi_1 = \phi_2$; at $r = a$

$$\sum_0^\infty A_n a^n P_n(\cos\theta) = \sum_0^\infty C_n a^{-(n+1)} P_n(\cos\theta) - E_0 a P_1(\cos\theta)$$

From the orthogonality of $P_n(\cos\theta)$ it is found that.

$$A_0 = \frac{1}{a} C_0$$
$$A_1 a = \frac{1}{a^2} C_1 - E_0 a$$
$$A_n a^n = C_n a^{-(n+1)} \quad \text{for } n \geq 2$$

B.C. (b) $\quad K \frac{\partial \phi_1}{\partial r} = \frac{\partial \phi_2}{\partial r} \Big|_{r=a}$.

$$K \sum_0^\infty n A_n a^{n-1} P_n(\cos\theta) = \sum_0^\infty -(n+1) C_n a^{-(n+2)} P_n(\cos\theta) - E_0 P_1(\cos\theta)$$

Again by the orthogonality principle.

$$0 = -C_0 a^{-2}$$
$$K A_1 = -2 C_1 a^{-3} - E_0$$
$$n K A_n a^{n-1} = -(n+1) C_n a^{-(n+2)} \quad \text{for } n \geq 2$$

Solving these two sets of boundary conditions

It is found that $\quad A_0 = C_0 = 0$

$$(2+K) A_1 = -3 E_0$$

RBH

$(K-2)\,c_1 = (1+K)E_0 a^3$

and for $n \geq 2$ $n = -(n+1)$; But this is not true so $A_n = C_n = 0$ for $n \geq 2$.

$$\therefore \quad \phi_1 = -3E_0\, r\cos\theta/(K+2)$$

$$\phi_2 = -E_0\, r\cos\theta\left\{1 - \frac{a^3(K-1)}{r^3(K+2)}\right\}$$

RBH

Worked on blackboard #7

Find the potential in the source free region $r<b$ given that $\phi=0$ on $r=b$ except for the part on the surface on the sphere cut out by a cone of angle α, on which $\phi=C$.

The solution to this problem is.

$$\phi = \sum A_n \left(\frac{r}{b}\right)^n P_n(\cos\theta)$$

$$\text{B.C.}\quad \left.\begin{array}{ll} \phi = C & \text{for } 0\le\theta<\alpha \\ \phi=0 & \text{for } \alpha<\theta\le\pi \end{array}\right\} \text{at } r=b.$$

$$\therefore \int_{\theta=0}^{\theta=\pi} \phi(b,\theta)\, P_n(\cos\theta)\sin\theta\, d\theta = A_n\left(\frac{2}{2n+1}\right)$$

$$\int_{\theta=0}^{\alpha} + C\, P_n(\cos\theta)\sin\theta\, d\theta + \int_{\alpha}^{\pi} +0\cdot P_n(\cos\theta)\sin\theta\, d\theta = A_n\left(\frac{2}{2n+1}\right)$$

for $n=0$, $P_n(\cos\theta) = 1$.,

$$A_0 = \frac{-1}{2} C(\cos\alpha - 1)$$

$$2A_n = \underline{2n+1}\int_{\cos\alpha}^{1} C\, P_n(x)\,dx = C\left\{P_{n+1}(x) - P_{n-1}(x)\right\}\Big|$$

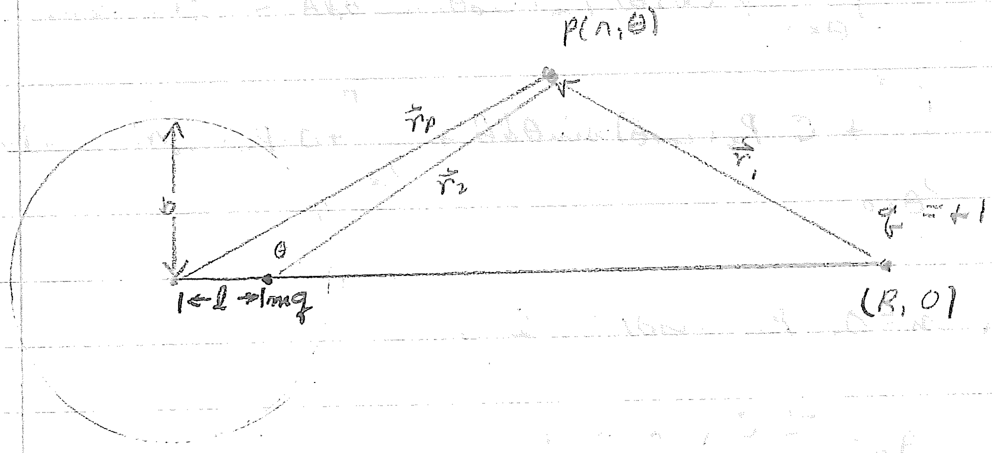
$$= C\left[P_{n+1}(1) - P_{n-1}(1) - P_{n+1}(\cos\alpha) + P_{n-1}(\cos\alpha)\right]$$

So,

$$\phi(r,\theta) = \frac{c}{2}\Big[1 - \cos\alpha - \sum_{1}^{\infty}\left\{P_{n+1}(\cos\alpha) - P_{n-1}(\cos\alpha)\right\}\left(\frac{r}{b}\right)^n P_n(\cos\theta)$$

8. A conducting sphere of radius b is brought into the neighborhood of a unit positive point charge. And then the sphere is grounded. Show that the potential due to the induced charge on the conductor at any point outside the conductor is:

$$\phi = \frac{-b}{Rr}\left[P_0(\cos\theta) + \frac{b^2}{Rr}P_1(\cos\theta) + \frac{b^4}{(Rr)^2}P_2(\cos\theta) + \dots\right]$$



This problem will be solved by the image method. A negative charge $-mq$ will be induced on the sphere and it is located at $(l, 0)$

RB11

The potential at the point P is.

$$V_p = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{m}{r_2}\right)$$

where $r_1 = (r_p^2 + R^2 - 2r_pR\cos\theta)^{1/2}$
$r_2 = (r_p^2 + l^2 - 2r_pl\cos\theta)^{1/2}$.

The boundary condition is that.

$V_p = 0$ at $r_p = b$ for all θ.

or, $\left(\frac{1}{r_1} = \frac{m}{r_2}\right)_{r=b}$

Squaring this last equation and equating the coefficients of 1 and $\cos\theta$, It is found that

$$l = b^2/R$$
$$m = b/R$$

The potential due to the induced charge is.

$$V_p = \frac{1}{4\pi\epsilon_0}\frac{-mq}{(r_p^2+l^2-2r_pl\cos\theta)^{1/2}}$$

$$= \frac{-mq}{4\pi\epsilon_0}\frac{1}{r_p}\frac{1}{\left[1+\frac{l^2}{r_p^2}-2\frac{l}{r_p}\cos\theta\right]^{1/2}} = \frac{-mq}{4\pi\epsilon_0}\frac{1}{r_p}\sum_0^\infty\left(\frac{l}{r_p}\right)^n P_n(\cos\theta)$$

or using the definitions of m and l.

$$V_p = -\frac{bq}{4\pi\epsilon_0 rR}\sum_0^\infty \left(\frac{b^2}{Rr}\right)^m P_m(\cos\theta) \quad \text{in MKS system.}$$

9. Find the gravitational potential of a homogeneous oblate spheroid both at points outside and within the spheroid (Poisson's equation inside).

The location of a point in orthogonate oblate spheroidal coordinates is (η, θ, ψ). The ~~ellipse~~ oblate spheroid is designated by η_0.

Outside the oblate spheroid.

$$\nabla^2\phi = 0.$$

WITHIN THE SPHEROID

$$\nabla^2\phi = -Q$$

Since there is axial symmetry,

$$\nabla^2\phi = \frac{1}{a^2(\cosh^2\eta - \sin^2\theta)}\left\{\frac{\partial^2\phi}{\partial\eta^2} + \tanh\eta\frac{\partial\phi}{\partial\eta} + \frac{\partial^2\phi}{\partial\theta^2} + \cot\theta\frac{\partial\phi}{\partial\theta}\right\}$$

RBH

The exterior solution is of the form

$$\phi_E = \sum_{n=0}^{\infty} \left[A_n P_n(i \sinh\eta) + B_n Q_n(i\sinh\eta) \right] P_n(\cos\theta)$$

The interior solution is.

$$\phi_I = \sum_{n=0}^{\infty} \left[C_n P_n(i\sinh\eta) + D_n Q_n(i\sinh\eta) \right] P_n(\cos\theta)$$
$$+ f_1(\eta) + f_2(\theta)$$
$$\cancel{- \frac{a^2 Q}{12}\left[\sinh\eta(\cosh^2\eta + \tfrac{1}{2}) - a\tfrac{3}{5}P_3(\cos\theta) + \tfrac{9}{10}P_1(\cos\theta)\right]}$$

The interior solution is found by separation and.

$$\frac{\partial^2\phi}{\partial\eta^2} + \tanh\frac{\partial\phi}{\partial\eta} + a^2 Q\cosh^2\eta = 0 \; ; \; \frac{1}{\cosh\eta}\left\{\frac{\partial}{\partial\eta}\left(\cosh\eta\frac{\partial\phi}{\partial\eta}\right)\right\} + a^2 Q\cosh^2\eta = 0$$

$$\frac{\partial^2\phi}{\partial\theta^2} + \cot\theta\frac{\partial\phi}{\partial\theta} - a^2 Q\sin^2\theta = 0 \; ; \; \frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)\right\} - a^2 Q\sin^2\theta = 0$$

These two are solved by letting
$$\phi = \Phi + f_1(\eta) + f_2(\theta)$$
where Φ is a solution of $\nabla^2\Phi = 0$

The particular solutions are

$$f_1(\eta) = -\frac{a^2 Q}{12}\left(\cosh^2\eta + \sinh^2\eta\right) - \frac{2}{3}a^2 Q \ln\cosh\eta$$

$$f_2(\theta) = -\frac{a^2 Q}{4}\left(\sin^2\theta - \cos^2\theta\right) - 2a^2 Q\ln\sin\theta.$$

However, since $f_2(\theta)$ is undefined for all θ, it will not be used.

Also, since $P_n(i\sinh\eta)$ goes to infinity for large arguments, the solutions are

$$V_E = \sum_0^\infty B_n Q_n(i\sinh\eta)\, P_n(\cos\theta)$$

$$V_I = \sum_0^\infty \left[C_n P_n(i\sinh\eta) + D_n Q_n(i\sinh\eta)\right] P_n(\cos\theta)$$

$$- \frac{a^2 Q}{12}(\cosh^2\eta + \sinh^2\eta) - \frac{2}{3}a^2 Q \ln\cosh\eta$$

B.C. (1) $V_E = V_I$ at $\eta = \eta_0$

$$\sum_0^\infty B_n Q_n(i\sinh\eta_0) P_n(\cos\theta) = \sum_0^\infty \left[C_n P_n(i\sinh\eta_0) + D_n Q_n(i\sinh\eta_0)\right] P_n(\cos\theta)$$

$$\left[- \frac{a^2 Q}{12}(\cosh^2\eta_0 + \sinh^2\eta_0) - \frac{2}{3}a^2 Q \ln\cosh\eta_0\right] P_0(\cos\theta)$$

By orthogonality principle,

$$B_0 Q_0(i\sinh\eta_0) = C_0 P_0(i\sinh\eta_0) + D_0 Q_0(i\sinh\eta_0) - a^2 Q\left[\frac{1}{12}(\cosh^2\eta_0 + \sinh^2\eta_0) + \frac{2}{3}\ln\cosh\eta_0\right]$$

$$B_l Q_l(i\sinh\eta_0) = C_l P_l(i\sinh\eta_0) + D_l Q_l(i\sinh\eta_0) \quad \text{for } l \geq 1$$

Since for the interior problem, $Q_n(iy)$ is well defined, the $Q_n(iy)$ can be expressed is terms of a Legendre polynomial expansion, and the interior solution can be written is a reduced form

$$V_I = \sum_0^\infty G_n P_n(i\sinh\eta)\, P_n(\cos\theta) - \frac{a^2 Q}{12}(\cosh^2\eta + \sinh^2\eta) - \frac{2}{3}a^2 Q \ln\cosh\eta)$$

RBH

Hence the boundary condition equation takes the form.

$$\cancel{\tfrac{F}{4}}\ B_0\, Q_0(i\sinh\eta_0) = \cancel{G_0 P_0(i\sinh\eta_0)}\ \cancel{+ G_2 P_2(i\sinh\eta_0)}$$
$$- a^2 Q\left[\tfrac{1}{12}(\cosh^2\eta_0 + \sinh^2\eta_0) + \tfrac{2}{3}\ln\cosh\eta_0\right]$$

$$B_\ell\, Q_\ell(i\sinh\eta_0) = G_\ell\, P_\ell(i\sinh\eta_0) \quad \text{for } \ell \geq 1$$

<u>FROM</u> BYERLY, "Elementary Treatise in Fourier Series + Spherical, Cylindrical and Ellipsoidal Harmonics"

Expressing interior of region enclosed by homogeneous oblate spheroidal shell of thickness $d\rho$ with parameter η, ~~an~~ where V_1 is a solution within the enclosed region and V_2 is the exterior solution, Poisson's equation takes the form.

$$4\pi\rho\kappa = \left(\frac{\partial u}{\partial n_+} - \frac{\partial u}{\partial n_-}\right),$$

and then expressing the total exterior solution as the sum over the oblate spheroid of all the potentials due to the shells, it is seen that

$$V_{EXT} = \frac{M}{a}\left[i\, Q_0(\sinh\eta) + i^3 P_2(\cancel{\sinh}\ \cos\theta)\, Q_2(i\sinh\eta)\right]$$

Hence, $B_0 = \dfrac{Mi}{a}$ $B_1 = 0$, $B_2 = \dfrac{-iM}{a}$, $B_\ell = 0$ for $\ell \geq 3$

$G_0 = \frac{Mi}{a} Q_0(i\sinh\eta_0) + a^2 Q\left[\frac{1}{2}(\quad) + \frac{2}{3}\ln\quad\right]$ $G_1 = 0$, $G_2 = -\frac{iM}{a} Q_2(\eta_0)$, $G_\ell = 0$ for $\ell \geq 3$

Ans. For a homogeneous oblate spheroid of mass M and density $\rho = \frac{Q}{4\pi}$,

The exterior solution is.

$$V_{EXT} = \frac{M}{a}\left[i Q_0(i \sinh\eta) - i P_2(\cos\theta) Q_2(i\sinh\eta)\right]$$

The interior solution is.

$$V_{INT} = \left\{ \frac{Mi}{a} Q_0(i\sinh\eta_0) + Qa^2\left[\frac{1}{12}(\cosh^2\eta_0 + \sinh^2\eta_0) + \frac{2}{3}\ln\cosh\eta_0\right]\right\} P_0(\cos\theta)$$

$$\left\{ -\frac{iM}{a}\frac{Q_2(i\sinh\eta_0)}{P_2(i\sinh\eta_0)}\right\} P_2(\cos\theta)$$

$$- Qa^2\left[\frac{1}{12}(\cosh^2\eta_0 + \sinh^2\eta_0) + \frac{2}{3}\ln\cosh\eta_0\right]$$

$$V_{INT} = \frac{M}{a}\left[i Q_0(i\sinh\eta_0) P_0(\cos\theta) + i^3 \frac{Q_2(i\sinh\eta_0)}{P_2(i\sinh\eta_0)} P_2(\cos\theta)\right]$$

$$+ Qa^2\left[\frac{1}{12}(\cosh^2\eta_0 + \sinh^2\eta_0) + \frac{2}{3}\ln\cosh\eta_0\right]$$

$$- Qa^2\left[\frac{1}{12}(\cosh^2\eta + \sinh^2\eta) + \frac{2}{3}\ln\cosh\eta\right]$$

RBH

Now $M = \frac{4}{3}\pi \rho a^3 \dfrac{\sinh^2 \eta_0}{\cosh^2 \eta_0}$

Using the relation.

$$Q_m(iz) = (-1)^m P_m(iz) \int_z^\infty \frac{dx}{(1+x^2)[P_m(x)]^2}$$

The expression solution ~~that~~ take the form:

$$V = \frac{M}{a} \left\{ \frac{\pi}{2} - \alpha' + \frac{1}{4}\left[\left(\frac{\pi}{2} - \alpha'\right)(3\tan^2\alpha' + 1) - 3\tan\alpha' \right] \right.$$

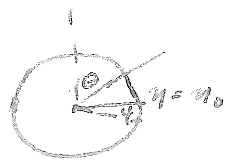
$$+ \left[3\tanh^2\beta - 1 \right]$$

where $\tan\alpha' = \sinh\eta$

$\tanh\beta = \cos\theta$.

Herrmann, Robert B.
Gph 201, Potential Theory
December 11, 1967

Find the gravitational potential of a homogeneous oblate spheroid both at points outside and within the spheroid.



The exterior problem is

$$\nabla^2 V_e = 0$$

The interior problem is

$$\nabla^2 V_i = -4\pi\sigma G$$

$x = a\cosh\eta\sin\theta\cos\phi$
$y = a\cosh\eta\sin\theta\sin\phi$
$z = a\sinh\eta\cos\theta$

Assuming axial symmetry, the solutions are of the forms,

for $\eta \geq \eta_0$

$$V_e = \sum_{n=0}^{\infty}\left\{A_n P_n(i\sinh\eta) + B_n Q_n(i\sinh\eta)\right\}P_n(\cos\theta)$$

for $\eta \leq \eta_0$

$$V_i = \sum_{n=0}^{\infty}\left\{C_n P_n(i\sinh\eta) + D_n Q_n(i\sinh\eta)\right\}P_n(\cos\theta)$$

$$-\pi G\sigma a^2 \sin^2\theta \cosh^2\eta$$

$$= \sum_0^{\infty}\left\{C_n P_n(i\sinh\eta) + D_n Q_n(i\sinh\eta)\right\}P_n(\cos\theta)$$

$$-\frac{2}{3}\pi G\sigma a^2\left[P_0(\cos\theta) - P_2(\cos\theta)\right]\cosh^2\eta$$

The condition that $\phi \to 0$ as $r \to \infty$ is equivalent to saying that $\phi \to 0$ as $\eta \to \infty$. Hence in the exterior solution, $A_n = 0$ $n = 0, 1, 2, \ldots,$

since $P_n(i\infty) \to \infty$ and $Q_n(i\infty) \to 0$

~~If~~

From the symmetry of the problem,

$$\phi(\eta, \theta) = \phi(\eta, 180^\circ - \theta),$$

it is required the the $P_n(\cos\theta)$ be even functions, or n = even numbers.

An integral representation of $Q_n(z)$ is

$$Q_n(z) = \frac{1}{2}\int_{-1}^{1} (z-t)^{-1} P_n(t)\,dt = (-1)^{n+1} Q_n(-z)$$

for z not on the real axis between -1 and ∞.

Hence, the value $z = 0$, is not defined.

So $D_n = 0$ for $n = 0, 1, 2, \ldots$

Recapping, the solutions are ....

$\eta > \eta_0$

$$V_e = \sum_{n=0}^{\infty} \{ B_{2n} Q_{2n} (i \sinh \eta) \} P_{2n}(\cos\theta)$$

$$V_i = \sum_{n=0}^{\infty} C_{2n} P_{2n}(i \sinh \eta) P_{2n}(\cos\theta)$$

$$- \frac{2}{3}\pi G \sigma a^2 [P_0(\cos\theta) - P_2(\cos\theta)] \cosh^2\eta$$

From Byerly, "Elementary Treatise on Fourier Series",

The potential at exterior points is:

$$V_{e_1} = \frac{4}{3}\pi G \sigma a^2 \sinh\eta_0 \cosh^2\eta_0 \left[i Q_0 (i \sinh\eta) + i^3 Q_2 (i \sinh\eta) P_2 (\cos\theta) \right]$$

From the B.C. $V_i = V_e$ at $\eta = \eta_0$

; $B_{2n} = C_{2n}$, $2n > 2$

$B_0 Q_0$

$$\frac{4}{3}\pi G \sigma a^2 \sinh\eta_0 \cosh^2\eta_0 \left[i Q_0 (i \sinh\eta_0) + i^3 Q_2 (i \sinh\eta_0) P_2(\cos\theta) \right]$$

$$= C_0 P_0(i \sinh\eta_0) P_0(\cos\theta) + C_2 P_2(i \sinh\eta_0) P_2(\cos\theta)$$
$$- \frac{2}{3}\pi G \sigma a^2 P_0(\cos\theta) \cosh^2\eta_0 + \frac{2}{3}\pi G \sigma a^2 P_2(\cos\theta) \cosh$$

Hence, using the orthogonality relationship of Legendre polynomials.

$$C_0 = \frac{1}{P_0(i\sinh\eta_0)}\left[\frac{4}{3}\pi G\sigma a^2 \sinh\eta_0\cosh^2\eta_0\, i\,Q_0(i\sinh\eta_0) + \frac{2}{3}\pi G\sigma a^2\cosh^2\eta_0\right]$$

$$= \frac{2}{3}\pi\frac{G\sigma a^2\cosh^2\eta_0}{P_0(i\sinh\eta_0)}\left[2\sinh\eta_0\, i\,Q_0(i\sinh\eta_0) + 1\right]$$

$$C_2 = \frac{1}{P_2(i\sinh\eta_0)}\left[\frac{4}{3}\pi G\sigma a^2\sinh\eta_0\cosh^2\eta_0\, i^3 Q_2(i\sinh\eta_0 - \frac{2}{3}\pi G\sigma a^2\cosh^2\eta_0\right]$$

$$= -\frac{2}{3}\pi\frac{G\sigma a^2\cosh^2\eta_0}{P_2(i\sinh\eta_0)}\left[2\sinh\eta_0\, i\,Q_2(i\sinh\eta_0) + 1\right]$$

$$V_e = \frac{4}{3}\pi G\sigma a^2\sinh\eta_0\cosh^2\eta_0\left[i\,Q_0(i\sinh\eta - i\,Q_2(i\sinh\eta)P_2(\cos\theta)\right]$$

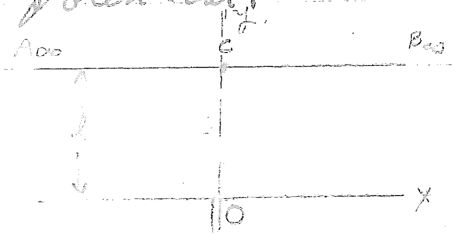
$$V_i = \frac{2}{3}\pi G\sigma a^2\cosh^2\eta_0\left\{(2\sinh\eta_0\, i\,Q_0(i\sinh\eta_0)+1)\frac{P_0(i\sinh\eta)}{P_0(i\sinh\eta_0)}P_0(\cos\theta) - (2\sinh\eta_0\, i\,Q_2(i\sinh\eta_0)\frac{P_2(i\sinh\eta)P_2(\cos\theta)}{P_2(i\sinh\eta_0)}\right\}$$

$$-\frac{2}{3}\pi G\sigma a^2\cosh^2\eta\left[P_0(\cos\theta) - P_2(\cos\theta\right].$$

Herrmann, Robert B.
Gph 201, Adv. Pot. Th.
January 8, 1967

Chap. 8.

#9. An infinite channel with parallel sides at a distance $l\pi$ apart has a narrow slit at a point O. A perfect fluid flows through a slit at a constant rate Q per unit length of the slit. Find the complex potential.



The point $(0, il)$ is seen to be a stagnation point, or $\left.\frac{d\Omega}{dz}\right|_{(0,il)} = 0$.

Also. $V_y = 0$ at $y = il$, $y = 0$

Apply a Schwarz-Christoffel transformation to the above figure. Transform. $z=0$ into $w=0$, (angle of π), $z=-\infty$ to $w=-1$ (angle of 0), $z=+\infty$ to $w=+1$ (angle of 0). $z = i\pi l$ to $w = \pm\infty$.

$$\frac{dz}{dw} = K(w+1)^{-1}(w-1)^{-1}(w)^{0}$$

or $$z = \frac{K'}{2} \ln \frac{1+w}{1-w} + C.$$

for $z=0$ when $w=0 \Rightarrow C=0$

from $z = i\pi l$, $\Rightarrow \frac{K'}{2} = l \cdot \frac{\pi}{l}$

At $w=+1$ and $w=-1$ there are sinks of strength $\frac{Q}{2\pi}$

At $w=0$ there is a source of strength $\frac{Q}{\pi}$

The complex potential in the w plane is.

$$\Omega = \frac{Q}{\pi}\ln w - \frac{Q}{2\pi}\ln(w-1) - \frac{Q}{2\pi}\ln(w+1)$$

(OVER)

the conformal transformation can be written as.

$$z = \frac{e^{w}-1}{e^{w}+1}$$

from which.

$$\underline{\underline{\Omega = \frac{Q}{\pi} \ln\left(\sinh \frac{z}{2}\right) + Qi}}$$

The complex velocity is

$$\frac{d\Omega}{dz} = \frac{Q}{2\pi} \coth\frac{z}{2} = \frac{Q}{2\pi}\left\{\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}}\right\}$$

It is seen that the point $z = i\pi$ is a stagnation point since $\left.\frac{d\Omega}{dz}\right|_{z=i\pi} = 0$

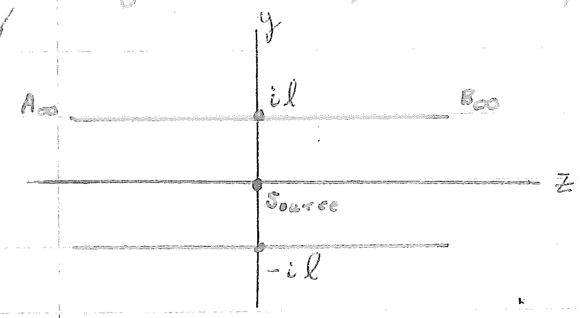
Here it was assumed that the channel height was π. It can be easily shown that if the channel height is l, then.

Ans. 9. $$\underline{\underline{\Omega = \frac{Q}{\pi} \ln\left(\sinh(\pi z/2l)\right) + Qi}}$$

OK

RBH

9(i) Find the complex potential for the problem of a source equidistant from two plane sides of an infinite channel.



Assume the source ~~the~~ emits Q units of velocity of fluid per length of source.

The problem is just a reflection together with the preceding problem.

The velocity from the previous problem is (~~cos~~

$$\Omega = \frac{Q}{\pi} \ln \sinh\left(\frac{\pi z}{2l}\right) + Qi$$

let $x' = \frac{\pi x}{2l}$, $y' = \frac{\pi y}{2l}$

Terms should be $\cosh y' \sinh x'$, $\sin y$, $\cos y$

$$V_x = \frac{Q}{2l}\left[\frac{\cos x' \sin x' \cosh^2 y' + \cosh x' \sin x' \sinh^2 y'}{-(\sin^2 x' \cosh^2 y' - \sinh^2 y' \cos^2 x')}\right]$$

$$V_y = \frac{Q}{2l}\left[\frac{\sin^2 x' \sinh y' \cosh y' - \cos^2 x' \sinh y' \cosh y'}{-(\sin^2 x' \cosh^2 y' - \sinh^2 y' \cos^2 x')}\right]$$

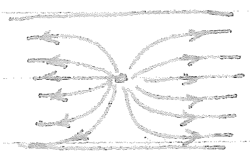
It is seen that $V_y = 0$ at $y = 0$ (no fluid passes plane $y=0$)

Hence, the solution from the lower region can be obtained by reflection. Also, the sign of V_x does not change for $\pm$ values of y. (which should be true by symmetry

(OVER)

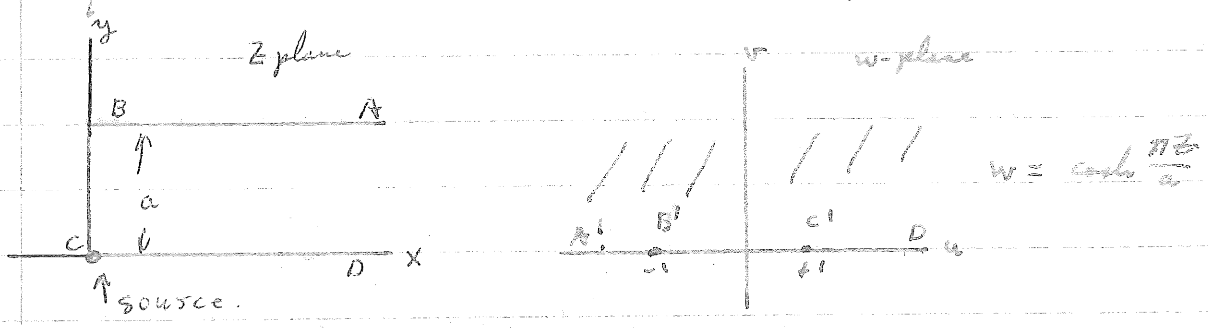
Ans 9(i) $\quad \Omega = \frac{Q}{\pi} \ln \sinh\left(\frac{\pi z}{2l}\right) + Qi$

OK. – your width is $2l$.



9(ii) A source at one vertex of a semi-infinite channel.

Depending on the orientation of the coordinate system, the solution could involve trigonometric or hyperbolic functions. I'll use the "cosh" function.



Let the source emit fluid at the rate Q.
Then the complex potential in the w plane is

$$\Omega = \frac{Q}{\pi} \ln (w-1) = \frac{Q}{\pi} \ln \left[1 - \cosh \frac{\pi z}{a}\right]$$

$$= \frac{Q}{\pi} \ln \left[-2 \sinh^2 \frac{\pi z}{2a}\right]$$

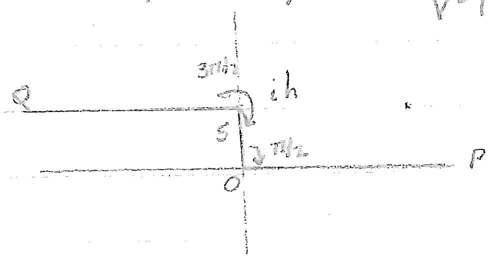
9(ii) $\quad \Omega = \frac{Q}{\pi}\left[\ln 2 + i\pi + 2 \ln \sinh \frac{\pi z}{2a}\right]$

which is again similar to the above solutions.

RBH

10. Find the complex potential for the horizontal flow, which is uniform at infinity with speed u, of a perfect fluid over a step of height h in the otherwise horizontal bed of a semi-infinite ocean.

For a perfect fluid, $\nabla^2\phi=0$, $\nabla^2\psi=0$ holds.



TRANSFORMATION:

z	w	α
$(0,0)$	$(+1,0)$	$\pi/2$
$(0,ih)$	$(-1,0)$	$3\pi/2$.

$$\frac{dz}{dw} = K(w-1)^{-1/2}(w+1)^{1/2}$$

Integrating,

$$z = K\left\{\sqrt{w^2-1} + 2\ln\left[\sqrt{w-1}+\sqrt{w+1}\right]\right\} + c_1$$

$$\equiv K\left\{\sqrt{w^2-1} + \ln\left(w+\sqrt{w^2-1}\right)\right\} + c_2$$

$$\equiv K\left\{\sqrt{w^2-1} + \cosh^{-1}(w)\right\} + c_3$$

Using the fact that $z=0$, $w=+1$, $z=ih$, when $w=-1$,

It is found that.

$$Z = \frac{h}{\pi}\left\{\sqrt{w^2-1} + \cosh^{-1}(w)\right\}$$

RBH

Now, to find Ω, the complex potential,

$$\frac{d\Omega}{dz} = V_x - iV_y = \frac{d\Omega}{dw} \cdot \frac{dw}{dz} = \frac{d\Omega}{dw}\frac{\pi}{h}\sqrt{\frac{w-1}{w+1}}$$ ✓

Now, the initial condition is that for $X \to \infty$, $V_x = -U$, $V_y = 0$. By the transformation, $z \to \infty$ is equivalent to $w \to \infty$. So that.

$$\left(\frac{d\Omega}{dz}\right)_{x \to \infty} = -U = \left(\frac{d\Omega}{dw}\right)_{w \to \infty} \cdot \frac{\pi}{h}$$

Also $$\left(\frac{d\Omega}{dz}\right)_{\substack{x \to -\infty \\ y \to ih}} = -U = \left(\frac{d\Omega}{dw}\right)_{w \to -\infty} \cdot \frac{\pi}{h}.$$

This shows that $\frac{d\Omega}{dw} = \text{constant} \ldots = \frac{hU}{\pi}$.

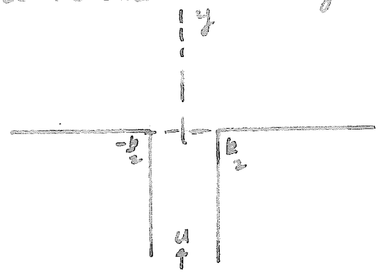
Also. $\Omega = k(u+iv)$ is a solution of $\nabla^2 u = \nabla^2 v = 0$.

Ans #10. $\underline{\Omega = -\frac{hU}{\pi} w}$ ✓

$\underline{\text{where } z = \frac{h}{\pi}\left\{\sqrt{w^2-1} + \cosh^{-1} w\right\}}$

RBH

14\. A semi-infinite ocean is bounded by the line $y=0$. A straight, parallel sided river of breadth b flows with uniform velocity U in the direction of the y-axis into the ocean. Write down the complex potential in the intermediate w-plane together with the associated complex potential.



MAPPING z	w	Angle.
$z=-\frac{b}{2}$	1	$3\pi/2$
$z=+\frac{b}{2}$	-1	$3\pi/2$
$z=-i\infty$	0	$\cancel{\pi/2}$ 0.

What about the points at infinity on the x-axis
I assume you mapped them at infinity.

Hence,

$$\frac{dz}{dw} = K\frac{\sqrt{1-w^2}}{w} = K'\frac{\sqrt{w^2-1}}{w}$$

Then $$z = K'\left\{\sqrt{w^2-1} - \cos^{-1}\left(\frac{1}{w}\right)\right\} + C'$$

$$\left[\cos^{-1}\left(\frac{1}{w}\right) = \frac{1}{i}\ln\left\{\frac{1+\sqrt{1-w^2}}{w}\right\}\right]$$

Applying mapping equivalences,

$$-\frac{b}{2} = K'\left\{\sqrt{0} - \cos^{-1}(1)\right\} + C'$$

$$+\frac{b}{2} = K'\left\{\sqrt{0} - \cos^{-1}(-1)\right\} + C'$$

$$-i\infty = K'\left\{\sqrt{-1} - \frac{1}{i}\ln\left\{\frac{2}{0}\right\}\right\} + C'$$

from which it is found that

$$C' = -\frac{b}{2}, \quad K' = -\frac{b}{\pi}.$$

$$\underline{\underline{z = -\frac{b}{\pi}\left[\sqrt{w^2-1} - \cos^{-1}\left(\frac{1}{w}\right)\right] - \frac{b}{2}}}$$

By this transformation, a source of strength m is placed at $w=0$. Then:

$$\Omega = m \log w.$$

$$\frac{d\Omega}{dz} = V_x - iV_y = \frac{d\Omega}{dw}\frac{dw}{dz} = \frac{m}{w}\cdot\left(\frac{-\pi}{b}\right)\frac{w}{\sqrt{w^2-1}}$$

Now at $z=-i\infty$, $\equiv$ $w=0$, $V_y = U_1$, $V_x=0$

Then,

$$-iU = \frac{-m\pi}{b}\frac{1}{i}$$

$$\text{or } m = -\frac{bU}{\pi}.$$

Ans 14. $\Omega = \left(-\frac{bU}{\pi}\right)\ln w$

where

$$z = -\frac{b}{\pi}\left[\sqrt{w^2-1} - \cos^{-1}\left(\frac{1}{w}\right)\right] - \frac{b}{2}$$

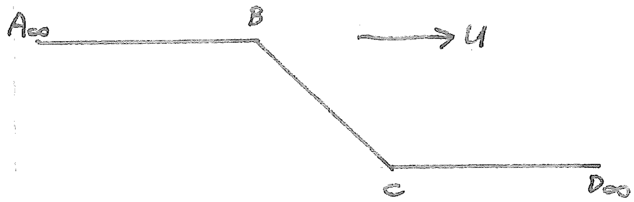
RBH.

17. Show that the complex potential Ω for a stream, uniform with velocity U at infinity in a horizontal direction $A_\infty B$, flowing over the bed $A_\infty BCD_\infty$ of a semi-infinite ocean is:

$$\Omega = (Ua\sqrt{2}/\pi)\ \ln\left\{\frac{1+w^4}{1-w^4}\right\}$$

$$z = (a/\pi\sqrt{2})\left[\log\left(\frac{1+w}{1-w}\right) + \frac{4w^3}{1-w^4} - 2\tan^{-1} w\right]$$

where $BC = a$...



TRANSFORMATION,

z	t	ANGLE
0	$t = +1$	$\frac{3}{4}\pi$
$\frac{a}{\sqrt{2}}(-1+i)$	$t = -1$	$\frac{5}{4}\pi$

The Schwarz-Christoffel Transformation is:

$$\frac{dz}{dt} = K\left(\frac{t+1}{t-1}\right)^{1/4}.$$

This is evaluated by using the relation found is Kober that if $\beta = p/q$ where p, q are integers and.

From Kober $\quad \frac{dz}{dt} = K\left(\frac{t+1}{t-1}\right)^{\beta}$, then $z = 2qK\int_{\xi}^{\infty} w^{p+q-1}(w^q-1)^{-2}\,dw$ where $\xi = \left(\frac{t+1}{t-1}\right)^{1}$

RBH

Thus, $\beta = 1/4 \quad p = 1,\ q = 4$

$$Z = 8K \int_{\xi}^{\infty} \frac{w^4}{(w^4-1)^2}\, dw \qquad , \text{ where } \xi = \left(\frac{t+1}{t-1}\right)^{1/4}$$

$$= -2K \int_{\xi}^{\infty} \frac{-4w^3 \cdot w}{(w^4-1)^2}$$

$$= -2K \left[\frac{w}{w^4-1}\Big|_{\xi}^{\infty} - \int_{\xi}^{\infty} \frac{dw}{w^4-1} \right]$$

$$= -2K \left[\frac{w}{w^4-1}\Big|_{\xi}^{\infty} - \int_{\xi}^{\infty} \left[\frac{-1/2}{w^2+1} - \frac{1/4}{t+1} \quad \frac{+1/4}{t-1} \right] dw \right]$$

$$= 2K \left[\frac{-w}{w^4-1}\Big|_{\xi}^{\infty} - \frac{1}{2}\tan^{-1} w\Big|_{\xi}^{\infty} + \frac{1}{4} \ln \frac{t-1}{t+1}\Big|_{\xi}^{\infty} \right]$$

$$Z = 2K \left[\frac{+\xi}{\xi^4-1} + \frac{\tan^{-1}\xi}{2} - \frac{1}{4} \ln\left(\frac{\xi-1}{\xi+1}\right) - \frac{\pi}{4} \right]$$

$$= 2K \left[\frac{\xi}{\xi^4-1} + \frac{1}{2}\tan^{-1}\xi + \frac{1}{4} \ln\left(\frac{\xi+1}{\xi-1}\right) - \frac{\pi}{4} \right]$$

To evaluate K,

$z = 0, \quad t = +1, \quad \xi = \infty$

$z = \frac{a}{\sqrt{2}}(-1+i)$ for $t = -1, \ \xi = 0$

$$\frac{a}{\sqrt{2}}(-1+i) = 2K\left[+\frac{1}{4}\ln(-1) - \frac{\pi}{4} \right] = \frac{2\pi}{4}K\left[i - 1 \right]$$

hence $\frac{a}{\sqrt{2}} = \frac{\pi K}{2}$, $\quad K = \frac{2a}{\sqrt{2}\,\pi}$

TRANSFORMATION:

$$Z = \frac{a}{\sqrt{2}\,\pi}\left[\ln\frac{(\xi+1)}{(\xi-1)} + 2\tan^{-1}\xi + \frac{4\xi}{\xi^4-1} - \frac{\pi}{2} \right]$$

RBH.

Now

$$\frac{d\Omega}{dz} = V_x - iV_y = \frac{d\Omega}{dt}\frac{dt}{dz}$$

at $z = \pm\infty$, $V_x = U$, $V_y = 0$ ...

Now $z = \pm\infty \Rightarrow t = \pm\infty$
also $\frac{dt}{dz} = \frac{1}{K}$ as $t \to \infty$

Thus

$$U = \left(\frac{d\Omega}{dt}\right)_{t\to\infty} \cdot \frac{1}{K}.$$

$$\text{or} \quad \left(\frac{d\Omega}{dt}\right)_{t\to\infty} = UK \therefore = \frac{2aU}{\sqrt{2}\,\pi} = \frac{Ua\sqrt{2}}{\pi}.$$

Now since $\xi = \left(\frac{t+1}{t-1}\right)^{1/4}$

$$t = \frac{\xi^4+1}{\xi^4-1}$$

Ans. 17.

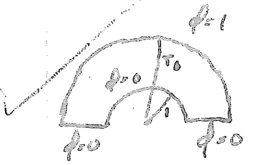
$$\Omega = \frac{Ua\sqrt{2}}{\pi}\, t = \frac{Ua\sqrt{2}}{\pi}\left(\frac{\xi^4+1}{\xi^4-1}\right)$$

$$z = \frac{a}{\sqrt{2}\,\pi}\left[\ln\left(\frac{\xi+1}{\xi-1}\right) + 2\tan^{-1}\xi + \frac{4\xi}{\xi^4-1} - \pi\right]$$

where $\xi = \left(\frac{t+1}{t-1}\right)^{1/4}$

RBH

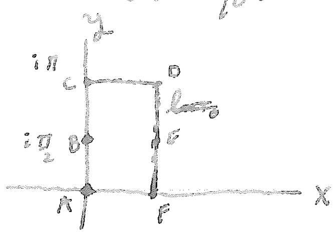
Board Problem #2

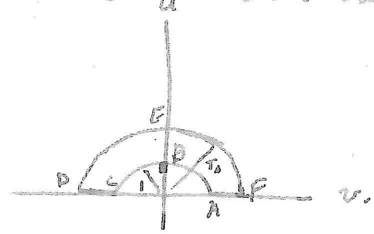


Show that the potential in the region $1 \le r \le r_0$, $0 \le \theta \le \pi$ is.

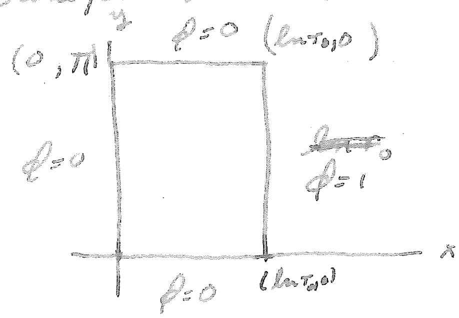
$$\phi = \frac{4}{\pi} \sum_{1}^{\infty} \frac{r^{2n-1} - r^{-(2n-1)}}{r_0^{2n-1} - r_0^{-(2n-1)}} \frac{\sin(2n-1)\theta}{2n-1}$$

The transformation $w = e^{z}$ does the following:





Hence the transformation. $z = \ln w$ will transform the above semi-ring to a rectangular region:



Since the mapping is conformal for the given region, $\nabla^2\phi = 0$ in the interior region...

The general solution is

$$\phi = \sum A_n \sinh kx \sin ky. \qquad \begin{matrix} x = \ln r \\ y = \theta \end{matrix}$$

THE B.C are. $y = \pi,\ \phi = 0. \Rightarrow k =$ ~~[illegible]~~ integer ≥ 1

$x = \ln r_0,\ \phi = 1 \Rightarrow k = (2n-1),\ A_n = \frac{4}{\pi}\frac{1}{(2n-1)\sinh(2n-1)\ln r_0}$

$$\phi = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sinh\{(2n-1)\ln r\} \sin(2n-1)\theta}{(2n-1)\sinh\{(2n-1)\ln r_0\}}$$

(OVER)

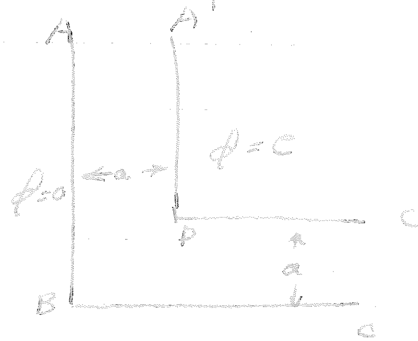
Now $\sinh\{(2n-1)\ln r\} = \dfrac{e^{(2n-1)\ln r} - e^{-(2n-1)\ln r}}{2}$

$$= \frac{r^{(2n-1)} - r^{-(2n-1)}}{2}$$

$$\therefore \quad \phi = \frac{4}{\pi} \sum_{1}^{\infty} \frac{r^{(2n-1)} - r^{-(2n-1)}}{r_0^{(2n-1)} - r_0^{-(2n-1)}} \, \frac{\sin(2n-1)\theta}{2n-1}$$

RBH

Board Problem #1



Show that the potential in the L-shaped region is

$$\phi = c\,\mathrm{Re}\left\{\frac{1}{\pi i}\ln w\right\}$$

where

$$\frac{z}{a} = \frac{2i}{\pi}\left[\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) - \tan^{-1} t\right] + 1 + i$$

and

$$+\left(\frac{w+1}{w-1}\right)^{1/2} = t$$

TRANSFORMATION!

POINT	Z	w	ANGLE.	t
A	$i\infty$	$+\infty$	0	1
B	0	+1	$\pi/2$	$+\infty$
C	∞	0	0	i
D	$a+ia$	-1	$3\pi/2$	0
A'	$i\infty+a$	$-\infty$	0	1

$$\frac{dz}{dw} = K\frac{1}{w}\sqrt{\frac{w+1}{w-1}}$$

By the substitution, $t = \sqrt{\frac{w+1}{w-1}}$

$$\frac{dz}{dt} = -2K\left[\frac{1}{t^2-1} + \frac{1}{t^2+1}\right]$$

$$\therefore \quad z = -2K\left[\frac{1}{2}\ln\frac{t-1}{t+1} + \tan^{-1} t\right] + C$$

$$= 2K\left[-\frac{1}{2}\ln\frac{1+t}{1-t} - \tan^{-1} t\right] + C$$

RBH.

EVALUATION OF CONSTANTS...

1. $z = a + ia$ at $w = 0$ $\equiv$ $t = 0$

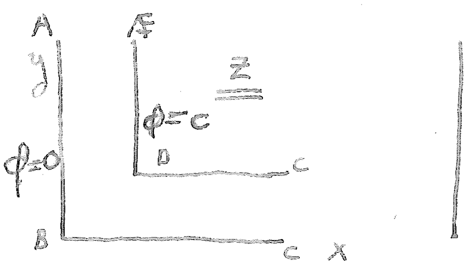
$\underline{\underline{a + ia = c\pi}}$

2. $z = 0$ at $w = 1$ $\equiv$ $t = \infty$

$$0 = 2K\left[\frac{i\pi}{2} - \frac{\pi}{2}\right] + a + ia$$

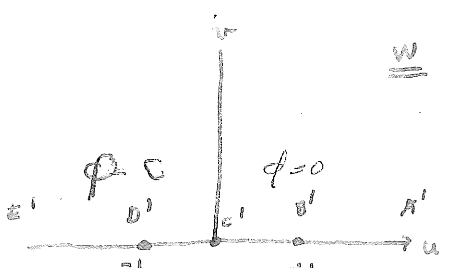
$$\underline{\underline{K = \frac{ia}{\pi}}}$$

Solution of the potential problem.



$$\frac{z}{a} = \frac{2i}{\pi}\left[\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) - \tan^{-1} t\right] + 1 + i$$

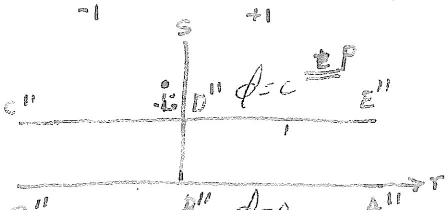
where $t = \left(\frac{w+1}{w-1}\right)^{1/2}$



$W = e^{\pi p}$

or $p = \frac{1}{\pi} \ln w$

The transformation $w = e^{p}$ will lead to the answer given with the problem



Now,

$\phi = \frac{c}{c}\, s$

and $s = \cancel{\mathrm{Re}}\ \mathrm{Im}\ "p"$

RBH.

ANS. $\frac{z}{a} = \frac{2i}{\pi}\left[\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) - \tan^{-1} t\right] + 1 + i$

$$\underline{\underline{\phi = \frac{c}{i}\,\mathrm{Im}\left\{\frac{1}{\pi}\ln w\right\}}}$$

Herrmann, Robert B.
Gph 284
January 17, 1967

Chapter 4.

#5. Determine the inverse Laplace transform with respect to x of

(i) $(3p-1)/(4p^2+8p+5)^2$

(ii) $e^{-2p}/\{(p-1)(p^2+1)\}$

(i) Expanding in partial fractions,

$$\frac{3p-1}{[4p^2+8p+5]^2} = \frac{3p-1}{[4(p+1)^2+1]^2} = \frac{3(p+1)}{16[(p+1)^2+\frac{1}{4}]^2} - \frac{4}{16[(p+1)^2+\frac{1}{4}]^2}$$

If. $f(p) = \mathcal{L}[F(t)]$, then $f(p-a) = \mathcal{L}[e^{at}F(t)]$

here $a = -1$,

$$F(t) = e^{-x}\left[\mathcal{L}^{-1}\left\langle \frac{3p}{16[p^2+\frac{1}{4}]^2} - \frac{4}{16[p^2+\frac{1}{4}]^2}\right\rangle\right]$$

$$= \frac{e^{-x}}{16}\left[3t\sin\frac{t}{2} - 16\sin t/2 + 8\cos t/2\right]$$

Ans (i)

(ii) $$\frac{e^{-2p}}{[(p-1)(p^2+1)]} = \frac{e^{-2p}}{2}\left[\frac{1}{p-1} - \frac{1}{p^2+1} - \frac{p}{p^2+1}\right]$$

If $g(p) = \mathcal{L}[f(x)]$, then $e^{-ap}g(p)$ for $a>0$ is the Laplace transform of $F(x)$, where

$$F(x) = \begin{cases} 0 & 0 < x < a \\ f(x-a) & x > a \end{cases}$$

in this case $g(p) = \dfrac{1}{(p-1)(p^2+1)}$ and

$$f(t) = \tfrac{1}{2}\left[e^{t} - \sin t + \cos t\right]$$

Ans ii $$F(t) = \begin{cases} 0 & 0 < x \leq 2 \\ \tfrac{1}{2}\left[e^{(t-2)} - \sin(t-2) + \cos(t-2)\right] & x \geqslant 0 \end{cases}$$

The equality can be used since $F(2) = 0$, whether approached from above or below.

Lap IV #9. Solve the following simultaneous equations using the Laplace transform.

$$\frac{d^2x}{dt^2} - \frac{dy}{dt} + \frac{dz}{dt} - 4x - 2y - 2z = 0$$

$$2\frac{dx}{dt} - \frac{d^2y}{dt^2} + \frac{d^2z}{dt^2} + 3y - 4z = 0$$

$$\frac{dx}{dt} \qquad \frac{d^2z}{dt^2} - 2x - y - 4z = 0,$$

given that $x = y = z = 1$, $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = 3$, $\frac{dz}{dt} = 1$ when $t = 0$.

Taking the Laplace transform,

$$(p^2-4)\bar{x} - (p+2)\bar{y} + (p-2)\bar{z} = p\,x(0) + \dot{x}(0) - y(0) + z(0)$$

$$2p\,\bar{x} - (p^2-3)\bar{y} + (p^2-4)\bar{z} = 2x(0) - p\,y(0) - \dot{y}(0) + p\,z(0) + \dot{z}(0)$$

$$(p-2)\bar{x} - \bar{y} + (p^2-4)\bar{z} = x(0) + p\,z(0) + \dot{z}(0).$$

Applying the initial conditions,

$$(p^2-4)\bar{x} - (p+2)\bar{y} + (p-2)\bar{z} = p+2$$

$$2p\,\bar{x} - (p^2-3)\bar{y} + (p^2-4)\bar{z} = 0$$

$$(p-2)\bar{x} - \bar{y} + (p^2-4)\bar{z} = p+2$$

On rearranging the system by various additions and subtractions,

$$\begin{pmatrix} (p+2)(p-1) & -(p+2)(p-1) & (p-2) \\ (p+1)(p^2+2p-4) & -(p+1)(p^2+p-1) & 0 \\ -1 & -(p-2) & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Solving this system by Cramers method, with

$$\Delta = (p+1)(p-1)(p-2)(p+3)(p-3)$$

$$\Delta_{\bar{x}} = +\overset{(p+1)(p+1)(p-2)(p-2)}{\cancel{(p+1)(p+1)(p-2)(p-2)}} \qquad \cancel{p^2-p-2}\ p^2-p-2$$

$$\Delta_{\bar{y}} = (p+1)(p-2)(p^2+2p-4)$$

$$\Delta_{\bar{z}} = (p-3)(p+2)(p-1)(p+1)$$

One obtains

$$\bar{x} = \frac{+\cancel{(p+1)(p-2)}}{(p-1)(p+3)(p-3)} = -\frac{1}{8}\frac{1}{p-1} + \frac{5}{24}\frac{1}{p+3} + \frac{22}{24}\frac{1}{p-3}$$

$$\bar{y} = \frac{p^2+2p-4}{(p-1)(p+3)(p-3)} = +\frac{1}{8}\frac{1}{p-1} - \frac{1}{24}\frac{1}{p+3} + \frac{22}{24}\frac{1}{p-3}$$

$$\bar{z} = \frac{(p+2)}{(p+3)(p-2)} = \frac{4}{5}\frac{1}{p-2} + \frac{1}{5}\frac{1}{p+3}$$

Taking inverse transform,

ANS. 9c

$$x = -\frac{1}{8}e^{t} + \frac{5}{24}e^{-3t} + \frac{11}{12}e^{3t}$$

$$y = \frac{1}{8}e^{t} - \frac{1}{24}e^{-3t} + \frac{11}{12}e^{3t}$$

$$z = \frac{4}{5}e^{2t} + \frac{1}{5}e^{-3t}.$$

Chap IV. 15. The equation governing the diffusion of electricity along a cable of length l is

$$\frac{\partial V}{\partial t} = \frac{1}{RC}\frac{\partial^2 V}{\partial x^2}. \qquad \kappa = 1/RC$$

Solve this equation, using two different transform methods, when V is subject to the boundary conditions

$$V = V_0(1 - x/l) \quad \text{when } t = 0\,;$$

and for $t > 0$, $V = 0$ for both $x = 0$ and $x = l$

(1) FINITE SINE TRANSFORM METHOD.

$$\bar{V}(m,t) = \int_0^l V(x,t)\,\sin(m\pi x/l)\,dx.$$

$$\int_0^l \frac{\partial^2 V}{\partial x^2}\sin(m\pi x/l)\,dx = \left[\frac{\partial V}{\partial x}\sin(m\pi x/l) - V(m\pi/l)\cos(m\pi x/l)\right]_0^l$$

$$- (m^2\pi^2/l^2)\,\bar{V} \quad = - (m^2\pi^2/l^2)\,\bar{V}$$

Also

$$\int_0^l \frac{\partial V}{\partial t}\,\sin(m\pi x/l)\,dx = \frac{d\bar{V}}{dt}$$

Thus,

$$\frac{d\bar{V}}{dt} + (\kappa m^2\pi^2/l^2)\,\bar{V} = 0$$

so that,

$$\bar{V} = B\exp\{-\kappa m^2\pi^2 t/l^2\}$$

The B.C. are transformed to

$$\bar{V}(m,0) = V_0\int_0^l (1 - x/l)\,\sin(m\pi x/l)\,dx \;=\; \frac{V_0 l}{m\pi}$$

First see the following page.

Chap IV #15 (2) Laplace transform method.

$$\frac{\partial V}{\partial t} = \frac{1}{RC}\frac{\partial^2 V}{\partial x^2}$$

$t=0 \quad V = V_0(1 - x/l) = \frac{2V_0}{\pi}\sum_1^\infty \frac{1}{n}\sin\frac{n\pi x}{l}$

$t>0 \quad V(0,t) = V(l,t) = 0$

let $\bar{V}(x,p) = \int_0^\infty e^{-pt} V(x,t)\,dt$

then.

$$\int_0^\infty e^{-pt}\frac{\partial V}{\partial t}dt = -V(x,0) + p\bar{V}(x,p)$$

$$\int_0^\infty e^{-pt}\frac{\partial^2 V}{\partial x^2}dt = \frac{d^2}{dx^2}\bar{V}(x,p).$$

the transform differential equation is.

$$\frac{1}{RC}\frac{d^2\bar{V}(x,p)}{dx^2} - p\bar{V}(x,p) = -V(x,0) = -\frac{2V_0}{\pi}\sum_1^\infty \frac{1}{n}\sin\frac{n\pi x}{l}$$

For a solution, let $\bar{V}(x,p) = \sum_1^\infty \frac{C_m}{m}\sin\frac{m\pi x}{l}$.

$$-\sum_1^\infty\left[\frac{m^2\pi^2}{RC\,l^2} + \frac{p}{\cancel{m}}\right]\frac{C_m}{m}\sin\frac{m\pi x}{l} = -\frac{2V_0}{\pi}\sum_1^\infty\frac{1}{n}\sin\frac{n\pi x}{l}$$

$$\text{or.}\quad C_m = \frac{2V_0}{\pi\left[\frac{m^2\pi^2}{RCl^2} + p\right]}$$

Inverting,

$$\underline{\underline{V(x,t) = \frac{2V_0}{\pi}\sum_1^\infty\left(\frac{1}{n}\right)\sin\left(\frac{n\pi x}{l}\right)\exp\left\{\frac{-n^2\pi^2 t}{RCl^2}\right\}}}$$

It is seen that $V(0,t) = V(l,t) = 0$.

$\therefore \; B = \dfrac{V_0 l}{\pi m}$

and $\quad \bar{V} = \dfrac{V_0 l}{m\pi} \exp\{-\kappa m^2 \pi^2 t/l^2\}$

Taking the inverse transform,

$$V = \frac{2}{l}\sum_1^{\infty} \sin\left(\frac{m\pi x}{l}\right)\int_0^l V \sin\left(\frac{m\pi x'}{l}\right)dx'$$

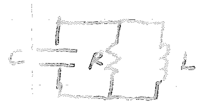
$$= \frac{2}{l}\sum_1^{\infty} \sin\left(\frac{m\pi x}{l}\right)\; \bar{V}(m)$$

$$= \frac{2V_0}{\pi}\sum_1^{\infty} (1/m)\sin\left(\frac{m\pi x}{l}\right)\exp\{-\kappa m^2\pi^2 t/l^2\}$$

where $\kappa = 1/RC$

Ans. 15. $V(x,t) = \dfrac{2V_0}{\pi}\displaystyle\sum_1^{\infty}\left(\frac{1}{m}\right)\sin\left(\frac{m\pi x}{l}\right)\exp\left\{\frac{-m^2\pi^2 t}{RCl^2}\right\}$

Chap IV #16. A closed circuit consists of an inductance L and a resistance R. At $t=0$ a capacitance bearing a charge Q is connected in parallel with the inductance and resistance. If q is the charge on the capacitor at a time t, then q satisfies.



$$\frac{d^2q}{dt^2} + \frac{1}{RC}\frac{dq}{dt} + \frac{1}{LC} q = 0$$

Determine q using the Laplace transform.

I.C. $\quad q = Q \quad$ at $t=0$ ...

$\quad \frac{dq}{dt} = 0 \quad$ at $t=0$ ...

Let $\bar{q} = \int_0^\infty e^{-pt} q(t) dt$, then.

$$\left(p^2 + \frac{1}{RC}p + \frac{1}{LC}\right)\bar{q} = pq(0) + \dot{q}(0) + \frac{1}{RC} q(0).$$

$$\bar{q} = \frac{Q\,(p + 1/RC)}{p^2 + \frac{1}{RC}p + \frac{1}{LC}}$$

Now. $\quad p^2 + \frac{1}{RC}p + \frac{1}{LC} = 0$ for

$p = \alpha_1, \alpha_2$ where α_1 and α_2 are

roots of $\alpha^2 + \frac{1}{RC}\alpha + \frac{1}{LC} = 0$

or $\alpha = \dfrac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{LC}}}{2}$

Thus,

$$\bar{q} = Q\left[\frac{p}{(p-\alpha_1)(p-\alpha_2)} + \frac{1}{RC}\frac{1}{(p-\alpha_1)(p-\alpha_2)}\right]$$

and

$$q = \frac{Q}{\alpha_1-\alpha_2}\left[(\alpha_1 e^{\alpha_1 t} - \alpha_2 e^{\alpha_2 t}) + \frac{1}{RC}(e^{\alpha_1 t} - e^{\alpha_2 t})\right]$$

$$= \frac{Q}{\alpha_1-\alpha_2}\left[(\alpha_1 + \tfrac{1}{RC})e^{\alpha_1 t} - (\alpha_2 + \tfrac{1}{RC})e^{\alpha_2 t}\right]$$

But

$$\alpha_{1,2} = \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{LC}}}{2}$$

$$\alpha_{1,2} + \frac{1}{RC} = \frac{+\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{LC}}}{2} = -\alpha_{2,1}$$

Ans. 16. $\quad q = \dfrac{Q}{\alpha_2-\alpha_1}\left[\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}\right]$

Herrmann, Robert B.
Adv. Potential Theory
January 23, 1967.

Chap X
16, 10. 4. Obtain the solution of the equation

$$\frac{\partial^2\phi}{\partial x^2} = \left(\frac{1}{k}\right)\frac{\partial\phi}{\partial t}, \quad 0 < x < l, \quad t > 0.$$

with,

$$\phi(0,t) = \phi(l,t) = 0 \qquad \text{for } t > 0$$

and for

$$t = 0, \quad \phi(x,0) = \begin{cases} x, & 0 \le x \le l/2 \\ l-x, & l/2 \le x \le l. \end{cases}$$

$$\text{or, } \phi(x,0) = \frac{4l}{\pi^2} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^2} \sin\frac{(2r+1)\pi x}{l}$$

By applying Laplace transform with respect to t to the above equation,

$$\bar{\phi}(x,p) = \int_0^\infty e^{-pt}\phi(x,t)\,dt$$

$$\int_0^\infty e^{-pt}\frac{\partial\phi}{\partial t}\,dt = e^{-pt}\phi\Big|_0^\infty + p\bar{\phi}$$

$$= -\phi(x,0) + p\bar{\phi}(x,p)$$

$$\int_0^\infty e^{-pt}\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\bar{\phi}}{\partial x^2}$$

Thus,

$$\frac{d^2\bar{\phi}}{dx^2} - \left(\frac{p}{k}\right)\bar{\phi} = \frac{-4l}{k\pi^2}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}\sin(2r+1)\pi x/l.$$

for a solution, let $\bar{\phi} = \sum \frac{(-1)^r}{(2r+1)^2} A_r \sin\left[(2r+1)\frac{\pi x}{l}\right].$

Then,

$$-\sum_{r=0}^{\infty}(-1)^r A_r \sin\frac{(2r+1)\pi x}{l}\left[\frac{\pi^2}{l^2} + \frac{p}{k(2r+1)^2}\right] = \frac{-4l}{k\pi^2}\sum_0^\infty\frac{(-1)^r}{(2r+1)^2}\sin\frac{(2r+1)\pi x}{l}$$

by orthogonality of sin,

$$A_r = \frac{4l}{\pi^2\left[\frac{\pi^2}{l^2}k(2r+1)^2 + p\right]}$$

The equation for $\bar{\phi}(x,p)$ is.

$$\bar{\phi}(x,p) = \frac{4\ell}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\left(\frac{(2n+1)\pi x}{\ell}\right)\frac{1}{\left[\frac{\pi^2 k^2}{\ell}(2n+1)^2+p\right]}$$

expanding the series, taking the inverse Laplace transform, and recombining with series notation,

$$\bar{\phi}(x,t) = \frac{4\ell}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\left\{\frac{(2n+1)\pi x}{\ell}\right\}\exp\left[-\frac{\pi^2 k(2n+1)^2}{\ell^2}t\right]$$

Chap X #6. If. $\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = \frac{1}{\kappa}\frac{\partial \phi}{\partial t}$, $r > a$, $t > 0$,

and ϕ satisfies the boundary conditions

$$\phi(r,0) = 0, \quad r > a; \quad \phi(a,t) = \phi_0 \text{ (a constant)}$$

show that $\bar{\phi}(p) = \dfrac{\phi_0 K_0\{r\sqrt{p/\kappa}\}}{p\, K_0\{a\sqrt{p/\kappa}\}}$

and hence that.

$$\phi = \phi_0 \left[1 + \frac{2}{\pi}\int_0^\infty \exp(-\kappa t \xi^2)\, J_0 \frac{(\xi r) Y_0(\xi a) - J_0(\xi a) Y_0(\xi r)}{J_0^2(\xi a) + Y_0^2(\xi a)} \frac{d\xi}{\xi} \right]$$

Let $\bar{\phi}(r,s) = \int_0^\infty e^{-pt}\phi(r,t)\,dt$

Then. $\int_0^\infty e^{-pt}\frac{\partial \phi}{\partial t}\,dt = e^{-pt}\phi\Big|_0^\infty + p\bar{\phi}$

$$= 0 - 0 + p\bar{\phi}$$

Spatial differentiation just carries through.

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{p}{\kappa}\right)\bar{\phi} = 0$$

$$\bar{\phi} = A_1 I_0\{r\sqrt{p/\kappa}\} + A_2 K_0\{r\sqrt{p/\kappa}\}.$$

Since the solution is desired in the region $r > a$,
$A_1 = 0$.

The B.C. $\phi(a,t) = \phi_0$ transforms to $\bar{\phi}(a,s) = \frac{\phi_0}{p}$.

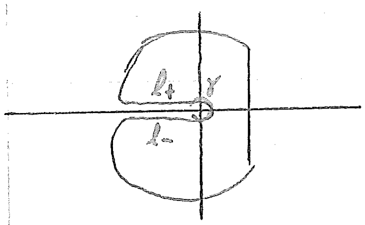
Thus, $\underline{\bar{\phi} = \dfrac{\phi_0}{p}\dfrac{K_0\{r\sqrt{p/\kappa}\}}{K_0\{a\sqrt{p/\kappa}\}}}$

The inverse Laplace transform is.

$$f(t) = \frac{1}{2\pi j}\int_{c-i\infty}^{c+i\infty} e^{\zeta x}\bar{f}(\zeta)d\zeta.$$

where $c>0$ is an arbitrary constant.

This integral is evaluated by contour integration.



Now $\bar{f}(\zeta) = \frac{\phi_0}{\zeta}\frac{K_0\{r\sqrt{\zeta/\kappa}\}}{K_0\{a\sqrt{\zeta/\kappa}\}}$

has a pole and branch point at $\zeta=0$.

Thus. $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\zeta x}\bar{f}(\zeta)d\zeta = -\frac{1}{2\pi j}\int_\gamma - \frac{1}{2\pi j}\int_{\infty}^{0}{}_{l_+} - \frac{1}{2\pi j}\int_{0}^{\infty}{}_{l_-}$$

The circular path $\int_{R\to\infty} e^{\zeta x}f(\zeta)d\zeta = 0.$ because $f(\zeta) = O(R^{-1})$ for large R

The path l_+
$$\zeta = \eta e^{i\pi} \quad d\zeta = -d\eta \quad \sqrt{\zeta} = i\sqrt{\eta}$$
On the path l_-
$$\zeta = \eta e^{-i\pi} \quad d\zeta = -d\eta \quad \sqrt{\zeta} = -i\sqrt{\eta}.$$
~~The residue at $\zeta=0$ is ϕ_0.~~
$$-\frac{1}{2\pi j}\int_\gamma = \phi_0$$

Thus,
$$f(t) = \phi_0 - \frac{1}{2\pi j}\int_\infty^0 \phi_0 e^{-\eta x}\frac{K_0\{ir\sqrt{\eta/\kappa}\}}{K_0\{ia\sqrt{\eta/\kappa}\}}\frac{d\eta}{\eta}$$
$$-\frac{1}{2\pi j}\int_0^\infty \phi_0 e^{-\eta x}\frac{K_0\{-ir\sqrt{\eta/\kappa}\}}{K_0\{-ia\sqrt{\eta/\kappa}\}}\frac{d\eta}{\eta}.$$

$$K_0(z) = \frac{1}{2}\pi i\, H_0^{(1)}(z e^{\pi i/2}) \qquad -\pi < \arg z \le \pi/2$$

$$K_0(z) = -\frac{\pi i}{2} H_0^{(2)}(z e^{-i\pi/2}) \qquad -\frac{\pi}{2} < \arg z \le \pi$$

where $\quad H_0^{(1)}(w) = J_0(w) + i\,Y_0(w)$

$\qquad\qquad H_0^{(2)}(w) = J_0(w) - i\,Y_0(w)$

Using these definitions,

$$K_0(R e^{i\pi/2}) = -\frac{i\pi}{2} H_0^{(2)}(R e^0) = -\frac{i\pi}{2}\left[J_0(R) - i\,Y_0(R)\right]$$

$$K_0(R e^{-i\pi/2}) = \frac{i\pi}{2} H_0^{(1)}(R e^0) = +\frac{i\pi}{2}\left[J_0(R) + i\,Y_0(R)\right]$$

Using these definitions,

$$\int_\infty^0 \phi_0 e^{-\eta x} \frac{K_0\{i r\sqrt{\eta/K}\}}{K_0\{i a\sqrt{\eta/K}\}} \frac{d\eta}{\eta} = \int_\infty^0 \phi_0 e^{-\eta x} \frac{\left[J_0(r\sqrt{\eta/K}) - i\,Y_0(r\sqrt{\eta/K})\right]\cdot\left[J_0(\;) + i\,Y_0(\;)\right]}{J_0^2(a\sqrt{\eta/K}) + Y_0^2(a\sqrt{\eta/K})} \frac{d\eta}{\eta}$$

$$\int_0^\infty \phi_0 e^{-\eta x} \frac{K_0\{-i r\sqrt{\eta/K}\}}{K_0\{-i a\sqrt{\eta/K}\}} \frac{d\eta}{\eta} = \int_0^\infty \phi_0 e^{-\eta x} \frac{\left[J_0(r\sqrt{\eta/K}) + i\,Y_0(r\sqrt{\eta/K})\right]\left[J_0(a\sqrt{\eta/K}) - i\,Y_0(a\sqrt{\eta/K})\right]}{J_0^2(a\sqrt{\eta/K}) + Y_0^2(a\sqrt{\eta/K})} \frac{d\eta}{\eta}$$

Also note that $\int_\infty^0 f(x)\,dx = -\int_0^\infty f(x)\,dx$

ADDING THE INTEGRALS,

$$f(t) = \phi_0\left[1 - \frac{1}{2\pi i}\int_0^\infty e^{-\eta t} \frac{2i\left[Y_0(r\sqrt{\eta/K})J_0(a\sqrt{\eta/K}) - Y_0(a\sqrt{\eta/K})J_0(r\sqrt{\eta/K})\right]}{J_0^2(a\sqrt{\eta/K}) + Y_0^2(a\sqrt{\eta/K})} \frac{d\eta}{\eta}\right.$$

The integral can be simplified by a change in the variable of integration.

Let $\cancel{\xi = \kappa\eta^2}$

$$\eta = \kappa \xi^2$$

$$\xi = \sqrt{\eta/\kappa}$$

$\cancel{d\xi} =$

$$d\eta = 2\xi \kappa \, d\xi$$

After this substitution, the desired answer is found to be.

$$\phi(r,t) = \phi = \phi_0 \left[1 + \frac{2}{\pi} \int_0^\infty e^{-(\kappa t \xi^2)} \frac{J_0(\xi r) Y_0(\xi a) - J_0(\xi a) Y_0(\xi r)}{J_0^2(\xi a) + Y_0^2(\xi a)} \frac{d\xi}{\xi} \right]$$

Chap X #10. By applying the Laplace transform to Bessel's ✓ equation of zero order, namely.

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

show that $\int_0^\infty e^{-px} J_0(x)\,dx = c/\sqrt{1+p^2}$

and also that $c=1$. Deduce the transform of $J_1(x)$.

$$\mathcal{L}\left(x\frac{d^2y}{dx^2}\right) = -\frac{d}{ds}\left[\mathcal{L}\left(\frac{d^2y}{dx^2}\right)\right] = -\frac{d}{ds}\left[s^2 Y(s) - sy(0) - y'(0)\right]$$

$$= -\left[s^2 Y'(s) + 2s Y(s) - y(0)\right] \quad ✓$$

$$\mathcal{L}\left(\frac{dy}{dx}\right) = s Y(s) - y(0). \quad ✓$$

$$\mathcal{L}(xy) = -\frac{d}{ds}\left[\mathcal{L}(y)\right] = -Y'(s) \quad ✓$$

Therefore, the transform of Bessel's equation is

$$(s^2+1) Y'(s) + s Y(s) = 0$$

or.

$$\frac{dY(s)}{Y(s)} = \frac{-s\,ds}{s^2+1} \quad ✓$$

Integration gives.

$$\ln Y(s) = -\frac{1}{2}\ln(s^2+1) + \ln c \quad ✓$$

$$= \ln\frac{c}{\sqrt{s^2+1}}$$

(OVER)

Hence $\quad Y(s) = \frac{c}{\sqrt{s^2+1}} = \frac{c}{s}\left(1+\frac{1}{s^2}\right)^{-1/2}$

using the binomial expansion,

$$Y(s) = c\left[\frac{1}{s} - \frac{1}{2}\frac{1}{s^3} + \frac{3\cdot 1}{2!\,2^2}\frac{1}{s^5} - \frac{5\cdot 3\cdot 1}{3!\,2^3}\frac{1}{s^7} + \dots\right]$$

Inverting for $y(x)$

$$y(x) = c\left[1 - \frac{1}{2!\,2}x^2 + \frac{3\cdot 1}{4!\,2!\,2^2}x^4 - \frac{5\cdot 3\cdot 1}{6!\,3!\,2^3}x^6 + \dots\right]$$

$$y(x) = c\sum_{K=0}^{\infty}\frac{(-1)^K}{(K!)^2}\left(\frac{x}{2}\right)^{2K} = c\,J_0(x)$$

Thus, $\mathcal{L}^{-1}[c\,J_0(x)] = \frac{c}{\sqrt{s^2+1}} = c\,\mathcal{L}[J_0(x)]$

$$\text{ANS.}\quad \int_0^{\infty} e^{-px}J_0(x)\,dx = \frac{1}{\sqrt{1+p^2}}$$

The differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$$

is transformed to

$$(s^2+1)Y''(s) + 3s\,Y'(s) + (1-n^2)Y(s) = 0$$

for $n=1$,

$$Y(s)_{n=1} = \frac{c_1 s}{\sqrt{s^2+1}} + c_2$$

The two constants can be evaluated by series expansion, inverting and comparing to the series expansion of $J_1(x)$.

A simpler way to obtain $J_1(x)$ is to use the relation.

$$\frac{d}{dx} J_0(x) = - J_1(x).$$

$$\int_0^\infty e^{-px} \frac{d}{dx} (J_0(x)) dx = - \int_0^\infty e^{-px} J_1(x) dx$$

$$= e^{-px} J_0(x) \Big|_0^\infty + p \int_0^\infty e^{-px} J_0(x) dx = - \int_0^\infty e^{-px} J_1(x) dx$$

But $J_0(0) = 1$, and $\int_0^\infty e^{-px} J_0(x) dx = \frac{1}{\sqrt{p^2+1}}$.

$$-1 + \frac{p}{\sqrt{1+p^2}} = - \int_0^\infty e^{-px} J_1(x) dx$$

$$\underline{\underline{\text{Ans. } \int_0^\infty e^{-px} J_1(x) dx = \frac{(\sqrt{1+p^2} - p)}{\sqrt{1+p^2}}}}$$

R. B. Herrmann
Adv. Potential Theory.
Jan 23, 1965

Optional Problem:

Given a halfspace with a circular disc current source. Show that $\phi(r,z) = \frac{\rho I}{\pi a}\int_0^\infty e^{-\lambda z} J_1(\lambda a) J_0(\lambda r)\frac{d\lambda}{\lambda}$ for $z \geq 0$

Show that for a point electrode, e.g. $\lim_{a\to 0}$,
$$\phi(r,z) = \frac{\rho I}{2\pi}\frac{1}{\sqrt{r^2+z^2}}$$

For this type of proplem, $\nabla^2\phi = 0$. A feature of the harmonic function ϕ is that
$$\nabla^2\left(\frac{\partial\phi}{\partial z}\right) = 0.$$
This $\frac{\partial\phi}{\partial z}$ is used since it is proportional to the (vertical) normal current in the medium.

Boundary Conditions.

(i) $\phi \to 0$ as $r, z \to \infty$

(ii) $-\frac{1}{\rho}\frac{\partial V}{\partial z} = 0$, at $z=0$ and $r > a$

(iii) $-\frac{1}{\rho}\frac{\partial V}{\partial z} = \frac{I}{\pi a^2}$, at $z=0$, $r \leq a$.

The solution to this problem, found by application of the Hankel transform with respect to r,
$$\frac{\partial V(r,z)}{\partial z} = \int_0^\infty \lambda\, \bar{F}(\lambda)\, e^{-\lambda z} J_0(\lambda r)\, d\lambda$$
where
$$\bar{F}(\lambda) = \int_0^\infty r\, F(r,0)\, J_0(\lambda r)\, d\lambda.$$
where
$$F(r,0) = \begin{cases} -\rho I/\pi a^2 & r \leq a \\ 0 & r > a \end{cases}$$

RBH.

Optional continued.

Now $\bar{F}(\lambda) = \int_0^a -\frac{\rho I}{\pi a^2} \, r \, J_0(\lambda r) dr + \int_a^\infty 0 \cdot r \cdot J_0(\lambda r) dr$

$= -\frac{\rho I}{\pi a} \cdot \frac{1}{\lambda} J_1(\lambda a).$

Thus, $\frac{\partial V}{\partial z}(r,z) = \int_0^\infty -\frac{\rho I}{\pi a} e^{-\lambda z} J_1(\lambda a) J_0(\lambda r) \, d\lambda$

Integrating with respect to z,

$$\underline{\underline{V_d(r,z) = \frac{\rho I}{\pi a} \int_0^\infty e^{-\lambda z} J_1(\lambda a) J_0(\lambda r) \frac{d\lambda}{\lambda}}}$$

Point source,

$$\frac{1}{a} J_1(\lambda a) = \frac{1}{a} \cdot \frac{\lambda a}{2} \left\{ 1 - \frac{(\lambda a)^2}{2^3} + \ldots \right\}$$

$$\lim_{a \to 0} \frac{1}{a} J_1(\lambda a) = \frac{\lambda}{2}$$

Thus, for point source, $V_p(r,z) = \lim_{a \to 0} V_d(r,z)$

$$\underline{\underline{V_p = \frac{\rho I}{2\pi} \int_0^\infty e^{-\lambda z} J_0(\lambda r) \, d\lambda = \frac{\rho I}{2\pi} \frac{1}{\sqrt{z^2 + r^2}}}}$$

~~(over)~~ ...

RBH.

It might be of some interest to know the potential distribution on the plane z=0.

$$V(r,z) = \frac{\rho I}{\pi a}\int_0^\infty e^{-\lambda z} J_1(\lambda a) J_0(\lambda r)\frac{d\lambda}{\lambda}$$

$$V(r,0) = \frac{\rho I}{\pi a}\int_0^\infty J_1(\lambda a) J_0(\lambda r)\frac{d\lambda}{\lambda}$$

Using formulae 11.4.33 and 11.4.34. from NBS <u>Handbook of Mathematical Functions</u>

for $0 < r < a$,

$$V(r,0) = \frac{\rho I}{\pi a}\ {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; \frac{r^2}{a^2}\right)$$

for $0 < a < r$

$$V(r,0) = \frac{\rho I}{2\pi a}\frac{a}{r}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{a^2}{r^2}\right)$$

where ${}_2F_1(a,b;c;d) = F(a,b;c;d)$ is the Gauss hypergeometric function.

and

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^\infty \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}$$

Both of the above solutions converge absolutely since $\Re(c-a-b) > 0$

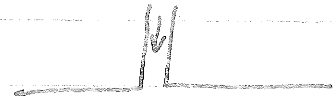
Some values are

$$V(0,0) = \frac{\rho I}{\pi a}\cdot\{1\}$$

$$V(a,0) = \frac{\rho I}{\pi a}\cdot\left\{\frac{2}{\pi}\right\}$$

$$V(\infty,0) = 0$$

Problem is analogous to flow from river into ocean



where instead of current u,
we analogously use J, the electric
current density.

RBH

Gph 204
Potential Theory

January 23, 1968
O. Nuttli

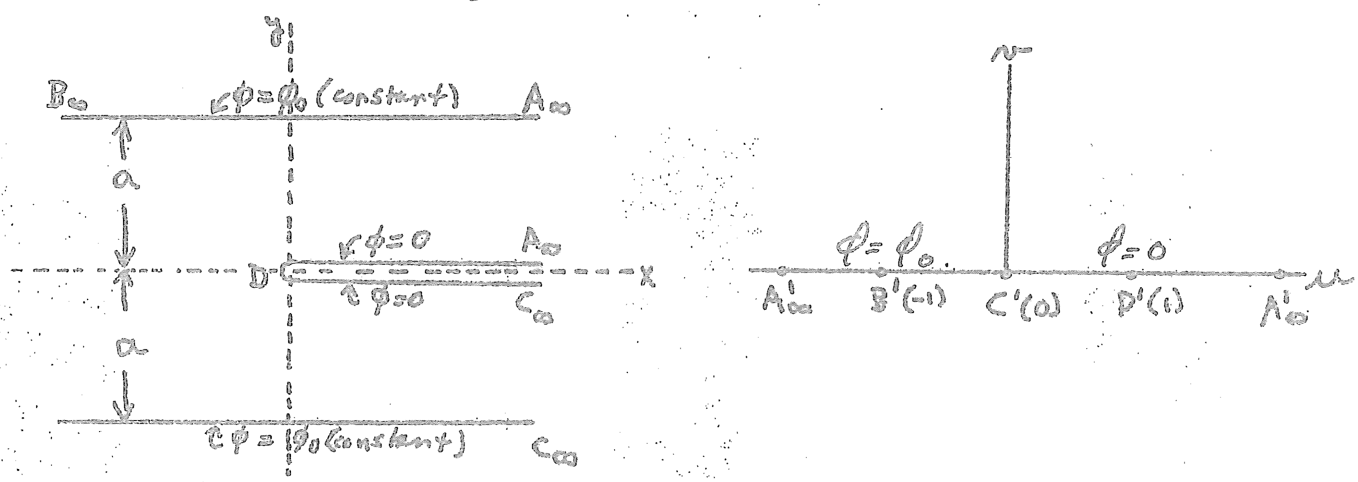
Semester Examination

1. Given $\nabla^2\phi = o$ in the interior of a spherical region of radius b, and $\frac{\partial\phi}{\partial n} = f(\theta)$ on $r = b$.

   a. Show that
$$\phi = \sum_{n=o}^{\infty} A_n r^n P_n(\cos\theta)$$
   where
$$A_n = \frac{2n+1}{2na^{n-1}} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta \, d\theta.$$

   b. If $f(\theta) = A\cos\theta$, find ϕ.

2. a. Show by using the Schwarz-Christoffel transformation that the figure in the z-plane is transformed to that in the w-plane by the transformation
$$z = \frac{a}{\pi}\left[\ln\frac{(w+1)^2}{w} - 2\ln 2\right].$$



   b. Find the potential ϕ in the region bounded by the surfaces $y = \pm a$, given that $\nabla^2\phi = o$ in the interior of that region.

3. Using the Fourier transform, show that the solution to the boundary value problem
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad -\infty<x<\infty, \quad 0<y<1$$

with

$$u(x,o) = e^{-2|x|}$$

$$u(x,1) = o$$

$u(x,y) \to o$ uniformly in y as $x \to \pm \infty$

is

$$u = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sinh[\xi(1-y)]\, e^{-i\xi x} d\xi}{(4+\xi^2)\sinh \xi}$$

Hint: you may use the relation

$$\int_0^{\infty} e^{-2x} \cos \xi x \, dx = \frac{2}{4+\xi^2}.$$

Robert B. Herrmann
Final Exam

100

1. ⓐ The solution of $\nabla^2\phi = 0$ in spherical coordinates is

$$\phi = \begin{Bmatrix} r^n \\ r^{-(n+1)} \end{Bmatrix} \begin{Bmatrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{\pm im\psi} \end{Bmatrix}$$

Since the boundary conditions are independent of ψ, the resulting function, by applying a fourier integral will be independent of ψ

Since this is an interior problem to be valid $0 \le \theta \le \pi$, and independent of ψ, since the B.C. does not involve ψ, the solution is of the form

$$\phi = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

B.C. $\dfrac{\partial\phi}{\partial n} = f(\theta)$ on $r = b$.

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$$

$$\frac{\partial\phi}{\partial r} = \sum_{n=0}^{\infty} n A_n r^{n-1} P_n(\cos\theta)$$

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=b} = f(\theta) = \sum_{n=0}^{\infty} n A_n b^{n-1} P_n(\cos\theta).$$

Multiplying through by $P_m(\cos\theta)\sin\theta\, d\theta$ and integrating from $0 \to \pi$, making use of the orthogonality relation of Legendre Polynomials,

$$\int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta\, d\theta = \begin{cases} 0 & n \ne m \\ \frac{2}{2n+1} & n = m \end{cases}$$

$\therefore \quad A_1 = \frac{3}{2b^{n-1}} \int_0^{\pi} A\, P_1^2(\cos\theta) \sin\theta \, d\theta =$

$$\frac{3}{2b^{n-1}} \cdot \frac{2A}{3} = \frac{A}{b^0} = A$$

$$A_2 = \frac{5}{4b} \int_0^{\pi} A P_1(\cos\theta) P_2(\cos\theta) \sin\theta \, d\theta = 0.$$

$\therefore \quad A_n = 0 \quad$ for $n \geq 2$.

$A_1 = 1.$

Ans b $\quad \phi = A r \cos\theta + C$

RBH

2. TRANSFORMATION.

Point	z	w	ANGLE.
B_∞	$-\infty$	-1	0
A_∞	$+\infty$	$-\infty$	0
D	0	1	2π
C_∞	$+\infty$	0	0

The Schwarz-Christoffel transformation is.
(taking care of infinite ~~point~~ by method in book)

$$\frac{dz}{dw} = K(w+1)^{-1}(w-1)^{-1}w^{-1}$$

$$\frac{dz}{dw} = K\frac{w-1}{w(w+1)} = K\left[\frac{-1}{w} + \frac{2}{w+1}\right]$$

$$z = K\left[\ln\frac{(w+1)^2}{w}\right] + c$$

$w = 1, \quad z = 0,$

$$0 = K\ln 4 + c \qquad c = -K\ln 4$$

$$z = K\left[\ln\frac{(w+1)^2}{w} - 2\ln 2\right]$$

To determine K, we make use of the fact that for the range ~~$-\infty < w < -1$~~, $-1 < w < 0$,
$Im(z) = \cancel{\infty}\, a$.

$$z = K\left[\ln(w+1)^2 - \ln w - 2\ln 2\right]$$

RBH

For negative values of w $-1<w<0$, the term $\ln(w+1)^2$ is real.

In this range $-1<w<0$, we can use.

$$w = r e^{i\pi}.$$

and

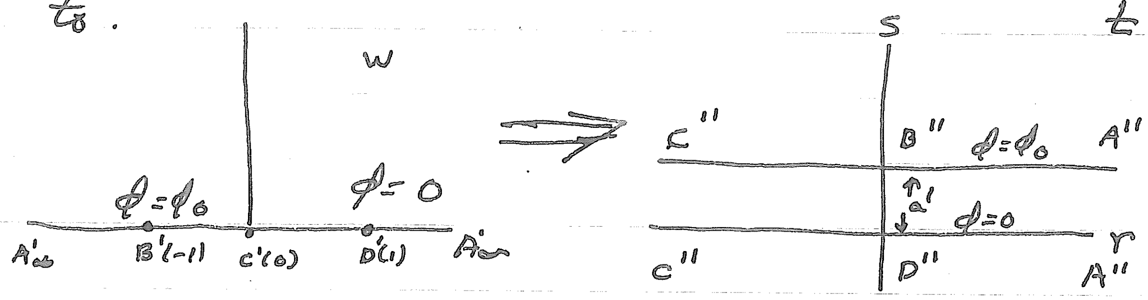
$$z = K\left[\ln(w+1)^2 - \ln r - i\pi - 2\ln 2\right]$$

$$\operatorname{Im}(z) = -a = -K\pi.$$

$$\therefore \quad K = \frac{a}{\pi}.$$

Ans. (a) $\quad z = \dfrac{a}{\pi}\left[\ln\dfrac{(w+1)^2}{w} - 2\ln 2\right]$

(b) The constant potentials of the figure are transformed to.



by the transformation.

$$w = e^{\pi t/a'}$$

Schaum's Complex Variables p205 #A-2

The solution to $\nabla^2\phi = 0$ in the t plane is:

$$\phi = \frac{\phi_0}{a'} \, Im(t)$$

if $t = \tau + is$ $Im(t) = s$.

Now

$$t = \frac{a'}{\pi} \ln w = \frac{a}{\pi} \left\{ \ln \sqrt{u^2+v^2} + i \tan^{-1} \frac{v}{u} \right\}$$

$$Im(t) = \frac{a'}{\pi} \tan^{-1} \frac{v}{u}.$$

Ans. $\underline{\phi = \frac{\phi_0}{\pi} \tan^{-1} \frac{v}{u}}$

where $\underline{w = u + iv}$

and $\underline{z = \frac{a}{\pi} \left[\ln \frac{(w+1)^2}{w} - 2\ln 2 \right]}$

or Ans. $\underline{\phi = \frac{\phi_0}{a'} Im(t).}$

$$\underline{t = \frac{a'}{\pi} \ln w}$$

$$\underline{z = \frac{a}{\pi} \left[\ln \frac{(w+1)^2}{w} - 2\ln 2 \right]}$$

RBH.

3. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$-\infty < x < \infty$

$0 < y < 1$

$u(x,0) = e^{-2|x|}$

$u(x,1) = 0$

$u(x,y) \to 0$ in y as $x \to \pm\infty$

Complex Fourier Transform:

$$\bar{f}(p) = \int_{-\infty}^{\infty} e^{+ipx} f(x) dx)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx} \bar{f}(p) dp.$$

$$\int_{-\infty}^{\infty} e^{ipx} \frac{\partial^2 u}{\partial x^2} dx = e^{ipx} \frac{\partial u}{\partial x}\Big|_{-\infty}^{\infty} - ip \int_{-\infty}^{\infty} e^{ipx} \frac{\partial u}{\partial x} dx$$

$$= e^{ipx} \frac{\partial u}{\partial x}\Big|_{-\infty}^{\infty} - ip e^{ipx} u(x,y)\Big|_{-\infty}^{\infty} - p^2 \bar{u}(p,y)$$

Assuming that $\partial u/\partial x$, $u \to 0$ as $x \to \pm\infty$,

$$\int_{-\infty}^{\infty} e^{ipx} \frac{\partial^2 u}{\partial x^2} dx = -p^2 \bar{u}(p,y)$$

$$\int_{-\infty}^{\infty} e^{ipx} \frac{\partial^2 u}{\partial y^2} dx = \frac{d^2 \bar{u}(p,y)}{dy^2}$$

The original P.D.E. is transformed into.

$$\frac{d^2 \bar{u}(p,y)}{dy^2} - p^2 \bar{u}(p,y) = 0$$

RBH

...ral solution to this transformed O.D.E is.

$$\bar{u} = A \sinh(py) + B \cosh(py).$$

Boundary conditions.

(1) $u(x,1)=0 \Rightarrow \bar{u}(p,1)=0.$

since

$$\bar{u}(p,1) = \int_{-\infty}^{\infty} e^{ipx} \cdot 0 \cdot dx = 0.$$

(2) $u(x,0) = e^{-2|x|} \Rightarrow \bar{u}(p,0) = \frac{4}{4+p^2}$

$$\bar{u}(p,0) = \int_{-\infty}^{\infty} e^{ipx} e^{-2|x|} dx$$

$$= \int_{-\infty}^{0} e^{ipx} \cdot e^{+2x} dx + \int_{0}^{\infty} e^{ipx} \cdot e^{-2x} dx$$

$$= \frac{1}{2+ip} - \frac{1}{ip-2} = \frac{4}{4+p^2}$$

Applying B.C.

$$0 = A \sinh p + B \cosh p$$

$$\frac{4}{4+p^2} = B$$

$$\therefore \quad B = \frac{4}{4+p^2}, \quad A = -\frac{4}{4+p^2} \coth p.$$

RBH

$$\bar{u}(p,y) = \frac{4}{4+p^2}\left[-\frac{\cosh p \sinh py + \sinh p \cosh py}{\sinh p}\right]$$

But $\sinh[p(1-y)] = \sinh p \cosh py - \sinh py \cosh p.$

$$\therefore \quad \bar{u}(p,y) = \frac{4}{4+p^2} \quad \frac{\sinh[p(1-y)]}{\sinh p}.$$

Taking the inverse transform,

ANS 3 $$f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{e^{-ipx} \sinh[p(1-y)]\,dp}{(4+p^2)\sinh p}.$$

✓

$f(x,y)$