Therefore (6.2) and (6.3) form our new boundary ~~condition~~ value problem.

Eq. (6.2) is easily solved. Its solution is

$$\bar{\theta} = Ae^{-\sqrt{p/\kappa}\,x} + Be^{+\sqrt{p/\kappa}\,x}$$

But $\bar{\theta}$ as well as $\theta$ must remain finite as $x \to \infty$. Therefore $B = 0$.

Using (6.3),

$$\bar{\theta}_{x=0} = \frac{\theta_0}{p} = Ae^{0} \quad \text{. Thus} \quad A = \frac{\theta_0}{p} \quad \text{and}$$

$$\bar{\theta}(p,x) = \frac{\theta_0}{p}\, e^{-\sqrt{p/\kappa}\,x} \quad . \qquad (6.4)$$

The solution of the problem now involves finding the inverse of (6.4).

~~If~~ Eq. (6.4) does not correspond to any of the functions given in Table I, pg 20; Either we must find a more complete table of Laplace transform pairs, or we must use a general expression, such as we shall see two classes from now. All we shall do at this time is write down the solution.

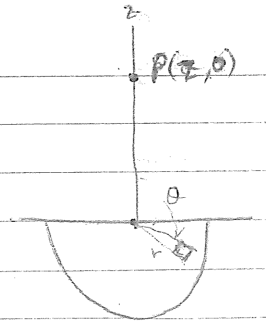
It is

$$\theta(t,x) = \theta_0 \int_{x/2\sqrt{\kappa t}}^{\infty} \frac{2}{\sqrt{\pi}}\, e^{-u^2}\, du \qquad (6.5)$$

The integral ~~multiplied~~ in (6.5) has been tabulated. It is called the complementary error function of $\frac{x}{2\sqrt{\kappa t}}$, and written as erfc $\frac{x}{2\sqrt{\kappa t}}$. Thus

$$\theta(x,t) = \theta_0\, \mathrm{erfc}\left\{\frac{x}{2\sqrt{\kappa t}}\right\} .$$

Consider $z > 0$.



$$\varphi(z,\theta) = 2\pi \int_0^b \int_{\pi/2}^{\pi} \frac{r^2 \sin\theta \, d\theta \, dr}{\sqrt{r^2+z^2-2zr\cos\theta}}$$

$$= 2\pi \int_0^b \left[ \frac{r}{z} \sqrt{r^2+z^2-2zr\cos\theta} \right]_{\pi/2}^{\pi} dr$$

$$= \frac{2\pi}{z} \int_0^b \left[ r(r+z) - r\sqrt{r^2+z^2} \right] dr$$

$$\varphi(z,0) = \frac{2\pi}{z} \left[ \frac{r^3}{3} + \frac{z r^2}{2} - \frac{1}{3}(r^2+z^2)^{3/2} \right]_0^b$$

$$= \frac{2\pi}{3z} \left[ b^3 + \frac{3zb^2}{2} - (b^2+z^2)^{3/2} + z^3 \right]$$

For $z > b$,

$$\varphi(z,0) = \frac{2}{3}\frac{\pi}{z}\left[ b^3 + \frac{3}{2}b^2 z + z^3 - z^3 \left\{ 1 + \frac{3}{2}\frac{b^2}{z^2} + \frac{3\cdot1}{2\cdot4}\frac{b^4}{z^4} - \frac{3\cdot1\cdot1}{2\cdot4\cdot6}\frac{b^6}{z^6} \cdots \right\} \right]$$

$$= \frac{2}{3}\pi \left[ \cancel{\frac{3}{2}b^2} + b^3/z - \cancel{\frac{3}{2}b^2} - \frac{3\cdot1}{2\cdot4}\frac{b^4}{z^2} + \frac{3\cdot1\cdot1}{2\cdot4\cdot6}\frac{b^6}{z^4} - \cdots \right]$$

$$\varphi(r,\theta) = \sum_{n=0}^{\infty} \left( A_n r^n + B_n r^{-(n+1)} \right) P_n(\cos\theta)$$

$\cancel{A_0 = \frac{2}{3}\pi \cdot \frac{3}{2}b^2 = \pi b^2}$ $\qquad A_n = 0$

$\cancel{A_1 = \frac{2}{3}\pi \cdot \left(-\frac{3}{2}b^2\right) = -\pi b^2}$

$B_0 = \frac{2}{3}\pi b^3$

$B_1 = \frac{2}{3}\pi\left(-\frac{3}{8}\right) b^4 = -\frac{\pi}{8} b^4$

$B_{2m} = 0$ for $m \geq 1$

$B_3 = \frac{\pi}{24} b^6$

$$\varphi(r,\theta) = \left[ \cancel{\pi b^2} + \frac{2}{3}\pi b^3/r \; P_0(\cos\theta) - \left(\cancel{\pi b^2} + \frac{\pi}{8} b^4/r^2\right) P_1(\cos\theta) + \pi \frac{b^6}{24}/r^4 \; P_3(\cos\theta) - \cdots \right]$$

For $z < b$

$$\varphi(z,0) = \frac{2}{3}\frac{\pi}{z}\left[ b^3 + \frac{3}{2}b^2 z + z^3 - b^3 \left\{ 1 + \frac{3}{2}\frac{z^2}{b^2} + \frac{3\cdot1}{2\cdot4}\frac{z^4}{b^4} - \frac{3\cdot1\cdot1}{2\cdot4\cdot6}\frac{z^6}{b^6} + \cdots \right\} \right]$$

$$= \frac{2}{3}\frac{\pi}{z}\left[ z^2 + \frac{3}{2}b^2 \cancel{z} - \frac{3}{2} b z^1 - \frac{3}{8}\frac{z^3}{b} + \frac{1}{16}\frac{z^5}{b^3} - \cdots \right]$$

$$\varphi(r,\theta) = \sum_{n=0}^{\infty} \left( A_n r^n + B_n r^{-(n+1)} \right) P_n(\cos\theta)$$

$A_0 = \pi b^2$, $A_1 = -\pi b$, $A_2 = \frac{2}{3}\pi$, $A_3 = -\frac{1}{4}\pi/b$, $A_5 = \frac{\pi}{24 b^3}$

$A_{2m} = 0$ for $m \geq 2$, $B_m = 0$

$$\varphi(r,\theta) = \pi b^2 - \pi b r P_1(\cos\theta) + \frac{2\pi}{3} r^2 P_2(\cos\theta) - \frac{\pi}{4b} r^3 P_3(\cos\theta) + \frac{\pi}{24 b^3}$$

7. The Finite Sine Transform.

Consider a function $f(x)$ defined in the interval $0 \le x \le a$ which satisfies the conditions which allow it to be expressed by a Fourier sine series in the interval $0 \le x \le a$. Then

$$f(x) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) \qquad (7.1)$$

with $$A_m = \frac{2}{a}\int_0^a f(x')\sin\left(\frac{m\pi x'}{a}\right)dx' .$$

Substituting the last expression into (7.1) gives

$$f(x) = \frac{2}{a}\sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right)\int_0^a f(x')\sin\left(\frac{m\pi x'}{a}\right)dx' . \qquad (7.2)$$

The integral in (7.2), which is a function of $m$, is called the finite sine transform of $f(x)$ and is indicated by $\bar{f}_s(m)$. A knowledge of $\bar{f}_s(m)$ will give us the value of $f(x)$, because

$$f(x) = \frac{2}{a}\sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right)\bar{f}_s(m). \qquad (7.3)$$

Example:

diff. eqn. $$\frac{\partial^2 V}{\partial x^2} = \frac{1}{K}\frac{\partial V}{\partial t} \qquad (7.4)$$

Bdy + initial conditions

1) $$V = \begin{cases} 2V_0 \dfrac{x}{l} & \text{for } 0 \le x \le l/2 \\ 2V_0\left(\dfrac{l-x}{l}\right) & \phantom{\text{for }} l/2 \le x \le l \end{cases} \quad \text{at } t=0$$

2) $V = 0$ at $x=0$ and at $x=l$ for all values of $t$.

We are to find $V$ for all $t>0$ anywhere in the interval $0 \le x \le l$.

As in the preceding example, obtain the transform of the diff. eqn. and the initial and bdy. condition equations. For the sine transform, we shall use the notation $\bar{V}_s(m,t) = \int_0^l V(x,t)\sin\frac{m\pi x}{l}dx$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{k}\frac{\partial \varphi}{\partial t}$$

for $0 \le x \le l$, $t > 0$

with 1) $\varphi = \begin{cases} 2\varphi_0 \frac{x}{l} & \text{for } 0 \le x \le l/2 \\ 2\varphi_0 \left(\frac{l-x}{l}\right) & \text{for } \frac{l}{2} \le x \le l \end{cases}$ at $t = 0$

2) $\varphi = 0$ at $x = 0$ and $x = l$ for all $t$.

$a = 1$, $b = c = 0$, $L = -\frac{1}{k}\frac{\partial}{\partial t}$, $f = 0$. Use finite sine transform $K = \sin\frac{n\pi x}{l}$

$\bar{f} = 0$

$$g = \left[K \frac{\partial \varphi}{\partial x} - \frac{\partial K}{\partial x}\varphi\right]_0^l$$

$$g = \left[\sin\frac{n\pi x}{l}\frac{\partial \varphi}{\partial x} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\,\varphi\right]_0^l = 0$$

$$(L + \lambda)\,\bar{\varphi}(n,t) = 0$$

$$\left(-\frac{1}{k}\frac{d}{dt} + \lambda\right)\bar{\varphi}(n,t) = 0$$ is transformed diff eq. where

$$\frac{\partial^2}{\partial x^2}\left(\sin\frac{n\pi x}{l}\right) = \lambda \sin\frac{n\pi x}{l}$$

$$-\left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l} = \lambda \sin\frac{n\pi x}{l} \quad \text{or} \quad \lambda = -\left(\frac{n\pi}{l}\right)^2$$

Therefore transformed diff. eq. is

$$\left[+\frac{1}{k}\frac{d}{dt} + \left(\frac{n\pi}{l}\right)^2\right]\bar{\varphi}(n,t) = 0$$

Solution is

$$\bar{\varphi}(n,t) = A e^{-k\left(\frac{n\pi}{l}\right)^2 t}$$

Transform of initial conditions (eq. 1) is

$$\bar{\varphi}(n,0) = \frac{2\varphi_0}{l}\left[\int_0^{l/2} x \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2\varphi_0}{l}\cancel{\left\{\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{l^2}{2n\pi}\cos n\pi - \frac{l^2}{n^2\pi^2}\left(-\sin\frac{n\pi}{2} - n\pi\cos n\pi\right)\right\}}$$

$$= \frac{2\varphi_0}{l}\left[2\left(\frac{l}{n\pi}\right)^2 \sin\frac{n\pi}{2}\right] = \frac{4\varphi_0 l}{(n\pi)^2}\sin\frac{n\pi}{2}$$

Thus

$$\bar{\varphi}(n,0) = \frac{4\varphi_0 l}{n^2\pi^2} \sin\frac{n\pi}{2}.$$

But

$$\bar{\varphi}(n,0) = A = \frac{4\varphi_0 l}{n^2\pi^2} \sin\frac{n\pi}{2} = (-1)^r \frac{4\varphi_0 l}{(2r+1)^2\pi^2}$$ with $n = 2r+1$, $r = 0, 1, 2, \ldots$

Therefore $$\bar{\varphi}(n,t) = (-1)^r \frac{4\varphi_0 l}{(2r+1)^2\pi^2} e^{-k\left(\frac{n\pi}{l}\right)^2 t}.$$

Taking the inverse transform

$$\varphi(x,t) = \frac{8\varphi_0}{\pi^2} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^2} \sin\left\{(2r+1)\frac{\pi x}{l}\right\} e^{-k\frac{\pi^2}{l^2}(2r+1)^2 t}$$

We have

$$\int_0^l \frac{\partial^2 V}{\partial x^2} \sin \frac{m\pi x}{l}\, dx = \frac{1}{K}\int_0^l \frac{\partial V}{\partial t} \sin \frac{m\pi x}{l}\, dx \qquad (7.4a)$$

Consider the term on the left hand side. Integration by parts gives

$$\int_0^l \frac{\partial^2 V}{\partial x^2} \sin \frac{m\pi x}{l}\, dx = \left[\frac{\partial V}{\partial x} \sin \frac{m\pi x}{l}\right]_0^l - \frac{m\pi}{l}\int_0^l \frac{\partial V}{\partial x} \cos \frac{m\pi x}{l}\, dx$$

Integration by parts a second

$$= 0 - 0 - \frac{m\pi}{l}\left[V \cos \frac{m\pi x}{l}\right]_0^l - \left(\frac{m\pi}{l}\right)^2 \int_0^l V \sin \frac{m\pi x}{l}\, dx$$

$$= -\frac{m\pi}{l}\underbrace{\left[\;0\;\right]}_{\text{because } V = 0 \text{ at } x = 0 \text{ and } x = l} - \left(\frac{m\pi}{l}\right)^2 \bar{V}\,.$$

Consider the term on the right hand side

$$\frac{1}{K}\int_0^l \frac{\partial V}{\partial t} \sin \frac{m\pi x}{l}\, dx = \frac{1}{K}\frac{d\bar{V}}{dt}\,.$$

Thus eq. (7.4a) becomes

$$-\left(\frac{m\pi}{l}\right)^2 \bar{V} = \frac{1}{K}\frac{d\bar{V}}{dt}$$

or

$$\frac{d\bar{V}}{dt} + K\left(\frac{m\pi}{l}\right)^2 \bar{V} = 0$$

whose solution is

$$\bar{V} = B e^{-K\left(\frac{m\pi}{l}\right)^2 t} \qquad (7.5)$$

The transform of the initial condition equation (1) is

$$\bar{V}(m,0) = \frac{2V_0}{l}\left\{\int_0^{l/2} x \sin \frac{m\pi x}{l}\, dx + \int_{l/2}^{l} (l-x)\sin\left(\frac{m\pi x}{l}\right) dx\right\}$$

$$= \frac{2V_0}{l}\left\{\frac{l^2}{m^2\pi^2}\left[\sin \frac{m\pi x}{l} - \frac{m\pi x}{l}\cos \frac{m\pi x}{l}\right]_0^{l/2} + \frac{l^2}{m\pi}\left[\cos \frac{m\pi x}{l}\right]_{l/2}^{l} - \frac{l^2}{m^2\pi^2}\left[\sin \frac{m\pi x}{l} - \frac{m\pi x}{l}\cos \frac{m\pi x}{l}\right]_{l/2}^{l}\right\}$$

which, upon integrating, becomes

$$\bar{V}(m,0) = \frac{2V_0}{l}\left\{\frac{l^2}{m^2\pi^2}\sin \frac{m\pi}{2}\right\} - \frac{l^2}{2m\pi}\cos m\pi - \frac{l^2}{m^2\pi^2}\left[-\sin \frac{m\pi}{2} - m\pi \cos m\pi\right]\}$$

$$\bar{V}(m,0) = \frac{4 l V_0}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right) \checkmark \qquad (7.6)$$

From (7.5),

$$\bar V(m,0) = B. \quad \text{Thus, from (7.6),}$$

$$B = \frac{4 l V_0}{m^2\pi^2} \sin\frac{m\pi}{2} = (-1)^r \frac{4 l V_0}{(2r+1)^2\pi^2} \quad \text{with } m = 2r+1,\ r = 0, 1, 2, \ldots$$

Thus, since from (7.5) $\bar V(m,t) \cancel{= \int_0^l V(x,t) \sin\frac{m\pi x}{l}\, dx} = B e^{-\frac{K m^2\pi^2}{l^2} t}$

$$\bar V(m,t) = (-1)^r \frac{4 l V_0}{(2r+1)^2\pi^2} e^{-\frac{(2r+1)^2 K\pi^2}{l^2} t}$$

and the solution of our boundary-value problem is

$$V(x,t) = \frac{8 V_0}{\pi^2} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^2} \sin\left\{(2r+1)\frac{\pi x}{l}\right\} e^{-(2r+1)^2\frac{K\pi^2}{l^2} t}.$$

## Finite cosine transform.

The finite cosine transform is similar to the sine transform.
The transform pair is given by

$$\bar f_c(m) = \int_0^a f(x) \cos\left(\frac{m\pi x}{a}\right) dx$$

and

$$f(x) = \frac{f(0)}{a} + \frac{2}{a}\sum_{m=1}^{\infty} \bar f_c(m) \cos\frac{m\pi x}{a}, \qquad (7.7)$$

where $f(0) = f(x)_{x=0}$.

## Double finite sine transform.

By an extension of the preceding ideas we can extend the analysis to a function $F(x,y)$ of two space variables. Suppose that $F(x,y)$, known in the region $0 \le x \le a,\ 0 \le y \le b$, can be expanded in a double Fourier sine series of the form

$$F(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad (7.8)$$

with

$$A_{mn} = \frac{4}{ab}\int_0^a\!\!\int_0^b F(x',y') \sin\left(\frac{m\pi x'}{a}\right) \sin\left(\frac{n\pi y'}{b}\right) dx'\,dy' \quad . \ (7$$

The transform pair ~~is readily~~ can be shown to be

$$\bar{F}(m,n) \equiv \int_0^a \int_0^b F(x',y') \sin\left(\frac{m\pi x'}{a}\right) \sin\left(\frac{n\pi y'}{b}\right) dx'dy' \qquad (7.13)$$

with $A_{mn} = \frac{4}{ab} \bar{F}(m,n)$

and
$$F(x,y) = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{F}(m,n) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right). \qquad (7.14)$$

Section III.

## Fourier Transform.

The complex form of the "infinite Fourier transform pair is

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\omega)\, e^{i\omega t}\, d\omega$$

and
$$\bar{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \,. \qquad (11.1)$$

As an example, consider the solution of the equation

$$\frac{d^2y}{dt^2} + \beta \frac{dy}{dt} + \omega_0^2 y = F(t) \qquad (11.2)$$

This equation appears frequently in problems involving oscillating systems with one degree of freedom, such as a mechanical seismograph.

Multiply both sides of (11.2) by $\frac{1}{\sqrt{2\pi}} e^{-i\omega t}$ and integrate with respect to $t$ from $-\infty$ to $+\infty$.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{d}{dt}\left(\frac{dy}{dt}\right) dt = \frac{1}{\sqrt{2\pi}} \left\{ \left[ e^{-i\omega t} \frac{dy}{dt} \right]_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty} (-i\omega)\, e^{-i\omega t} \frac{dy}{dt}\, dt \right\}$$

There is no limiting value, in the sense we are accustomed to think of, of the expression in brackets. We shall assume that for large $t$ its value is zero. Integrate $\frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{dy}{dt}\, dt$ by parts. This gives

We might expect this if $\frac{dy}{dt} \to 0$ as $t \to \infty$. We have a damping term $\beta \frac{dy}{dt}$ in the diff. eqn.

$$\frac{i\omega}{\sqrt{2\pi}}\left\{\left[e^{-i\omega t}\,y\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} y\,(-i\omega)\,e^{-i\omega t}\,dt\right\}.$$

Again *assume* the term in brackets is zero.

[We might expect $y \to 0$ as $|t| \to \infty$ because of the presence of the "damping" term $\beta$ in the diff. eqn.)

Thus

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d^2y}{dt^2}\,e^{-i\omega t}\,dt = -\omega^2\,\bar{y}(\omega)$$

Similarly,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\beta\,\frac{dy}{dt}\,e^{-i\omega t}\,dt = i\beta\omega\;\bar{y}(\omega).$$

Therefore

$$(\omega_0^2 + i\beta\omega - \omega^2)\,\bar{y}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(t)\,e^{-i\omega t}\,dt \equiv A(\omega),$$ where

~~$A(\omega)$ is called the frequency spectrum of $F(t)$~~.

Thus

$$\bar{y}(\omega) = \frac{A(\omega)}{\omega_0^2 + i\beta\omega - \omega^2} = \frac{A(\omega)}{Z(\omega)}$$

where $Z(\omega) \equiv \omega_0^2 + i\beta\omega - \omega^2$ is called the complex impedance of the system.

Therefore,

$$y(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{y}(\omega)\,e^{i\omega t}\,d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{A(\omega)}{Z(\omega)}\,e^{i\omega t}\,d\omega.$$

# Method of Integral Transforms

Sneddon "Elements of Partial Diff. Eqs" pp 126-129

Suppose we have a ~~function u which we wish to evaluate, and it depends on the n var~~ linear partial differential equation in n independent variables $(x_1, x_2, \cdots x_n)$. Let the dependent variable, which we wish to evaluate, be $u$. The diff. eq. may be written as

$$a(x_1)\frac{\partial^2 u}{\partial x_1^2} + b(x_1)\frac{\partial u}{\partial x_1} + c(x_1)u + Lu = f(x_1, x_2, x_3, \cdots x_n) \tag{1}$$

where $L$ is a <u>linear</u> differential operator in the variables $(x_2, x_3, \cdots x_n)$ and the range of variation of $x_1$ is $\alpha \le x_1 \le \beta$.

We shall define the transform $\bar{u}(\xi, x_2, \cdots x_n)$ to be

$$\bar{u}(\xi, x_2, x_3, \cdots x_n) \equiv \int_\alpha^\beta u(x_1, x_2, \cdots, x_n)\, K(\xi, x_1)\, dx_1 \tag{2}$$

The quantity $K(\xi, x_1)$ is known as the kernel of the transform.

see pg. 129 next.

Multiply the first three terms on the left-hand side of the equation by $K(\xi, x_1)$ and integrate this expression with respect to $x_1$, with $x_1$ varying from $\alpha$ to $\beta$. Let us integrate each separately, using integration by parts. This gives

$$\int_\alpha^\beta a(x_1)\, K(\xi, x_1)\frac{\partial^2 u}{\partial x_1^2}\, dx_1 = \left[aK\frac{\partial u}{\partial x_1}\right]_{x_1=\alpha}^{x_1=\beta} - \int_\alpha^\beta \frac{\partial(aK)}{\partial x_1}\frac{\partial u}{\partial x_1}\, dx_1 .$$

Integrate the last expression on the right by parts again. This gives

$$\int_\alpha^\beta \frac{\partial(aK)}{\partial x_1}\frac{\partial u}{\partial x_1}\, dx_1 = \left[\frac{\partial(aK)}{\partial x_1}u\right]_\alpha^\beta - \int_\alpha^\beta u\,\frac{\partial^2(aK)}{\partial x_1^2}\, dx_1 .$$

Thus

$$\int_\alpha^\beta a(x_1)\, K(\xi, x_1)\, \frac{\partial^2 u}{\partial x_1^2}\, dx = \left[ aK \frac{\partial u}{\partial x_1} - \frac{\partial (aK)}{\partial x_1}\, u \right]_\alpha^\beta + \int_\alpha^\beta u\, \frac{\partial^2 (aK)}{\partial x_1^2}\, dx_1$$

Integrating by parts,

$$\int_\alpha^\beta b(x_1)\, K(\xi, x_1)\, \frac{\partial u}{\partial x_1}\, dx_1 = \Big[ bKu \Big]_{x_1=\alpha}^{x_1=\beta} - \int_\alpha^\beta u\, \frac{\partial (bK)}{\partial x_1}\, dx_1$$

~~and~~

$$\cancel{\int_\alpha^\beta K c(x_1)\, u\, dx}$$

$$\cancel{\int_\alpha^\beta c(x_1)\, K(\xi, x_1)\, u\, dx_1 = \int}$$

Thus

$$\int_\alpha^\beta K(\xi, x_1) \left\{ a(x_1) \frac{\partial^2 u}{\partial x_1^2} + b(x_1) \frac{\partial u}{\partial x_1} + c(x_1)\, u \right\} dx_1$$

$$= \left[ aK \frac{\partial u}{\partial x_1} + \left\{ bK - \frac{\partial (aK)}{\partial x_1} \right\} u \right]_\alpha^\beta + \int_\alpha^\beta u \left\{ \frac{\partial^2 (aK)}{\partial x_1^2} - \frac{\partial (bK)}{\partial x_1} + cK \right\} dx_1$$

$$= g(\xi, x_2, x_3, \cdots, x_n) + \int_\alpha^\beta u \left\{ \frac{\partial^2 (aK)}{\partial x_1^2} - \frac{\partial (bK)}{\partial x_1} + cK \right\} dx_1$$

where $g(\xi, x_2, x_3, \cdots, x_n) \equiv \left[ aK \frac{\partial u}{\partial x_1} + u \left\{ bK - \frac{\partial (aK)}{\partial x_1} \right\} \right]_\alpha^\beta$.

Choose the function $K(\xi, x_1)$ so that

$$\frac{\partial^2 (aK)}{\partial x_1^2} - \frac{\partial (bK)}{\partial x_1} + cK = \lambda K \qquad (3)$$

where $\lambda$ is a constant.

(v) Multiply eq. (1) by $K(\xi, x_1)$ and integrate with respect to $x_1$ from $\alpha$ to $\beta$. This gives

$$\int_\alpha^\beta \left\{ a(x_1)\frac{\partial^2 u}{\partial x_1^2} + b(x_1)\frac{\partial u}{\partial x_1} + c(x_1)u + Lu \right\} K(\xi, x_1)\, dx_1 = \int_\alpha^\beta K f(x_1, x_2, \cdots, x_n)\, dx_1$$

Now the left hand side by eq. (2) can be written as

$$L.H.S. = g(\xi, x_2, x_3, \cdots, x_n) + \int_\alpha^\beta (\lambda + L) u K\, dx_1$$

$$R.H.S. = \int_\alpha^\beta K f(x_1, x_2, \cdots, x_n)\, dx_1 = \bar{f}(\xi, x_2, x_3, \cdots, x_n)$$

We might note that in the equation L.H.S.,

$$\int_\alpha^\beta (\lambda + L) u K\, dx_1 = (\lambda + L)\int_\alpha^\beta u K\, dx_1$$

because $\lambda$ is a constant and $L$ is <u>not</u> a function of $x_1$.

Equating L.H.S. and R.H.S., we get

$$(L + \lambda)\bar{u} + g = \bar{f}$$

and

$$(L + \lambda)\bar{u} = F(\xi, x_2, x_3, \cdots, x_n) \qquad (4)$$

where

$$F(\xi, x_2, x_3, \cdots, x_n) \equiv \bar{f}(\xi, x_2, x_3, \cdots, x_n) - g(\xi, x_2, x_3, \cdots, x_n).$$

$\bar{u}$ is called the integral transform of $u$ corresponding to the kernel $K(\xi, x_1)$.

The effect of employing the integral transform defined by eqs. (2) and (3) is therefore to reduce the partial differential equation (1) in $n$ independent variables $x_1, x_2, x_3, \cdots, x_n$ to one in $(n-1)$ independent variables $\{x_2, x_3, \cdots, x_n\}$ and a parameter $\xi$. By successive use of integral transforms of this type the diff. eqn. may be eventually reduced to an ordinary one, and possibly ... an algebraic equa

— Alternate to pp 127 – 130 —

Method of Integral Transforms for Solving Linear Partial Differential Equations (Sneddon "Elements of Partial Diff. Eqns", pg 126 ff)

Illustrate method for a general second order linear partial differential operation. Consider the equation

$$a(x_1)\frac{\partial^2 u}{\partial x_1^2} + b(x_1)\frac{\partial u}{\partial x_1} + c(x_1)u + L u = f(x_1, x_2, \dots, x_n) \qquad (1)$$

in which $L$ is a linear differential operator in the variables $x_2, x_3, \dots, x$ and the range of variation of $x_1$ is $\alpha \le x_1 \le \beta$. The solution of (1) is the function $u$ of the $n$ independent variables $x_1, x_2, \dots, x_n$.

Define $\bar{u}(\xi, x_2, x_3, \dots, x_n) \equiv \int_\alpha^\beta u(x_1, x_2, \dots, x_n)\, K(\xi, x_1)\, dx_1 \qquad (2)$

where we call $\bar{u}$ the integral transform of $u$ ~~with~~ corresponding to the kernel $K(\xi, x_1)$.

Consider the first three terms on the left-hand side of eq (1). multiply them by $K(\xi, x_1)$ and integrate with respect to $x_1$ from $\alpha$ to $\beta$.

This gives

$$\int_\alpha^\beta \left\{ a(x_1)\frac{\partial^2 u}{\partial x_1^2} + b(x_1)\frac{\partial u}{\partial x_1} + c(x_1)u \right\} K(\xi, x_1)\, dx_1$$

which, on integration by parts of the first two terms, equals

$$= \left[ a\frac{\partial u}{\partial x_1}K(\xi, x_1) + b\,u\,K(\xi, x_1)\right]_\alpha^\beta - \int_\alpha^\beta \left\{ \frac{\partial u}{\partial x_1}\frac{\partial (aK)}{\partial x_1} + u\frac{\partial (bK)}{\partial x_1} - ucK \right\} dx_1$$

which, on integrating by parts the first term in the integral of the last expression, equals

$$= \left[ a\frac{\partial u}{\partial x_1}K(\xi, x_1) + b\,u\,K(\xi, x_1) - u\frac{\partial (aK)}{\partial x_1}\right]_\alpha^\beta - \int_\alpha^\beta \left\{ -u\frac{\partial^2 (aK)}{\partial x_1^2} + u\frac{\partial (bK)}{\partial x_1} - ucK \right\} dx_1$$

$$= g(\xi, x_2, \dots, x_n) + \int_\alpha^\beta u\left\{ \frac{\partial^2 (aK)}{\partial x_1^2} - \frac{\partial (bK)}{\partial x_1} + cK \right\} dx_1$$

where $g(\xi, x_2, \dots, x_n) \equiv \left[ a\frac{\partial u}{\partial x_1}K + u\left\{ bK - \frac{\partial}{\partial x_1}(aK)\right\}\right]_\alpha^\beta$.

$$\int_\alpha^\beta a(x_1)\,\frac{\partial^2 u}{\partial x_1^2}\,K(\xi, x_1)\,d x_1 = a\,\frac{\partial u}{\partial x_1}\,K(\xi, x_1)\Big]_\alpha^\beta - \int_\alpha^\beta \frac{\partial u}{\partial x_1}\,\frac{\partial (aK)}{\partial x_1}\,dx_1$$

$$= \left[a\,\frac{\partial u}{\partial x_1}\,K(\xi, x_1)\right]_\alpha^\beta - \left[u\,\frac{\partial (aK)}{\partial x_1}\right]_\alpha^\beta + \int_\alpha^\beta u\,\frac{\partial^2 (aK)}{\partial x_1^2}\,dx_1$$

$$\int_\alpha^\beta b(x_1)\,\frac{\partial u}{\partial x_1}\,K(\xi, x_1)\,dx_1 = \Big[b\,u\,K(\xi, x_1)\Big]_\alpha^\beta - \int_\alpha^\beta u\,\frac{\partial (bK)}{\partial x_1}\,dx_1$$

Left hand side becomes, after transformation,

$$\left[aK\frac{\partial u}{\partial x_1} + bKu - u\,\frac{\partial (aK)}{\partial x_1}\right]_\alpha^\beta + \int_\alpha^\beta \Big[u\,\frac{\partial^2 (aK)}{\partial x_1^2} - u\,\frac{\partial (bK)}{\partial x_1} + cKu \quad dx_1$$

— Alternate to pp 127–130 —

Let us choose the function $K(\xi, x_1)$ so that

$$\frac{\partial^2}{\partial x_1^2}(aK) - \frac{\partial (bK)}{\partial x_1} + cK = \lambda K \qquad (3)$$

where $\lambda$ is a constant. Then if we multiply the entire eq (1) by $K(\xi, x_1)$ and integrate with respect to $x_1$ from $\alpha$ to $\beta$ we obtain

$$\int_\alpha^\beta u\,\lambda K\,dx_1 + g(\xi, x_2, \cdots, x_n) + \int_\alpha^\beta Lu \cdot K\,dx_1 = \int_\alpha^\beta f(x_1, x_2, \cdots, x_n)\,K\,dx_1$$

Using eq (2),

$$\lambda \bar{u} + L\bar{u} = \bar{f}(\xi, x_2, \cdots, x_n) - g(\xi, x_2, \cdots, x_n)$$

or

$$(\lambda + L)\,\bar{u}(\xi, x_2, \cdots, x_n) = F(\xi, x_2, \cdots, x_n) \qquad (4)$$

where $F(\xi, x_2, \ldots, x_n) \equiv \bar{f}(\xi, x_2, \cdots, x_n) - g(\xi, x_2, \cdots, x_n)$.

The effect of employing the integral transform defined by eqs (2) and (3) is to reduce the partial diff. eq. (1) in ~~n~~ independent variables $x_1, x_2, \cdots, x_n$ to one in $n-1$ independent variables $x_2, \cdots, x_n$ and the parameter $\xi$. By successive use of integral transforms of this type the given partial diff. eq. can eventually be reduced to an ordinary diff. eqn., or even to an algebraic equation. However, in doing this we are still faced with the difficulty of solving integral equations of the type

$$\bar{u}(\xi, x_2, \cdots, x_n) = \int_\alpha^\beta u(x_1, x_2, \cdots, x_n)\,K(\xi, x_1)\,dx_1$$

if we are to find $u$ when $\bar{u}$ has been determined.

See pg 130

In the process of solving the partial diff. eqns. we have to solve integral equations of the type

$$\bar{u}(\xi, x_2, x_3, \cdots, x_n) = \int_\alpha^\beta u(x_1, x_2, \cdots, x_n)\, K(\xi, x_1)\, dx$$

when $\bar{u}$ is found previously, K is selected, and $u(x_1, x_2, \cdots, x_n)$ is the <u>unknown</u>. For certain kernels which frequently appear in mathematical physics it is possible to solve the integral eqn above in the ~~way~~ manner

$$u(x_1, x_2, \cdots, x_n) = \int_\gamma^\delta \bar{u}(\xi, x_2, x_3, \cdots, x_n)\, H(\xi, x_1)\, d\xi \qquad (5)$$

Eq. (5) is an inversion theorem. We can list some of the more common transform pairs.

| Name of Transform | $(\alpha,\beta)$ | $K(\xi,x)$ | $(\gamma,\delta)$ | $H(\xi,x)$ |
|---|---|---|---|---|
| Fourier | $(-\infty,\infty)$ | $\frac{1}{\sqrt{2\pi}} e^{i\xi x}$ | $(-\infty,\infty)$ | $\frac{1}{\sqrt{2\pi}} e^{-i\xi x}$ |
| Fourier cosine | $(0,\infty)$ | $\sqrt{\frac{2}{\pi}} \cos(\xi x)$ | $(0,\infty)$ | $\sqrt{\frac{2}{\pi}} \cos(\xi x)$ |
| Fourier sine | $(0,\infty)$ | $\sqrt{\frac{2}{\pi}} \sin(\xi x)$ | $(0,\infty)$ | $\sqrt{\frac{2}{\pi}} \sin(\xi x)$ |
| Laplace | $(0,\infty)$ | $e^{-\xi x}$, $R(\xi)>c$ | $(\gamma-i\infty, \gamma+i\infty)$ | $\frac{1}{2\pi i} e^{\xi x}$, $\gamma>c$ |
| Mellin | $(0,\infty)$ | $x^{\xi-1}$ | $(\gamma-i\infty, \gamma+i\infty)$ | $\frac{1}{2\pi i} x^{-\xi}$ |
| Hankel | $(0,\infty)$ | $x J_\nu(\xi x)$, $\nu \geq -\frac{1}{2}$ | $(0,\infty)$ | $\xi J_\nu(\xi x)$ |
| Hilbert | $(-\infty,\infty)$ | $\frac{1}{\pi(\xi-x)}$ | $(-\infty,\infty)$ | $\frac{1}{\pi(x-\xi)}$ |
| ~~finite Hilbert~~ | $(-1,1)$ | | $(-1,1)$ | |

Sneddon (pg. 127) has summarized the procedure to be follo in applying the theory of integral transforms to the solution of partial differential equations. It is:

(a) Calculate the function $\bar{f}(\xi, x_2, x_3, \cdots, x_n)$ by integration

$$\bar{f}(\xi, x_2, x_3, \cdots, x_n) = \int_\alpha^\beta f(x_1, x_2, \cdots, x_n)\, K(\xi, x_1)\, dx_1$$

(b) Construct eq. (4) for the transform $\bar{u}$.

$$(L+\lambda)\, \bar{u}(\xi, x_2, \cdots\cdots, x_n) = \bar{f}(\xi, x_2, x_3, \cdots, x_n) - g(\xi, x_2, x_3, \cdots, x_n)$$

(c) Solve the equation in (b) for $\bar{u}$.

(d) Find the inverse of $\bar{u}$ by the inversion theorem.

Example: (taken from Sneddon, Elements of Part. Diff. Eqns, pg 128, ex. 11)

Find the solution of the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{for } r \geq 0, \; z \geq 0$$

which satisfies the conditions

(A) $V \to 0$ as $z \to \infty$ and/or as $r \to \infty$

(B) $V = F(r)$ on $z = 0$, $r \geq 0$.

Let us use the general method of solving a part. diff. eq., such as (1) on pg. 127 of notes. The method of solution is outlined on the top of this page. Comparing the p.d.e. in our problem with the general one (1) we see that $a = 1$, $b = \frac{1}{r}$, $c = 0$, $L = \frac{\partial^2}{\partial z^2}$, $f = 0$.

Step (a), pg. 131 — find $\bar{f}$

~~We~~ We are given that $f = 0$. Therefore $\bar{f} = 0$.

Step (b) — Construct the equation $(L+\lambda)\,\bar{u}(\xi, x_2, \cdots, x_n) = \bar{f}(\xi, x_2, \cdots, x_n) - g(\xi, x_2, \cdots, x_n)$

Note that $x_2 = z$, $x_3 = x_4 = \cdots = x_n = 0$

Let us first find $g(\xi, x_2, x_3, \cdots, x_n)$. From pg. 128,

$$g(\xi, x_2, x_3, \cdots, x_n) = g(\xi, z) = \left[K\frac{\partial V}{\partial r} + V\left\{\frac{1}{r}K - \frac{\partial K}{\partial r}\right\}\right]_{\alpha}^{\beta}$$

We shall use the Hankel transform, where $K = r J_0(\xi r)$, $\alpha = 0$, $\beta = \infty$.

Thus

$$g(\xi, z) = \left[r J_0(\xi r)\frac{\partial V}{\partial r} + V\left\{\frac{1}{r} r J_0(\xi r) - \frac{\partial}{\partial r}\left(r J_0(\xi r)\right)\right\}\right]_0^{\infty}$$

$$= \left[r J_0(\xi r)\frac{\partial V}{\partial r} + V J_0(\xi r) - V J_0(\xi r) + V r \xi J_1(\xi r)\right]_0^{\infty}$$

where we use the relation $\frac{\partial}{\partial r} J_0(\xi r) = -\xi J_1(\xi r)$.

Therefore

$$g(\xi, z) = \left[r J_0(\xi r)\frac{\partial V}{\partial r} + V r \xi J_1(\xi r)\right]_0^{\infty}$$

Now $\left|\frac{\partial V}{\partial r}\right| = O\left(\frac{1}{r^2}\right)$ for large $r$, so $\lim_{r\to\infty}\left(r\frac{\partial V}{\partial r}\right) = 0$ and

$\lim_{r\to\infty}\left[r J_0(\xi r)\frac{\partial V}{\partial r}\right] = 0$.

Also, for large $r$, $rV$ is finite and $\lim_{r\to\infty}(rV) = $ constant.

But $J_1(\xi r) \to 0$ as $r \to \infty$. Therefore $\lim_{r\to\infty} V r \xi J_1(\xi r) = 0$.

Next consider the lower limit. $\lim_{r\to 0} J_0(\xi r) = 0$. We ~~shall assume~~

that $V$ is such that $r\frac{\partial V}{\partial r}$ ~~$= 0$~~ is finite when $r = 0$ ~~We can probably justify this~~ ~~is not a source or si~~

~~because $\nabla^2 V = 0$ at $r=0$, which means that $r = 0$~~ ~~and neither $V$ nor $\partial V/\partial r = \infty$ at $r = 0$~~

For $\lim_{r\to 0}\left[V r \xi J_1(\xi r)\right]$ we may note that $J_1(\xi r) \to 0$ as $r \to 0$ and

$V$ is ~~at most of order $\frac{1}{r}$~~ not infinite at $r = 0$.

Thus we conclude that ~~the~~ both the upper and lower limits of the in brackets ~~are~~ are zero, and therefore $g(\xi, z) = 0$.

Step (b) reduces to

$$\left(\frac{d^2}{dz^2} + \lambda\right) \bar{V}_0(\xi, z) = 0 .$$

The next thing we must do is to evaluate the constant $\lambda$.
We have from eq.(3), p. 128 of notes, that

$$\frac{\partial^2 (aK)}{\partial x_1^2} - \frac{\partial (bK)}{\partial x_1} + cK = \lambda K .$$

In the notation of our problem,

$$\frac{\partial^2 [r J_0(\xi r)]}{\partial r^2} - \frac{\partial [J_0(\xi r)]}{\partial r} = \lambda r J_0(\xi r) .$$

Thus
$$\lambda r J_0(\xi r) = \frac{\partial}{\partial r}[J_0(\xi r) - r\xi J_1(\xi r)] - \frac{\partial}{\partial r}[J_0(\xi r)]$$
$$= -\frac{\partial}{\partial r}[r\xi J_1(\xi r)]$$
$$= -\xi[J_1(\xi r) + r\frac{\partial}{\partial r} J_1(\xi r)]$$
$$= -\xi^2 r J_0(\xi r)$$ ← this relation is derived from

$$n J_n(x) + x J_n'(x) = x J_{n-1}(x)$$

or

$$J_1(\xi r) + \xi r\left[\frac{\partial J_1(\xi r)}{\partial r}\frac{\partial r}{\partial(\xi r)}\right] = \xi r J_0(\xi r)$$

Therefore $\lambda = -\xi^2$. Step (b) finally gives

$$\left(\frac{d^2}{dz^2} - \xi^2\right) \bar{V}(\xi, z) = 0 .$$

This ordinary diff. eq. has solutions

$$\bar{V} = C_1 e^{\xi z} + C_2 e^{-\xi z} .$$

From boundary condition (A), $V \to 0$ as $z \to \infty$. Thus

$$(\bar{V})_{z=\infty} = \lim_{z\to\infty}\int_0^\infty V r J_0(\xi r)\, dr$$ . ~~Now $V$ and $J_0(\xi r)$~~ [We shall ~~assume that~~ $(\bar{V})_{z=\infty} = 0$.]

-over-

~~$\lim_{z\to\infty} (V) = C, \lim_{r\to\infty}$~~

Now

$$(\bar{V})_{z=\infty} = \lim_{z\to\infty} \int_0^\infty V r \, J_0(\xi r) \, dr =$$

$$= \lim_{\epsilon\to 0} \epsilon \int_0^\infty r \, J_0(\xi r) \, dr \qquad \text{because } V \to 0 \text{ as } z \to \infty$$

$$= \lim_{\substack{\epsilon\to 0 \\ a\to\infty}} \epsilon \int_0^a r \, J_0(\xi r) \, dr$$

But $\int_0^a x^{m+1} J_m(\xi x) \, dx = \frac{a^{m+1}}{\xi} J_{m+1}(\xi a)$

Thus $\int_0^a r \, J_0(\xi r) \, dr = \frac{a}{\xi} J_1(\xi a)$

and

$$(\bar{V})_{z=\infty} = \lim_{\substack{\epsilon\to 0 \\ a\to\infty}} \frac{\epsilon a}{\xi} J_1(\xi a)$$

Now $\lim_{a\to\infty} a J_1(\xi a) = C$

Thus $\lim_{\substack{\epsilon\to 0 \\ a\to\infty}} \frac{\epsilon a}{\xi} J_1(\xi a) = 0$

Then $\lim_{z\to\infty} \bar{V}(\xi, z) = 0$ and

$$\bar{V}(\xi, z) = C_2 e^{-\xi z} \qquad \text{because } C_1 = 0.$$

Boundary cond. (B) gives, at $z=0$,

$$\bar{V}(\xi, 0) = \left[C_2 e^{-\xi z}\right]_{z=0} = C_2 = \int_0^\infty F(r) \cdot r\, J_0(\xi r)\, dr \equiv \bar{F}(\xi).$$

Thus

$$\bar{V}(\xi, z) = \bar{F}(\xi)\, e^{-\xi z}.$$

Our solution is obtained from the inverse transform. It is:

$$V(r,z) = \int_0^\infty \xi J_0(\xi, r)\, \bar{V}(\xi, z)\, d\xi$$

$$= \int_0^\infty \xi \bar{F}(\xi)\, e^{-\xi z}\, J_0(\xi r)\, d\xi.$$

Example (from Sneddon, pg 129, ex. 12)

Determine the solution of $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial y^2} = 0$ for $-\infty \le x \le \infty$, $y \ge 0$

satisfying the conditions

1) $z$ and its partial derivatives tend to zero as $x \to \pm\infty$

2) $\left.\begin{array}{l} z = f(x) \\ \frac{\partial z}{\partial y} = 0 \end{array}\right\}$ on $y = 0$.

Solution: use the complex ~~Fourier~~ Fourier transform. Note that $L = \frac{\partial^2}{\partial y^2}$, $f = 0$, $x_1 = x$, $x_2 = y$, $x_3 = x_n = 0$

$a_4 = 1$, $b_k = c_k = d_k = 0$,

Step (a) $\bar{f} = 0$ because $f = 0$

Step (b) $g(\xi, z) = \left[K \frac{\partial z}{\partial x} + z\left\{ - \frac{\partial K}{\partial x}\right\}\right]_{-\infty}^{\infty} = 0$ by bdy cond (1)

Therefore $(L + \lambda)\bar{Z} = 0$, or $\left(\frac{d^2}{dy^2} + \lambda\right)\bar{Z} = 0$

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$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad \text{Multiply the eqn by } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x}\,dx$$

$$\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \frac{\partial^4 z}{\partial x^4} e^{i\xi x}\,dx = \cancel{\frac{1}{\sqrt{2\pi}}}\left\{\left[\frac{1}{\sqrt{2\pi}}\frac{\partial^3 z}{\partial x^3} e^{i\xi x} - \frac{i\xi}{\sqrt{2\pi}}\int \frac{\partial^3 z}{\partial x^3} e^{i\xi x}\,dx\right]\right\}_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[\frac{\partial^3 z}{\partial x^3} e^{i\xi x} - i\xi \frac{\partial^2 z}{\partial x^2} e^{i\xi x} + (i\xi)^2 \int \frac{\partial^2 z}{\partial x^2} e^{i\xi x}\,dx\right\}_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{\partial^3 z}{\partial x^3} e^{i\xi x} - i\xi \frac{\partial^2 z}{\partial x^2} + (i\xi)^2 \frac{\partial z}{\partial x} e^{i\xi x} - (i\xi)^3 \int \frac{\partial z}{\partial x} e^{i\xi x}\,dx\right\}_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{\partial^3 z}{\partial x^3} e^{i\xi x} - i\xi \frac{\partial^2 z}{\partial x^2} + (i\xi)^2 \frac{\partial z}{\partial x} e^{i\xi x} - (i\xi)^3 z e^{i\xi x} + (i\xi)^4 \int z e^{i\xi x}\,dx\right\}_{-\infty}^{\infty}$$

$$= \frac{(i\xi)^4}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z e^{i\xi x}\,dx = \xi^4\, \bar{Z}(\xi, y)$$

Also

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 z}{\partial y^2} e^{i\xi x}\,dx = \frac{d^2 \bar{Z}}{dy^2}$$

The diff. eqn is

$$\left(\frac{d^2}{dy^2} + \xi^4\right)\bar{Z} = 0$$

Next find $\lambda$.

$$\frac{\partial^4 K}{\partial x^4} = \lambda K \qquad K = \frac{1}{\sqrt{2\pi}} e^{i\xi x}$$

~~for~~

$$\cancel{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x} \frac{\partial^4 K}{\partial x^4}\, dx = \lambda \bar{K}}$$

$$\frac{\partial^4}{\partial x^4}\left[\frac{1}{\sqrt{2\pi}} e^{i\xi x}\right] = \frac{\lambda}{\sqrt{2\pi}} e^{i\xi x}$$

Differentiating, $\xi^4 e^{i\xi x} = \lambda e^{i\xi x}$ or $\lambda = \xi^4$.

This gives

$$\left(\frac{d^2}{dy^2} + \xi^4\right)\bar{z} = 0 .$$

The solution of this ordinary differential equation is

$$\bar{z} = C_1 e^{i\xi^2 y} + C_2 e^{-i\xi^2 y} .$$

Bdy. cond. 2) gives us (A) $z = f(x)$ on $y = 0$. Therefore, using A
(B) $\frac{\partial z}{\partial y} = 0$

$$(\bar{z})_{y=0} = C_1 + C_2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\xi x}\, dx = \bar{f}(\xi)$$ where $\bar{f}(\xi)$ is the Fourier transform of $f(x)$.

Next consider (B).

$$\left(\frac{\partial \bar{z}}{\partial y}\right)_{y=0} = i\xi^2 C_1 - i\xi^2 C_2 = i\xi^2 (C_1 - C_2) .$$ Therefore

$$\left(\frac{\partial \bar{z}}{\partial y}\right)_{y=0} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(\frac{\partial z}{\partial y}\right)_{y=0} e^{i\xi x}\, dx = 0 \quad \text{or} \quad \cancel{i\xi^2}(C_1 - C_2) = 0$$

which tells us that $C_1 = C_2$, and also that $\bar{f}(\xi) = 2C_1$. Thus

$$\bar{z} = \frac{1}{2}\bar{f}(\xi)\left[e^{i\xi^2 y} + e^{-i\xi^2 y}\right] = \bar{f}(\xi)\cos(\xi^2 y) .$$

Using the inversion theorem,

$$z(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{z}(x,y)\, e^{-i\xi x}\, d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(\xi)\cos(\xi^2 y)\, e^{-i\xi x}\, d\xi .$$

To go any further, we must know the particular form of $f(x)$ at $y = 0$. Any substituting

Another example, illustrating use of double Fourier transform. It is problem 5, pg. 130 of Sneddon "Part. Diff. Equa."

Given:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \quad \text{for } t \geq 0,\ -\infty \leq \begin{matrix} x \\ y \end{matrix} \leq \infty \qquad (1)$$

and the initial conditions
$$\left.\begin{matrix} 1)\ z = F(x,y) \\ 2)\ \partial z/\partial t = 0 \end{matrix}\right\} \text{at } t = 0 .$$

3) $z$ and $\frac{\partial z}{\partial x} = 0$ at $|x| = \infty$

4) $z$ and $\frac{\partial z}{\partial y} = 0$ at $|y| = \infty$

Find: $z(x, y, t)$ for $t > 0$.

Solution: In this problem $a_1 = 1$, $b_1 = c_1 = 0$, $L_1 = \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$, $f(x,y,t) = 0$

Step (a) $\quad \bar{f}(\xi, y, t) = 0 \quad$ because $f(x,y,t) = 0$.

Step (b) $\quad g(\xi, y, t) = \left[\frac{\partial z}{\partial x} K(\xi, x) - z\frac{\partial K(\xi,x)}{\partial x}\right]_{x=-\infty}^{\infty}$ where $K(\xi,x) = \frac{e^{i\xi x}}{\sqrt{2\pi}}$.

Now we shall <u>assume</u> that $z$ and $\frac{\partial z}{\partial x} = 0$ at $|x| = \infty$.

Then $\quad g(\xi, y, t) = 0$, and

$$(L_1 + \lambda_1)\,\bar{z}(\xi, y, t) = 0 .$$

We must find $\lambda_1$, knowing that in general

$$\frac{\partial^2(aK)}{\partial x^2} - \frac{\partial(bK)}{\partial x} + cK = \lambda K .$$

Using the values of $a_1$, $b_1$, $c_1$, $K(\xi, x)$, we have

$$-\xi^2 K(\xi, x) = \lambda_1 K(\xi, x) \quad \text{or} \quad \lambda_1 = -\xi^2 .$$

Thus our partial diff. equa. (1) is reduced to the part. diff. equa.

$$\frac{\partial^2 \bar{z}(\xi,y,t)}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 \bar{z}(\xi,y,t)}{\partial t^2} - \xi^2\,\bar{z}(\xi,y,t) = 0 . \qquad (2)$$

[illegible] top pg 137

We can use various methods to solve the part. diff. equa. (2). I shall use a second Fourier Transform, for illustrative purposes.

We have that

$$\bar z(\xi,y,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z(x,y,t)\,e^{i\xi x}\,dx\,;$$

and

$$z(x,y,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar z(\xi,y,t)\,e^{-i\xi x}\,d\xi\,.$$

Thus in order to ~~evaluate~~ solve for $z(x,y,t)$ we must first solve eq. (2) for $\bar z(\xi,y,t)$.

We shall now solve eq. (2). In it, $a_2 = 1$, $b_2 = c_2 = 0$,

$$L_2 = -\frac{1}{c^2}\frac{d^2}{dt^2}\,, \quad f(\xi,y,t) = \xi^2\,\bar z(\xi,y,t)\,.$$

{ (can also let $c_2 = -\xi^2$ and $f_2 = 0$.

Step (a) gives

$$\bar f(\xi,\eta,t) = \cancel{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}} f(\xi,y,t)\,e^{i\eta y}\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,t)\,e^{i(\xi x+\eta y)}\,dx\,dy}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \xi^2\,\bar z(\xi,y,t)\,e^{i\eta y}\,dy = \xi^2\,\bar z(\xi,\eta,t)$$

Step (b):

$$g(\xi,\eta,t) = \left[\frac{\partial \bar z(\xi,y,t)}{\partial y}K(\eta,y) - \bar z(\xi,y,t)\frac{\partial K(\eta,y)}{\partial y}\right]_{-\infty}^{\infty} = 0$$

if we assume that $\frac{\partial}{\partial y}\left(\bar z(\xi,y,t)\right) = \bar z(\xi,y,t) = 0$ when $|y| = \infty$.

Therefore,

$$(L_2 + \lambda_2)\,\bar z(\xi,\eta,t) = \bar f(\xi,\eta,t) = \xi^2\,\bar z(\xi,\eta,t)\,.$$

Substituting from above into the expression which gives $\lambda_2$,

$$\frac{\partial^2}{\partial y^2}\left[K(\eta,y)\right] = \lambda_2\,K(\eta,y)$$

or

$$\frac{-\eta^2}{\sqrt{2\pi}}\,e^{i\eta y} = \lambda_2\,\frac{e^{i\eta y}}{\sqrt{2\pi}}\,.$$

or $\lambda_2 = -\eta^2$.

Thus

$$\left[-\frac{1}{c^2}\frac{d^2}{dt^2} - \eta^2 - \xi^2\right]\bar z(\xi,\eta,t) = 0$$

This ordinary diff. eq. ~~(2)~~ may be written as

$$\left[\frac{d^2}{dt^2} + c^2(\xi^2+\eta^2)\right]\bar{z}(\xi,\eta,t) = 0 \qquad (3)$$

The solution of (3) is

$$\bar{z}(\xi,\eta,t) = k_1 e^{ic\sqrt{\xi^2+\eta^2}\,t} + k_2 e^{-ic\sqrt{\xi^2+\eta^2}\,t} \qquad (4)$$

From ~~the~~ initial condition 1), $z(x,y,0) = F(x,y)$. Thus

$$\bar{z}(\xi,\eta,0) = k_1 + k_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\, e^{i(\xi x+\eta y)}\,dx\,dy. \qquad (5)$$

The other initial condition 2) tells us that $\left[\frac{\partial z(x,y,t)}{\partial t}\right]_{t=0} = 0$. Use this in eq. (4), giving

$$\left[\frac{\partial \bar{z}(\xi,\eta,t)}{\partial t}\right]_{t=0} = \left[k_1 ic\sqrt{\xi^2+\eta^2}\, e^{ic\sqrt{\xi^2+\eta^2}\,t} - k_2 ic\sqrt{\xi^2+\eta^2}\, e^{-ic\sqrt{\xi^2+\eta^2}\,t}\right]_{t=0}$$

$$= ic\sqrt{\xi^2+\eta^2}\,(k_1-k_2) = \left[\frac{1}{2\pi}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z(x,y,t)\, e^{i(\xi x+\eta y)}\,dx\,dy\right]_{t=0}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\left[\frac{\partial z(x,y,t)}{\partial t}\right]_{t=0}}_{=0 \text{ from initial conditions}} e^{i(\xi x+\eta y)}\,dx\,dy = 0$$

Therefore $k_1 = k_2$, and eq (5) becomes

$$\bar{z}(\xi,\eta,0) = 2k_1 = \bar{F}(\xi,\eta) \qquad \text{where } \bar{F}(\xi,\eta) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\, e^{i(\xi x+\eta y)}\,dx\,dy \qquad (6)$$

and

$$\bar{z}(\xi,\eta,t) = \frac{\bar{F}(\xi,\eta)}{2}\left[e^{ic\sqrt{\xi^2+\eta^2}\,t} + e^{-ic\sqrt{\xi^2+\eta^2}\,t}\right] = \bar{F}(\xi,\eta)\cos\{c\sqrt{\xi^2+\eta^2}\,t\}$$

Next take the inverse transform twice. The first operation gives

$$\bar{z}(\xi,y,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{F}(\xi,\eta)\cos\{c\sqrt{\xi^2+\eta^2}\,t\}\, e^{-i\eta y}\,d\eta$$

and the second operation gives

$$z(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{F}(\xi,\eta)\cos\{c\sqrt{\xi^2+\eta^2}\,t\}\, e^{-i(\xi x+\eta y)}\}\,d\xi\,d\eta$$

where $\bar{F}(\xi,\eta)$ is defined in eq. (6).

Diff eqns.

$$A(x_1)\frac{\partial^m u}{\partial x_1^m} + B(x_1)\frac{\partial^{m-1} u}{\partial x_1^{m-1}} + \cdots + M(x_1)\frac{\partial u}{\partial x_1} + N(x_1)u + \mathcal{L}u = f(x_1, x_2, \ldots, x_n)$$

$$\int_\alpha^\beta A(x_1)\,K(\xi,x_1)\frac{\partial^m u}{\partial x_1^m}\,dx_1 = \Big[ A K \frac{\partial^{m-1} u}{\partial x_1^{m-1}} - \frac{\partial(AK)}{\partial x_1}\frac{\partial^{m-2} u}{\partial x_1^{m-2}} + \cdots$$

$$+ \frac{\partial^{m-2}(AK)}{\partial x_1^{m-2}}\frac{\partial u}{\partial x_1} - \frac{\partial^{m-1}(AK)}{\partial x_1^{m-1}}\,u\Big]_\alpha^\beta - \int_\alpha^\beta u\,\frac{\partial^m(AK)}{\partial x_1^m}\,dx_1$$

$$\int_\alpha^\beta K(\xi,x_1)\Big\{A(x_1)\frac{\partial^m u}{\partial x_1^m} + B(x_1)\frac{\partial^{m-1} u}{\partial x_1^{m-1}} + \cdots + N(x_1)u\Big\}dx_1$$

$$= \Big[ A K\frac{\partial^{m-1} u}{\partial x_1^{m-1}} + \Big\{BK - \frac{\partial(AK)}{\partial x_1}\Big\}\frac{\partial^{m-2} u}{\partial x_1^{m-2}} + \Big\{CK - \frac{\partial(BK)}{\partial x_1} + \frac{\partial^2(AK)}{\partial x_1}\Big\}\frac{\partial^{m-3} u}{\partial x_1^{m}}$$

$$+ \cdots + \Big\{MK - \frac{\partial(LK)}{\partial x_1} + \frac{\partial^2(KK)}{\partial x_1^2} - \cdots$$

$$+ \frac{\partial^{m-2}(BK)}{\partial x_1^{m-2}} - \frac{\partial^{m-1}(AK)}{\partial x_1^{m-1}}\Big\}u\Big]_\alpha^\beta$$

$$+ \int_\alpha^\beta u\Big\{\frac{\partial^m(AK)}{\partial x_1^m} - \frac{\partial^{m-1}(BK)}{\partial x_1^{m-1}} + \cdots - \frac{\partial(MK)}{\partial x_1} + NK\Big\}dx_1$$

The part in brackets is defined as $g(\xi, x_2, x_3, \ldots x_n)$ &

the part in braces in the integrand is defined as $\lambda K$.

Diff eqns.

$$A(x_1)\frac{\partial^m u}{\partial x_1^m} + B(x_1)\frac{\partial^{m-1} u}{\partial x_1^{m-1}} + \cdots + M(x_1)\frac{\partial u}{\partial x_1} + N(x_1)u + \mathcal{L}u = f(x_1, x_2, \ldots, x_m)$$

$$\int_\alpha^\beta A(x_1)\, K(\xi, x_1)\frac{\partial^m u}{\partial x_1^m}\,dx_1 = \Big[ A K \frac{\partial^{m-1} u}{\partial x_1^{m-1}} - \frac{\partial (AK)}{\partial x_1}\frac{\partial^{m-2} u}{\partial x_1^{m-2}} + \cdots$$

$$+ \frac{\partial^{m-2}(AK)}{\partial x_1^{m-2}}\frac{\partial u}{\partial x_1} - \frac{\partial^{m-1}(AK)}{\partial x_1^{m-1}}u \Big]_\alpha^\beta - \int_\alpha^\beta u\frac{\partial^m (AK)}{\partial x_1^m}\,dx_1$$

$$\int_\alpha^\beta K(\xi, x_1)\Big\{ A(x_1)\frac{\partial^m u}{\partial x_1^m} + B(x_1)\frac{\partial^{m-1} u}{\partial x_1^{m-1}} + \cdots + N(x_1)u \Big\}\,dx_1$$

$$= \Big[ A K \frac{\partial^{m-1} u}{\partial x_1^{m-1}} + \Big\{ B K - \frac{\partial (AK)}{\partial x_1}\Big\}\frac{\partial^{m-2} u}{\partial x_1^{m-2}} + \Big\{ C K - \frac{\partial (BK)}{\partial x_1} + \frac{\partial^2 (AK)}{\partial x_1}\Big\}\frac{\partial^{m-3}}{\partial x_1^{m-3}}$$

$$+ \cdots + \Big\{ M K - \frac{\partial (LK)}{\partial x_1} + \frac{\partial^2 (KK)}{\partial x_1^2} - \cdots$$

$$+ \frac{\partial^{m-2}(BK)}{\partial x_1^{m-2}} - \frac{\partial^{m-1}(AK)}{\partial x_1^{m-1}}\Big\} u \Big]_\alpha^\beta$$

$$+ \int_\alpha^\beta u \Big\{ \frac{\partial^m (AK)}{\partial x_1^m} - \frac{\partial^{m-1}(BK)}{\partial x_1^{m-1}} + \cdots - \frac{\partial (MK)}{\partial x_1} + NK \Big\}\,dx_1$$

The part in brackets is defined as $g(\xi, x_2, x_3, \ldots x_n)$ so

the part in braces in the integrand is defined as $\bar{\partial} K$.

Use cap A, B, C.

$$A(x_1)\frac{\partial^4 u}{\partial x_1^4} + B(x_1)\frac{\partial^3 u}{\partial x_1^3} + C(x_1)\frac{\partial^2 u}{\partial x_1^2} + D(x_1)\frac{\partial u}{\partial x_1} + E(x_1) + Lu = f(x_1, \ldots$$

$$\int_\alpha^\beta a(x_1)K(\xi, x_1)\frac{\partial^4 u}{\partial x_1^4}dx_1 = \left[aK\frac{\partial^3 u}{\partial x_1^3}\right]_\alpha^\beta - \int_\alpha^\beta \frac{\partial (aK)}{\partial x_1}\frac{\partial^3 u}{\partial x_1^3}dx_1$$

$$= \left[aK\frac{\partial^3 u}{\partial x_1^3}\right]_\alpha^\beta - \left[\frac{\partial(aK)}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2}\right]_\alpha^\beta + \int_\alpha \frac{\partial^2(aK)}{\partial x_1^2}\frac{\partial^2 u}{\partial x_1^2}dx_1$$

$$= \left[aK\frac{\partial^3 u}{\partial x_1^3} - \frac{\partial(aK)}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2(aK)}{\partial x_1^2}\frac{\partial u}{\partial x_1}\right]_\alpha^\beta - \int_\alpha^\beta \frac{\partial^3(aK)}{\partial x_1^3}\frac{\partial u}{\partial x_1}dx_1$$

$$= \left[aK\frac{\partial^3 u}{\partial x_1^3} - \frac{\partial(aK)}{\partial x_1}\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2(aK)}{\partial x_1^2}\frac{\partial u}{\partial x_1} - \frac{\partial^3(aK)}{\partial x_1^3}u\right]_\alpha^\beta - \int_\alpha^\beta u\frac{\partial^4(aK)}{\partial x_1^4}dx_1$$

$$\int_\alpha^\beta A(x_1)K(\xi, x_1)\frac{\partial^m u}{\partial x_1^m}dx_1 = \Big[AK\frac{\partial^{m-1}u}{\partial x_1^{m-1}} - \frac{\partial(AK)}{\partial x_1}\frac{\partial^{m-2}u}{\partial x_1^{m-2}} + \cdots$$

$$+ \frac{\partial^{m-2}(AK)}{\partial x_1^{m-2}}\frac{\partial u}{\partial x_1} \; \frac{\partial^{m-1}(AK)}{\partial x_1^{m-1}}u\Big]_\alpha^\beta - \int_\alpha^\beta u\frac{\partial^m(aK)}{\partial x_1^m}dx_1$$

$$\int_\alpha^\beta K(\xi, x_1)\left\{A(x_1)\frac{\partial^m u}{\partial x_1^m} + B(x_1)\frac{\partial^{m-1}u}{\partial x_1^{m-1}} + \cdots M(x_1)u\right\}dx_1$$

$$= \Big[AK\frac{\partial^{m-1}u}{\partial x_1^{m-1}} + \left\{BK - \frac{\partial(AK)}{\partial x_1}\right\}\frac{\partial^{m-2}u}{\partial x_1^{m-2}} + \left\{CK - \frac{\partial(BK)}{\partial x_1} + \frac{\partial(AK)}{\partial x_1}\right\}\frac{\partial^{m-3}u}{\partial x_1^{m-3}}$$

$$+ \cdots + \left\{MK - \frac{\partial(LK)}{\partial x_1} + \frac{\partial(KK)}{\partial x_1} - \frac{\partial(JK)}{\partial x_1}\right.$$

$$\left. + \cdots + \frac{\partial^{m-2}(BK)}{\partial x_1^{m-2}} - \frac{\partial^{m-1}(AK)}{\partial x_1^{m-1}}\right\}u\Big]_\alpha^\beta$$

$$+ \int_\alpha^\beta u\left\{\frac{\partial^m(AK)}{\partial x_1^m} - \frac{\partial^{m-1}(BK)}{\partial x_1^{m-1}} + \cdots - \frac{\partial(MK)}{\partial x_1} + MK\right\}d\ldots$$

... in brackets is defined as $g(\xi, x_2, x_3 \ldots x_n$

... integrand $\equiv \lambda K$.

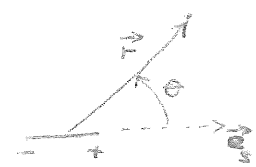
$$\int_{\alpha}^{\beta} u \left\{ \frac{\partial^m (AK)}{\partial x_1^m} - \cdots + \cdots + NK \right\} = \lambda K$$

This is the diff equ

$$A(x_1) \frac{\partial^m u}{\partial x_1^m} + B(x_1) \frac{\partial^{m-1} u}{\partial x_1^{m-1}} + \cdots + M(x_1) \frac{\partial u}{\partial x_1} + N(x_1) u + \lambda u =$$

I <u>Review of Gph 104</u>:

1. Newtonian law for particles — a central force field whose magnitude obeys inverse square law
2. Field intensity - force exerted on a unit hypothetical test particle
3. Condition that field intensity (or any vector quantity) has a scalar potential - curl is zero or $\left\{ \begin{array}{l} \text{line integral } \int_{P_1}^{P_2} \underline{f}\cdot d\underline{r} \text{ is independent of path} \\ \text{line integral } \oint \underline{f}\cdot d\underline{r} \text{ along any and every closed path is zero} \end{array} \right.$
4. Newtonian potential of a particle — $\varphi = \frac{Km}{r}$
5. Extension of Newt. potential to a number of particles, linear distribution, surface distribution, and volume distribution — <u>write formulae</u>
6. Logarithmic potential and 2-dimensional bodies — for 2-D point, $\varphi_L = 2k\sigma_L \ln \frac{1}{r}$
   Extend formula to surfaces and volumes.
7. Potential of dipoles (doublets) — $\varphi_{dipole} = \frac{K\mu}{r^2}\frac{dr}{ds} = \frac{K\mu}{r^2}\cos\theta$




K = constant
$\mu$ = dipole moment

We can build up lines, surfaces, or volumes of dipoles. e.g. for lines

−+ −+ −+ or $\overset{+}{-}\ \overset{+}{-}\ \overset{+}{-}\ \overset{+}{-}\ \overset{+}{-}\ \overset{+}{-}\ \overset{+}{-}\ \overset{+}{-}$ . Such a surface is called a double-layer $\begin{array}{c} + + + + + \\ \hline - - - - - \end{array}$

The potential is

$$\varphi_{\text{dipole distribution}} = K\int_{C,S,V} \frac{M_{L,S,V}}{r^2}\cos\theta \, d(s,S,V)$$

$$= K\int_{C,S,V} M_{L,S,V}\left[\nabla_{\xi,\eta,\zeta}\left(\frac{1}{r}\right)\cdot\vec{e}_s\right] d(s,S,V)$$ where $\xi,\eta,\zeta$ are the rectangular coordinates of some point in the element of $\left\{ \begin{array}{l} \text{length} \\ \text{surface} \\ \text{volume} \end{array} \right.$

Also 2-dimensional dipole fields. Basic formula for potential is

$$\varphi_{2\text{-D dipole}} = \frac{2K M_L \cos\theta}{r}$$ where $M_L$ is magnetic dipole moment per unit length.

3.

8. Field intensity — in all cases, $E = \nabla\varphi$

9. Properties of Newtonian potential — a) values at infinity $\lim_{r\to\infty}\varphi = 0$ $\lim_{r\to\infty}(\varphi r) = Km$

b) $\varphi$ and its ~~first~~ first derivatives are always finite and continuous for a volume ~~or~~ ~~surface~~ distribution of matter, provided that the quantity of matter is finite.

c) Laplace's and Poisson's equations

$$\nabla^2\varphi = 0 \quad \text{or} \quad \nabla^2\varphi = -4\pi K\sigma$$

Neither applies at points where $\sigma$ is discontinuous.

-13-

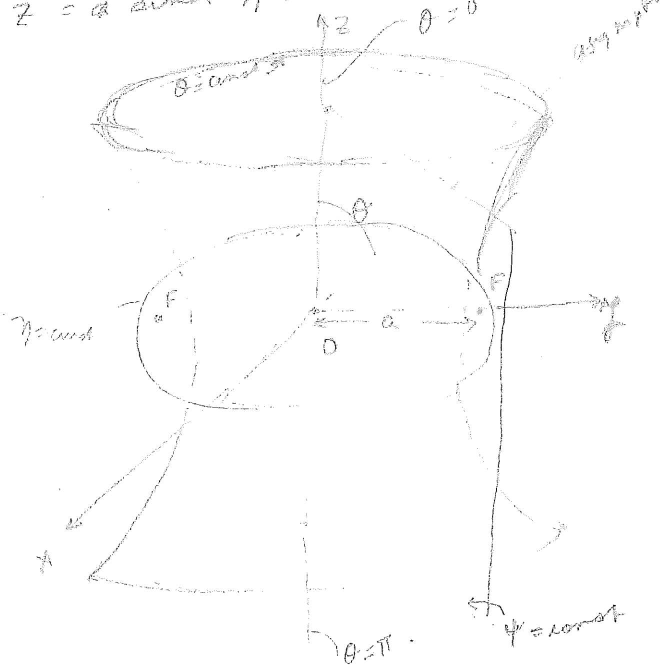
(8)

Oblate spheroidal coordinates

$x = a \cosh\eta \sin\theta \cos\psi$

$y = a \cosh\eta \sin\theta \sin\psi$

$z = a \sinh\eta \cos\theta$




The coordinate surfaces are oblate spheroids, half-planes, and hyperboloids of one sheet.

The surfaces $\eta$ = constant are: (oblate spheroids)

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $b = a\cosh\eta$, $c = a\sinh\eta$

$0 \le \eta \le \infty$

The surfaces $\theta$ = constant are (hyperboloids of one sheet)

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

where $b = a\sin\theta$, $c = a\cos\theta$

$0 \le \theta \le \pi$

The surfaces $\psi$ = constant are half-planes containing the z-axis. ($\psi$ measured from x-axis)

$0 \le \psi \le 2\pi$

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial\eta}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2 + \left(\frac{\partial z}{\partial\eta}\right)^2}$$

$\frac{\partial x}{\partial\eta} = a\sinh\eta\sin\theta\cos\psi$, $\frac{\partial y}{\partial\eta} = a\sinh\eta\sin\theta\sin\psi$, $\frac{\partial z}{\partial\eta} = a\cosh\eta\cos\theta$

$\frac{\partial x}{\partial\theta} = a\cosh\eta\cos\theta\cos\psi$, $\frac{\partial y}{\partial\theta} = a\cosh\eta\cos\theta\sin\psi$, $\frac{\partial z}{\partial\theta} = -a\sinh\eta\sin\theta$

$\frac{\partial x}{\partial\psi} = -a\cosh\eta\sin\theta\sin\psi$, $\frac{\partial y}{\partial\psi} = a\cosh\eta\sin\theta\cos\psi$, $\frac{\partial z}{\partial\psi} = 0$

Thus

$$h_1 = a\sqrt{\sinh^2\eta(\sin^2\theta(\cos^2\psi + \sin^2\psi)) + \cosh^2\eta\cos^2\theta}$$

$$= a\sqrt{\sinh^2\eta\sin^2\theta + \cosh^2\eta\cos^2\theta}$$

$$= a\sqrt{\cosh^2\eta\sin^2\theta - \sin^2\theta + \cosh^2\eta\cos^2\theta}$$

$$= a\sqrt{\cosh^2\eta - \sin^2\theta}$$

See next page for continuation

$$h_2 = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2}$$

$$h_2 = a\sqrt{\cosh^2\eta \cos^2\theta + \sinh^2\eta \sin^2\theta}$$

$$= a\sqrt{\cosh^2\eta - \sin^2\theta}$$

$$h_3 = \sqrt{\left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \psi}\right)^2}$$

$$= a\sqrt{\cosh^2\eta \ \sin^2\theta(\sin^2\psi + \cos^2\psi)}$$

$$h_3 = a\cosh\eta \sin\theta$$

Then

$$\nabla^2 V = \frac{\frac{\partial}{\partial \eta}\left[(a\cosh\eta\sin\theta)\frac{\partial V}{\partial \eta}\right] + \frac{\partial}{\partial \theta}\left[(a\cosh\eta\sin\theta)\frac{\partial V}{\partial \theta}\right] + \frac{\partial}{\partial \psi}\left[\frac{a(\cosh^2\eta - \sin^2\theta)}{\cosh\eta\sin\theta}\frac{\partial V}{\partial \psi}\right]}{a^3(\cosh^2\eta - \sin^2\theta)\cosh\eta\sin^2\theta}$$

$$= \frac{1}{a^2(\cosh^2\eta - \sin^2\theta)}\left[\frac{\partial^2 V}{\partial \eta^2} + \tanh\eta\frac{\partial V}{\partial \eta} + \frac{\partial^2 V}{\partial \theta^2} + \cot\theta\frac{\partial V}{\partial \theta}\right] + \frac{1}{a^2\cosh^2\eta\sin^2\theta}\frac{\partial^2 V}{\partial \psi^2}$$

Morning & Mullineux

Chap. 1. Introduction to Partial Differential Equations

1. Introduction

We shall be concerned with boundary-value problems, i.e., partial diff. eqns with specified boundary or initial conditions. These are

$\nabla^2 \phi = 0$ Laplace's eqn.

$\nabla^2 \phi = -K$ Poisson's eqn.

$\nabla^2 \phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$ Wave eqn.

$\nabla^2 \phi = \frac{1}{k}\frac{\partial \phi}{\partial t}$ Diffusion eqn.

$\nabla^2 \phi + k^2 \phi = 0$ Wave eqn. for sinusoidal time dependence, diff. eqn. for exponential time dependence

c and K need not be constants ... for P ... sources.

Our principal concern will be the first two of these equations. ~~For~~ For two dimensional fields the expression for $\nabla^2 \phi$ reduces to

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{(Cartesian coordinates)}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad \text{(Plane polar coordinates)}.$$

order — power of highest derivative. $\frac{\partial^n \phi}{\partial x^n}$ is of nth order. $\left(\frac{\partial \phi}{\partial x}\right)^n$ is of nth degree.

~~And~~ We can observe that all of these equations are linear partial differential equations, i.e., none of the terms contains $\left(\frac{\partial \phi}{\partial x_i}\right)^2$, ~~product~~ or higher powers of $\frac{\partial \phi}{\partial x_i}$, or of the higher derivatives of $\phi$. (A P.D.E. is linear if it is of the first degree in the unknown function and its derivatives. (Churchill, Fourier Series, p. ...)) If $\phi$ happens to be a vector, a linear partial diff. eqn. will also not contain products of components such as $\frac{\partial \phi}{\partial x_1}\frac{\partial \phi}{\partial x_2}$. For a partial diff. eqn. $\phi$ may appear as $\phi$ and the eqn. is still called linear.

Check ... to ... of initial ... linearity

These equations, with the exception of Poisson's equation, are homogeneous. A homogeneous diff. eqn. contains no term which does not involve the dependent variable or its derivatives. As we know, if $\phi_1, \phi_2, \ldots \phi_n$ are a set of solutions of a homogeneous linear differential equation, then $\sum_{i=1}^{n} \phi_i$ is also a solution.

For a non-homogeneous linear equation, we write the solution as $\phi = \phi_H + \phi_i$, where $\phi_H$ is a solution of the corresponding homogeneous equation, and $\phi_i$ is a particular solution to the non-homo. equation.

... that there are an infinite number of particular solutions to the p.d.e., and ... we must select from them those which satisfy the initial & bdy. conditions

By a solution of a partial differential equation of mathematical physics with initial and boundary conditions we mean a functional relationship between the ... and independent variables which not only satisfies all the conditions i.e. ~~and~~, but is the only stable solution. By stable we mean

15c DELETE

## Harmonic Functions

Any solution of Laplace's eqn. is a harmonic function. A function may be ~~usually~~ harmonic only in a specified region of space. For example, the Newtonian potential is a harmonic function at all places which are not sources or sinks of the field.

Let us look for some harmonic functions, and first of all consider rectangular coordinates. Some obvious particular solutions of $\nabla^2 \varphi = 0$ are:

$\varphi = A$

$\varphi = Bx$

$\varphi = Cy$

$\varphi = Dz$

$\varphi = Exy$

$\varphi = Fxz$

$\varphi = Gyz$ $\quad \varphi = Hxyz$ $\qquad \varphi = K(x^3 - \frac{3}{2}xy^2)$ etc.

~~$\varphi = Hx^2$~~ $\quad \varphi = J(x^2 - y^2)$ $\qquad$ etc

~~$\varphi = Jy^2$~~ $\quad$ etc

~~$\varphi = Kz^2$~~.

Any sum of these solutions is also a solution of $\nabla^2\varphi = 0$ and thus is a harmonic function.

We also know from Elements of Potential Theory that

$\varphi = \dfrac{C}{\sqrt{x^2+y^2+z^2}}$ is a harmonic function except at $x=y=z=0$. We also showed that any derivative of this quantity with respect to the space coordinates $x, y, z$ is also a harmonic function. Thus

$$\frac{\partial}{\partial x}\left(\frac{C}{\sqrt{x^2+y^2+z^2}}\right) \text{ and } \frac{\partial^5}{\partial x \partial y^2 \partial z^2}\left(\frac{C}{\sqrt{x^2+y^2+z^2}}\right) \text{ are harmonic}$$

functions, except at the origin.

For two-dimensional problems, the corresponding solutions of $\dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} = 0$ are $\varphi_2 = -K \ln\sqrt{x^2+y^2}$. except at $x=y=0$. Derivatives of this solution with respect to $x$ and $y$ are also solutions of $\dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} = 0$ except at $x=y=0$.

(15 d) Let us now look at some simple solutions of the Dirichlet and Neumann problems.

As an example, consider the interior Dirichlet problem for which $\varphi$ is specified to equal $K_1 + K_2 y$ on a circle of radius $b$ and center at the origin (circle is in x-y plane).

Because we are in the interior of the circular region, we ~~must~~ exclude all solutions with $(x^2+y^2)$ in the denominator because they become infinite at $x = y = 0$. We can write a solution to Laplace's equation

$$\varphi = A + Bx + Cy + Dxy + Ex^2 + Fy^2$$

where the coefficients are to be determined by the boundary conditions

Now

$$\varphi_{r=b} = (K_1 + K_2 y)_{r=b} = K_1 + K_2 b \sin\theta = A + Bb\cos\theta + Cb\sin\theta + Db^2 \sin\theta\cos\theta$$
$$+ Eb^2\cos^2\theta + Fb^2\sin^2\theta$$

By the orthogonality properties of the sin and cos functions in the interval $-\pi$ to $\pi$, we can equate coefficients of $\sin n\theta$ and $\cos n\theta$.

Now $\sin\theta\cos\theta = \dfrac{\sin 2\theta}{2}$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}.$$

Thus

$$K_1 + K_2 b\sin\theta = A + Bb\cos\theta + Cb\sin\theta + \frac{D}{2}b^2\sin 2\theta$$
$$+ Eb^2\left(\frac{1+\cos 2\theta}{2}\right) + Fb^2\left(\frac{1-\cos 2\theta}{2}\right)$$
$$= A' + Bb\cos\theta + Cb\sin\theta + D'b^2\sin 2\theta + E'b^2\cos 2\theta$$

and $K_1 = A'$, $K_2 = C$, $B = D' = E' = 0$.

The solution therefore is

$$\varphi = K_1 + K_2 y = K_1 + K_2 r\sin\theta.$$

Another example. Find a solution to the interior Neumann problem $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$ within a circle of radius $b$, and on $b$ it is given that $\left(\frac{\partial \varphi}{\partial n}\right)_{r=b} = C$

For interior Neumann problems to have a solution, it is necessary that $\int_{\text{Circle}} f \, ds = 0$ where $f$ is $\left(\frac{\partial \varphi}{\partial n}\right)_{r=b}$ in our problem.

Now $\int_{-\pi}^{\pi} C b \, d\theta = 2Cb\pi \neq 0$. Thus the problem is inconsistent, that is, there is no function which satisfies $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$ in the interior of the circle $r=b$ and whose normal derivative on the bounding circle equals a constant.

As a further example, let

$$\int_{-\pi}^{\pi} K(x^2-y^2)\, b\, d\theta = Kb^3 \int_{-\pi}^{\pi} \cos 2\theta \, d\theta = \frac{Kb^3}{2} \left[\sin 2\theta\right]_{-\pi}^{\pi} = 0$$

$$f = \left(\frac{\partial \varphi}{\partial n}\right)_{r=b} = K(x^2 - y^2)$$

Write $\varphi = A + Br\cos\theta + Cr\sin\theta + Dr^2 \sin 2\theta + Er^2 \cos 2\theta$

$$\left(\frac{\partial \varphi}{\partial n}\right)_{r=b} = \left(\frac{\partial \varphi}{\partial r}\right)_{r=b} = B\cos\theta + C\sin\theta + 2Db\sin 2\theta + 2Eb\cos 2\theta$$

$$= Kb^2(\cos^2\theta - \sin^2\theta)$$

$$= \frac{Kb^2}{2}\left[1 + \cos 2\theta - (1 - \cos 2\theta)\right] = Kb^2 \cos 2\theta$$

Equating coefficients of $\cos n\theta$,

$B = C = D = 0$, $E = \frac{Kb^2}{2b} = \frac{Kb}{2}$.

Thus $\varphi = A + \frac{Kb}{2} r^2 \cos 2\theta$

15 f

Repeat the Dirichlet problem example of fig 15d, but make it an exterior problem.

We have that $\phi$ is harmonic outside the circle $r=b$,

$(\phi)_{r=b} = K_1 + K_2 y$, and $\lim_{r\to\infty} \phi = 0$.

We cannot use the ~~form~~ type of harmonic functions selected in the interior problems because they do not vanish at infinity. Also $\phi = -A \ln\sqrt{x^2+y^2}$ does not vanish at infinity. But all derivatives of the latter expression with respect to $x$ and/or $y$ go to zero at infinity, such as $\frac{\partial}{\partial x}(\ln\sqrt{x^2+y^2})$, $\frac{\partial}{\partial y}(\ln\sqrt{x^2+y^2})$, $\frac{\partial^2}{\partial x \partial y}(\ln\sqrt{x^2+y^2})$, $\frac{\partial^2}{\partial x^2}(\ln\sqrt{x^2+y^2})$, etc.

Let us write down a few of these harmonic functions

$$\frac{\partial}{\partial x}(\ln\sqrt{x^2+y^2}) = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

$$\frac{\partial}{\partial y}(\ln\sqrt{x^2+y^2}) = \frac{y}{x^2+y^2} = \frac{\sin\theta}{r}$$

$$\frac{\partial^2}{\partial x^2}(\ln\sqrt{x^2+y^2}) = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\cos^2\theta - \sin^2\theta}{r^2}$$

$$\frac{\partial^2}{\partial x \partial y}(\ln\sqrt{x^2+y^2}) = \frac{-xy}{(x^2+y^2)^2} = -\frac{\sin\theta\cos\theta}{r^2}$$

The boundary condition can be written as

$$(\phi)_{r=b} = K_1 + K_2 b \sin\theta .$$

The harmonic function which we shall select to match this boundary condition is $\phi = \frac{A\sin\theta}{r}$, such that $(\phi)_{r=b} = \frac{A\sin\theta}{b}$. Thus $A = K_2 b^2$, and the solution is

$$\phi = \frac{K_2 b^2 \sin\theta}{r} + K_1 .$$

It may be argued that the presence of $K_1$ in the solution does not allow $\phi \to 0$ as $r \to \infty$. This is true, but it is only an additive

(15g) Poisson's equation.

Example: Given $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 1$ in the interior of a circle with center at the origin and radius $b$, and $(\phi)_{r=b} = 0$. Find $\phi$ in r's.

For this problem there is circular symmetry, so ~~Laplace~~ Poisson's equation reduces to

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}=1.$$

This is a linear 2nd order diff. eqn. with variable coefficients. To solve it, reduce it to a first order diff. eqn. by writing $\frac{d\phi}{dr}=p$. Then

$$\frac{dp}{dr}+\frac{p}{r}=1 \quad \text{or} \quad \cancel{dp+\frac{1}{r}p\,dr = dr}.$$

The last equation is of the form $\frac{dp}{dr}+Qp=S$ where $Q$ and $S$ are functions of $r$ only. It has the solution

$$Rp=\int RS\,dr + C \quad , \text{ where } R=e^{\int Q\,dr}.$$

For our problem, $R=e^{\int \frac{1}{r}dr}=r$ and

$$rp=\int r\,dr + C$$

or

$$r\frac{d\phi}{dr}=\frac{r^2}{2}+C.$$

The variables in this equation are separable. Thus

$$\phi=\int\left(\frac{\frac{r^2}{2}+C}{r}\right)dr = \frac{r^2}{4}+C\ln r + C'.$$

We have no point sources or sinks in our problem, so $\phi$ must be finite everywhere inside the circle, including the origin. Therefore $C=0$. To evaluate $C'$, use $(\phi)_{r=b}=0$. This gives

$$0=\frac{b^2}{4}+C' \quad \text{or} \quad C'=-\frac{b^2}{4}.$$

Thus the solution to the boundary value problem is

$$\phi=\frac{r^2-b^2}{4}.$$

Prob 31

Given $\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = A$ inside $r = b$, $\left(\frac{d\phi}{dr}\right)_{r=b} = B$

The diff. equn. becomes

$$r\frac{d\phi}{dr} = \frac{Ar^2}{2} + C$$

$$\phi = \int\left(\frac{A}{2}r + \frac{C}{r}\right)dr = \frac{Ar^2}{4} + C\ln r + C'$$

$C = 0$ because $\phi$ is bounded within the circle

$$\phi = \frac{Ar^2}{4} + C'$$

$$\frac{\partial\phi}{\partial r} = \frac{Ar}{2}, \quad \left(\frac{\partial\phi}{\partial r}\right)_{r=b} = \frac{Ab}{2} = B$$

Thus $\phi = \frac{Ar^2}{4} + C$ ~~where~~ if $\frac{Ab}{2} = \frac{B}{b}$

$$= \frac{Br^2}{2b} + C.$$

## 9. Laplace's equn. in 3 dimensions

a. Cartesian coordinates: $\dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} + \dfrac{\partial^2\varphi}{\partial z^2} = 0$

Let $\varphi = X(x)\, Y(y)\, Z(z)$. Substitution gives:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0, \quad \text{or}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -p^2 = -\left(\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2}\right).$$

This gives the equations

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = p^2 - \frac{1}{Z}\frac{d^2Z}{dz^2} = -q^2$$

$$\frac{d^2X}{dx^2} + p^2 X = 0$$

$$\frac{d^2Y}{dy^2} + q^2 Y = 0 \qquad \text{where} \quad q^2 = -\frac{1}{Y}\frac{d^2Y}{dy^2}.$$

$$\frac{d^2Z}{dz^2} - (p^2+q^2) Z = 0.$$

These equations have solutions of the type:

$$\varphi = \left.\begin{matrix} e^{\sqrt{p^2+q^2}\,z} \\ e^{-\sqrt{p^2+q^2}\,z}\end{matrix}\right\} \left.\begin{matrix}\cos px \\ \sin px\end{matrix}\right\} \left.\begin{matrix}\cos qy \\ \sin qy\end{matrix}\right\}$$

$$\varphi = \left.\begin{matrix}\cosh \\ \sinh\end{matrix}\sqrt{p^2+q^2}\,z\right\} \left.\begin{matrix}\cos \\ \sin\end{matrix}(px+qy)\right\} \left.\begin{matrix}\cos \\ \sin\end{matrix}(px-qy)\right\}.$$

Other particular solutions can be obtained by using $e^{\pm ipx}$ and/or $e^{\pm iqy}$ and/or $e^{\pm\sqrt{p^2+q^2}\,z}$. We may also interchange the coefficients of $x, y, z$ for other particular solutions. Finally, any sum of such particular linear solutions is also a solution.

Example: Laplace's equation in 3-D Cartesian coordinates

Given: $\left\{ \begin{array}{l} \phi = 0 \text{ on } x=0,\ x=a,\ y=0,\ y=b,\ z=0 \\ \phi = f(x,y) \text{ on } z = c. \end{array} \right\}$ ~~the region~~

these plane surfaces.

Find $\phi$ in the region bounded by these plane surfaces.

Let us look at particular solutions of the form

$$\phi = \left.\begin{matrix}\cos \\ \sin\end{matrix}\ px\right\} \left.\begin{matrix}\cos \\ \sin\end{matrix}\ qy\right\} \left.\begin{matrix}\cosh \\ \sinh\end{matrix}\ \sqrt{p^2+q^2}\, z\right\}.$$

~~Solut~~ We did not use $e^{\pm\sqrt{p^2+q^2}\,z}$ because neither is zero for $z=0$

1) The boundary condition $\phi = 0$ on $z=0$ implies that the coefficient of $\cosh\sqrt{p^2+q^2}\, z = 0$.

2) The bdy. cond. $\phi = 0$ on $x = 0$ implies that the coeff. of $\cos px = 0$.

3) " " " $\phi = 0$ on $y = 0$ " " " " $\cos qy = 0$

4) " " " $\phi = 0$ on $x = a$ implies that ~~the~~ $pa = m\pi$, $m$ is an integer

5) " " " $\phi = 0$ on $y = b$ implies that $qb = n\pi$, $n$ is an integer.

Therefore

$$\phi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh(s_{mn} z) \quad \text{where } s = \sqrt{p^2+q^2}$$

or $s_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$.

The boundary condition $\phi = f(x,y)$ on $z = c$ gives

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh(s_{mn} c).$$

The last expression may be considered to be a Fourier series

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \quad F_{mn} \equiv A_{mn}\sinh(s_{mn}c),$$

~~where~~ and $$F_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy, \quad {m \atop n} = 1,2,3,\cdots$$

Finally,

$$\phi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{F_{mn}}{\sinh(s_{mn}c)} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh(s_{mn} z)$$

where ${n \atop m} = 1,2,3,\cdots$ and $s_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$.

Mann & Spencer chap. 11
Morse & Feshbach chap.

# Conditions for Separability of Laplace's Equation

Although separation of variables is a powerful tool in solving the equations of mathematical physics, it cannot always be used. There are some partial differential in which the variables cannot be separated. More importantly, particular equations such as the wave equation or Laplace's equation will only separate in certain coordinate systems. We shall look at the conditions we must place on the coordinate systems so that Laplace's equation can be satisfied.

In curvilinear, orthogonal coordinates, Laplace's equation is

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial w}\right)\right]=0.$$

Let $\phi = U(u)\,V(v)\,W(w)$. Then

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{1}{U}\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{dU}{du}\right)+\frac{1}{V}\frac{\partial}{\partial v}\left(\frac{h_1h_3}{h_2}\frac{\partial V}{\partial v}\right)+\frac{1}{W}\frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{dW}{dw}\right)\right]=0$$

A necessary, but not sufficient, condition for separability is

$$\frac{h_2h_3}{h_1} = f_1(u)\,F_1(v,w), \quad \frac{h_1h_3}{h_2} = f_2(v)\,F_2(u,w), \quad \frac{h_1h_2}{h_3} = f_3(w)\,F_3(u,v)$$

which gives

(1)
$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{F_1(v,w)}{U}\frac{d}{du}\left(f_1(u)\frac{dU}{du}\right)+\frac{F_2(u,w)}{V}\frac{d}{dv}\left(f_2(v)\frac{dV}{dv}\right)+\frac{F_3(u,v)}{W}\frac{d}{dw}\left(f_3(w)\frac{dW}{dw}\right)\right]=0.$$

Special cases would correspond to A) $\frac{h_2h_3}{h_1}$, $\frac{h_1h_3}{h_2}$, $\frac{h_1h_2}{h_3}$ being constant,

or B) $\frac{h_2h_3}{h_1} = f_1(u)$, $\frac{h_1h_3}{h_2} = f_2(v)$, $\frac{h_1h_2}{h_3} = f_3(w)$.

Separating variables in the expression for Laplace's equation, we have

$$\frac{F_1}{U}\frac{d}{du}\left(f_1\frac{dU}{du}\right)+\frac{F_2}{V}\frac{d}{dv}\left(f_2\frac{dV}{dv}\right) = -\frac{F_3}{W}\frac{d}{dw}\left(f_3\frac{dW}{dw}\right) = \alpha_2$$ ?? ($\alpha_2$ is a separation constant)

$$\frac{F_1}{U}\frac{d}{du}\left(f_1\frac{dU}{du}\right) - \alpha_2 = -\frac{F_2}{V}\frac{d}{dv}\left(f_2\frac{dV}{dv}\right) = \alpha_3.$$ $F_3$ is a $\alpha_3$ ~~function of~~

~~name the equations can be separated, and~~

Call $\alpha_2$ and $\alpha_3$ the separation constants for eq. (1).

(2)

These depend only on the scale factors of the coordinates

The U's $(U, V, W)$ are a function of the $\alpha$'s, whereas the F's and f's are not.

If we differentiate Eq (1) with respect to $\alpha_2$ and $\alpha_3$, we obtain

$$F_1 \frac{\partial}{\partial \alpha_2}\left[\frac{1}{U}\frac{d}{du}\left(f_1 \frac{dU}{du}\right)\right] + F_2 \frac{\partial}{\partial \alpha_2}\left[\frac{1}{V}\frac{d}{dv}\left(f_2 \frac{dV}{dv}\right)\right]$$
$$+ F_3 \frac{\partial}{\partial \alpha_2}\left[\frac{1}{W}\frac{d}{dw}\left(f_3 \frac{dW}{dw}\right)\right] = 0$$

and

$$F_1 \frac{\partial}{\partial \alpha_3}\left[\frac{1}{U}\frac{d}{du}\left(f_1 \frac{dU}{du}\right)\right] + F_2 \frac{\partial}{\partial \alpha_3}\left[\frac{1}{V}\frac{d}{dv}\left(f_2 \frac{dV}{dv}\right)\right] + F_3 \frac{\partial}{\partial \alpha_3}\left[\frac{1}{W}\frac{d}{dw}\left(f_3 \frac{dW}{dw}\right)\right] =$$

Define $$\Phi_{is}(u_i) \equiv \frac{1}{f_i(u_i)} \frac{\partial}{\partial \alpha_s}\left[\frac{1}{U_i}\frac{d}{du_i}\left(f_i \frac{dU_i}{du_i}\right)\right]$$

Substitute into the preceding equations. Then

$$f_1 F_1 \Phi_{12}(u) + f_2 F_2 \Phi_{22}(v) + f_3 F_3 \Phi_{32}(w) = 0$$
$$f_1 F_1 \Phi_{13}(u) + f_2 F_2 \Phi_{23}(v) + f_3 F_3 \Phi_{33}(w) = 0 .$$

Divide both equations by $f_1 F_1$. This gives

$$\Phi_{22}(v) \frac{f_2 F_2}{f_1 F_1} + \Phi_{32}(w) \frac{f_3 F_3}{f_1 F_1} = -\Phi_{12}(u)$$
$$\Phi_{23}(v) \frac{f_2 F_2}{f_1 F_1} + \Phi_{33}(w) \frac{f_3 F_3}{f_1 F_1} = -\Phi_{13}(u) .$$

These equations may be looked upon as equations in the unknowns $\frac{f_2 F_2}{f_1 F_1}$ and $\frac{f_3 F_3}{f_1 F_1}$. Solving for them, we obtain

$$\frac{f_2 F_2}{f_1 F_1} = \frac{\begin{vmatrix} -\Phi_{12}(u) & \Phi_{32}(w) \\ -\Phi_{13}(u) & \Phi_{33}(w) \end{vmatrix}}{\begin{vmatrix} \Phi_{22}(v) & \Phi_{32}(w) \\ \Phi_{23}(v) & \Phi_{33}(w) \end{vmatrix}} \equiv \frac{M_{21}(u,w)}{M_{11}(v,w)}$$

$$M_{21}(u,w) \equiv \begin{vmatrix} -\Phi_{12}(u) & \Phi_{32}(w) \\ -\Phi_{13}(u) & \Phi_{33}(w) \end{vmatrix} .$$

etc.

(2)
$$\frac{f_3 F_3}{f_1 F_1} = \frac{\begin{vmatrix} \Phi_{22}(v) & -\Phi_{12}(u) \\ \Phi_{23}(v) & -\Phi_{13}(u) \end{vmatrix}}{\begin{vmatrix} \Phi_{22}(v) & \Phi_{32}(w) \\ \Phi_{23}(v) & \Phi_{33}(w) \end{vmatrix}} \equiv \frac{M_{31}(u,v)}{M_{11}(v,w)} .$$

From before, $\dfrac{h_2 h_3}{h_1} = f_1 F_1$ , $\dfrac{h_1 h_3}{h_2} = f_2 F_2$ , $\dfrac{h_1 h_2}{h_3} = f_3 F_3$ .

Thus $\dfrac{f_2 F_2}{f_1 F_1} = \dfrac{h_1^2}{h_2^2}$ and $\dfrac{f_3 F_3}{f_1 F_1} = \dfrac{h_1^2}{h_3^2}$ . These are called the first condition for separability. We have ~~shown~~ seen they are necessary for Laplace's eqn. to be separable.

Also, from (2)

$$f_1(u)\left[\frac{F_1(v,w)}{M_{11}(v,w)}\right] = f_2(v)\left[\frac{F_2(u,w)}{M_{21}(u,w)}\right] = f_3(w)\left[\frac{F_3(u,v)}{M_{31}(u,v)}\right].$$

These equations can be satisfied only if each of the three terms is equal to $f_1 f_2 f_3$ , as can be seen by inspection. Therefore

$$\frac{f_1 F_1}{M_{11}} = f_1 f_2 f_3 = \frac{h_2 h_3}{h_1 M_{11}}$$

or

$$\frac{h_2 h_3}{h_1} = f_1 f_2 f_3 M_{11} .$$

Similarly

$$\frac{h_1 h_3}{h_2} = f_1 f_2 f_3 M_{21}$$

$$\frac{h_1 h_2}{h_3} = f_1 f_2 f_3 M_{31} .$$

These last three equations are called the second condition for separability. We have proved that they are necessary conditions.

It can also be proved that the 1st and 2nd conditions of separability are sufficient conditions for Laplace's eqn. to be separable. From this proof comes the formula

$$\frac{1}{f_i}\frac{d}{du_i}\left(f_i \frac{dU_i}{du_i}\right) + U_i \sum_{j=1}^{3} \alpha_j \Phi_{ij} = 0 \quad \text{where } \alpha_1 = 0 .$$

[illegible] the 3 separated equations directly, without first substituting

(4)

Some examples showing that the coordinate systems give a separable Laplace equation:

2) Circular cylindrical coordinates $(h_1 = 1,\ h_2 = r,\ h_3 = 1)$

$$\frac{M_{21}}{M_{11}} = \frac{1}{r^2} \qquad \frac{M_{21}}{M_{11}} = \frac{f_2 f_3}{f_1 f_3} = \frac{h_1 h_3 / h_2}{h_2 h_3 / h_1} = \frac{h_1^2}{h_2^2} = \frac{1}{r^2}$$

$$\frac{M_{31}}{M_{11}} = 1$$

$$\frac{h_2 h_3}{h_1} = r = f_1^{(r)} f_2^{(\theta)} f_3^{(z)} M_{11}(\theta, z) \longrightarrow f_2 = f_3 = M_{11} = 1, \quad f_1 = r$$

$$\frac{h_1 h_3}{h_2} = \frac{1}{r} = f_1 f_2 f_3 M_{21} \longrightarrow M_{21} = \frac{1}{r^2}$$

$$\frac{h_1 h_2}{h_3} = r = f_1 f_2 f_3 M_{31} \longrightarrow M_{31} = 1$$

Thus the ~~other~~ first two conditions are also satisfied, and ~~the~~ $\nabla^2 \phi = 0$ is separable in this coordinate system.

To obtain the separated equations, use $\frac{1}{f_i} \frac{d}{du_i}\left(f_i \frac{dU_i}{du_i}\right) + U_i \sum_{j=1}^{3} \alpha_j \Phi_{ij} = 0$ with $\alpha_1 = 0$.

This gives

$$\frac{1}{r}\frac{d}{dr}\left(r \frac{dR}{dr}\right) + R(\alpha_2 \Phi_{12} + \alpha_3 \Phi_{13}) = 0$$

$$\frac{d}{d\theta}\left(\frac{d\Theta}{d\theta}\right) + \Theta(\alpha_2 \Phi_{22} + \alpha_3 \Phi_{23}) = 0$$

$$\frac{d}{dz}\left(\frac{dZ}{dz}\right) + Z(\alpha_2 \Phi_{32} + \alpha_3 \Phi_{33}) = 0.$$

Now

$$\begin{vmatrix} -\Phi_{12}(u) & \Phi_{32}(w) \\ -\Phi_{13}(u) & \Phi_{33}(w) \end{vmatrix} = M_{21} = \frac{1}{r^2} \longrightarrow \Phi_{12} = \frac{1}{r^2},\ \Phi_{33} = 1, \text{ either } \Phi_{13} \text{ or } \Phi_{32} \text{ or both} = 0$$

$$\begin{vmatrix} \Phi_{22}(v) & \Phi_{32}(w) \\ \Phi_{23}(v) & \Phi_{33}(w) \end{vmatrix} = M_{11} = 1 \longrightarrow \Phi_{22} = 1, \text{ either } \Phi_{23} \text{ or } \Phi_{32} \text{ or both} = 0$$

$$\begin{vmatrix} \Phi_{22}(v) & -\Phi_{12}(u) \\ \Phi_{23}(v) & -\Phi_{13}(u) \end{vmatrix} = M_{31} = 1 \longrightarrow \Phi_{13} = -1,\ \Phi_{23} = 0$$

From the first and third lines, because $\Phi_{13} \neq 0$, $\Phi_{32} = 0$

(5)

Therefore the separated equations become

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + R\left[-\frac{\alpha_2}{r^2} - \alpha_3\right] = 0$$

$$\frac{d^2\Theta}{d\theta^2} + \Theta\,\alpha_2 = 0$$

$$\frac{d^2 Z_1}{dz^2} + Z_1\,\alpha_3 = 0 .$$

(In our earlier work we used $q^2$ instead of $\alpha_2$ and $-p^2$ instead of $\alpha_3$.

For this set of coordinates the above method of obtaining the separated equations is much more difficult than by substituting $\phi = R(r)\,\Theta(\theta)\,Z_1(z)$.

However, for some coordinate systems this is not the case.

(3) elliptic cylindrical

$x = a\cosh\eta\cos\psi$, $y = a\sinh\eta\sin\psi$, $z = z$ $\left\}\begin{array}{l} 0 \le \eta \le \infty \\ -\pi < \psi \le \pi \\ -\infty < z < \infty \end{array}\right.$

$h_1 = \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2} = a\sqrt{\sinh^2\eta\cos^2\psi + \cosh^2\eta\sin^2\psi}$

$h_2 = h_1$, $h_3 = 1$

$$\frac{M_{21}}{M_{11}} = 1$$

$$\frac{M_{31}}{M_{11}} = a^2(\sinh^2\eta\cos^2\psi + \cosh^2\eta\sin^2\psi)$$

$$\frac{h_2h_3}{h_1} = 1 = f_1f_2f_3M_{11} \longrightarrow f_1 = f_2 = f_3 = M_{11} = 1$$

$$\frac{h_1h_3}{h_2} = 1 = f_1f_2f_3M_{21} \longrightarrow M_{21} = 1$$

$$\frac{h_1h_2}{h_3} = a^2(\sinh^2\eta\cos^2\psi + \cosh^2\eta\sin^2\psi) = f_1f_2f_3M_{31} \longrightarrow M_{31} = a^2(\sinh^2\eta\cos^2\psi + \cosh^2\eta\sin^2\psi),$$

(4) parabolic cylindrical

$x = \frac{1}{2}(\mu^2 - \nu^2)$, $y = \mu\nu$, $z = z$

$h_1 = \sqrt{\mu^2 + \nu^2}$, $h_2 = h_1$, $h_3 = 1$

$$\frac{M_{21}}{M_{11}} = 1$$

$$\frac{M_{31}}{M_{11}} = \mu^2 + \nu^2$$

$$\frac{h_2h_3}{h_1} = 1 = f_1f_2f_3M_{11} \rightarrow f_1 = f_2 = f_3 = M_{11} = 1$$

$$\frac{h_1h_3}{h_2} = 1 = f_1f_2f_3M_{21} \rightarrow M_{21} = 1$$

$$\frac{h_1h_2}{h_3} = \mu^2 + \nu^2 = f_1f_2f_3M_{31} \rightarrow M_{31} = \mu^2 + \nu^2$$

(5) spherical

$$\frac{M_{21}}{M_{11}} = \frac{h_1^2}{h_2^2} = \frac{1}{r^2}$$

$$\frac{M_{31}}{M_{11}} = \frac{h_1^2}{h_3^2} = \frac{1}{r^2\sin^2\theta}$$

$f_1 = r^2$, $f_2 = \sin\theta$, $f_3 = 1$

$$\frac{h_2h_3}{h_1} = r^2\sin\theta = f_1f_2f_3M_{11} \rightarrow \cancel{f_1 = f_2 = f_3 = 1}, \quad M_{11} = \cancel{r^2\sin\theta}\ 1$$

$$\frac{h_1h_3}{h_2} = \sin\theta = \overset{r^2\sin\theta\, M_{21}}{f_1f_2f_3M_{21}} \rightarrow M_{21} = \cancel{\sin\theta}\ 1/r^2$$

$$\frac{h_1h_2}{h_3} = \frac{1}{\sin\theta} = \overset{r^2\sin\theta}{f_1f_2f_3}M_{31} \rightarrow M_{31} = \frac{1}{r^2\sin^2\theta}$$

(1) There are only 11 coordinate systems in which the wave equation is separable. Laplace's eqn. is separable in all of these, in addition to some others. The eleven are

1) rectangular $u = x$, $v = y$, $w = z$, $x = x$, $y = y$, $z = z$, $h_1 = 1$, $h_2 = 1$, $h_3 = 1$
$-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$

2) circular cylindrical $u = r$, $v = \theta$, $w = z$, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $h_1 = 1$, $h_2 = r$, $h_3 = 1$
$0 \le r < \infty$, $0 \le \theta < 2\pi$, $-\infty < z < \infty$

3) elliptic cylindrical $u = \eta$, $v = \phi$, $w = z$, $x = a\cosh\eta\cos\phi$, $y = a\sinh\eta\sin\phi$, $z = z$
$0 \le \eta < \infty$, $-\pi < \phi \le \pi$, $-\infty < z < \infty$ $h_1 = h_2 = a\sqrt{\sinh^2\eta\cos^2\phi + \cosh^2\eta\sin^2\phi}$, $h_3 = 1$.
or $0 \le \phi < 2\pi$

4) parabolic cylindrical $u = \mu$, $v = \nu$, $w = z$, $x = \frac{1}{2}(\mu^2 - \nu^2)$, $y = \mu\nu$, $z = z$
$0 \le \mu < \infty$, $0 \le \nu < \infty$, $-\infty < z < \infty$ $h_1 = h_2 = \sqrt{\mu^2 + \nu^2}$, $h_3 = 1$

5) spherical coordinates $u = r$, $v = \theta$, $w = \phi$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
$0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$ $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$

6) parabolic coordinates $u = \mu$, $v = \nu$, $w = \phi$, $x = \mu\nu\cos\phi$, $y = \mu\nu\sin\phi$, $z = \frac{1}{2}(\mu^2 - \nu^2)$
$0 \le \mu < \infty$, $0 \le \nu < \infty$, $0 \le \phi < 2\pi$ $h_1 = h_2 = \sqrt{\mu^2 + \nu^2}$, $h_3 = \mu\nu$

7) prolate spheroidal coordinates $u = \eta$, $v = \theta$, $w = \phi$, $x = a\sinh\eta\sin\theta\cos\phi$, $y = a\sinh\eta\sin\theta\sin\phi$, $z = a\cosh\eta\cos$
$0 \le \eta < \infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$ $h_1 = h_2 = a\sqrt{\sinh^2\eta + \sin^2\theta}$, $h_3 = a\sinh\eta\sin\theta$

8) oblate spheroidal coordinates $u = \eta$, $v = \theta$, $w = \phi$, $x = a\cosh\eta\sin\theta\cos\phi$, $y = a\cosh\eta\sin\theta\sin\phi$, $z = a\sinh\eta\cos$
$0 \le \eta < \infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$ $h_1 = h_2 = a\sqrt{\cosh^2\eta - \sin^2\theta}$, $h_3 = a\cosh\eta\sin\theta$

9) Ellipsoidal coordinates

10) paraboloidal coordinates

11) conical coordinates

1)

In addition to the eleven coordinate systems ~~given~~ considered, Laplace's equation is separable in any orthogonal cylindrical coordinate system in which $\phi$ is independent of $z$. (Stated without proof.)

Example: bicylindrical coordinates $u=\eta$, $v=\theta$, $w=z$.

$$x = \frac{a \sinh \eta}{\cosh \eta - \cos \theta}$$

$$y = \frac{a \sin \theta}{\cosh \eta - \cos \theta}$$

$$z = z$$

$$h_1 = h_2 = \frac{a}{\cosh \eta - \cos \theta} \quad , h_3 = 1 .$$

$$\frac{M_{21}}{M_{11}} = \frac{h_1^2}{h_2^2} = 1$$

$$\frac{M_{31}}{M_{11}} = \frac{h_1^2}{h_3^2} = \frac{a^2}{(\cosh \eta - \cos \theta)^2}$$

$$\frac{h_2 h_3}{h_1} = 1 = f_1 f_2 f_3 M_{11}(v,w)$$

$$\frac{h_1 h_3}{h_2} = 1 = f_1 f_2 f_3 M_{21}(u,w) ,$$

$$\frac{h_1 h_2}{h_3} = \frac{a^2}{(\cosh \eta - \cos \theta)^2} = f_1 f_2 f_3 M_{31}(u,v)$$

This would indicate that $M_{31}(\eta,\theta) = \frac{a^2}{(\cosh \eta - \cos \theta)^2}$. But this is not a permissible form for $M_{31}$. Consider its expansion:

$$M_{31}(\eta,\theta) = \begin{vmatrix} \Phi_{12}(\eta) & \Phi_{13}(\eta) \\ \Phi_{22}(\theta) & \Phi_{23}(\theta) \end{vmatrix} .$$

But no sum of products of functions of $\eta$ only and of $\theta$ only will have the form $\frac{1}{(\cosh \eta - \cos \theta)^2}$. Thus the variables cannot be separated

If $\phi \neq \phi(z)$, then the equation in bicylindrical coordinates can be separated. ~~In this case $h_3 = 0$, and~~ This follows from the theorems which we stated earlier without proof. It might also be $\cancel{\frac{M_{21}}{M_{11}} = \frac{h_1^2}{h_2^2} = 1}$ shown by looking at Laplace's eqn. directly. In these coordinates, with $\phi$ independent of $z$,

$$\nabla^2 \phi = 0 = \left(\frac{\cosh\eta - \cos\theta}{a}\right)^2 \left(\frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial^2 \phi}{\partial \theta^2}\right).$$

<u>R-Separability</u> Definition: If the assumption $\phi = \frac{U(u)V(v)W(w)}{R(u,v,w)}$ permits the separation of the partial differential equation into three ordinary differential equations, and if $R$ is not a constant, then the equation is said to be R-separable.

Although the wave equation is never R-separable, there are several coordinate systems in which Laplace's equation is R-separable.

As an example, consider toroidal coordinates $(\eta, \theta, \psi)$.

$$x = \frac{a \sinh\eta \cos\psi}{\cosh\eta - \cos\theta}, \quad y = \frac{a \sinh\eta \sin\psi}{\cosh\eta - \cos\theta}, \quad z = \frac{a \sin\theta}{\cosh\eta - \cos\theta}.$$

Then
$$h_1 = h_2 = \frac{a}{\cosh\eta - \cos\theta}, \quad h_3 = \frac{a \sinh\eta}{\cosh\eta - \cos\theta}.$$

It can be shown that $R = 1/\sqrt{\cosh\eta - \cos\theta}$

Laplace's equation is

$$\nabla^2 \phi = 0 = \frac{(\cosh\eta - \cos\theta)^3}{a^3 \sinh\eta} \left[\frac{\partial}{\partial\eta}\left(\frac{a \sinh\eta}{\cosh\eta - \cos\theta}\frac{\partial\phi}{\partial\eta}\right) + \frac{\partial}{\partial\theta}\left(\frac{a\sinh\eta}{\cosh\eta - \cos\theta}\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial\psi}\left(\frac{a}{\sinh\eta(\cosh\eta - \cos\theta)}\frac{\partial\phi}{\partial\psi}\right)\right]$$

$$\frac{\partial}{\partial \eta}\left(\frac{a \sinh \eta}{\cosh \eta - \cos\theta}\,\frac{\partial \varphi}{\partial \eta}\right) + \frac{\partial}{\partial \theta}\left(\frac{a \sinh \eta}{\cosh \eta - \cos\theta}\,\frac{\partial \varphi}{\partial \theta}\right) + \frac{\partial}{\partial \psi}\left(\frac{a}{\sinh\eta(\cosh \eta - \cos\theta)}\,\frac{\partial \varphi}{\partial \psi}\right) = 0$$

$$\frac{\partial}{\partial \eta}\left(\frac{a \sinh \eta}{\cosh \eta - \cos\theta}\,\frac{\partial}{\partial \eta}\sqrt{\cosh \eta - \cos\theta}\; H\,\theta\,\psi\right)$$

$$= \frac{\partial}{\partial \eta}\left\{\frac{a \sinh \eta}{\cosh \eta - \cos\theta}\left(\sqrt{\cosh \eta - \cos\theta}\; H'\theta\psi + \frac{\sinh\eta}{2\sqrt{\cosh \eta - \cos\theta}}\, H\theta\psi\right)\right\}$$

$$= \frac{a\cosh\eta(\cosh\eta-\cos\theta) - a\sinh^2\eta}{(\cosh\eta - \cos\theta)^2}\left\{\sqrt{\cosh \eta - \cos\theta}\; H'\theta\psi + \frac{\sinh\eta}{2\sqrt{\cosh \eta - \cos\theta}}\, H\theta\psi\right\}$$

$$+ \frac{a \sinh \eta}{(\cosh\eta - \cos\theta)}\left\{\frac{1}{2}\frac{\sinh\eta}{\sqrt{\cosh \eta - \cos\theta}}\, H'\theta\psi + \sqrt{\cosh \eta - \cos\theta}\; H''\theta\psi + \frac{\sinh\eta}{2\sqrt{\cosh \eta - \cos\theta}}\, H'\theta\psi \right.$$

$$\left. + \frac{1}{2}\,\frac{\sqrt{\cosh\eta-\cos\theta}\cdot\cosh\eta - \frac{1}{2}\sinh^2\eta/\sqrt{\cosh\eta-\cos\theta}}{(\cosh\eta-\cos\theta)}\, H\theta\psi\right\}$$

$$= \left\{\frac{a\left[\cosh\eta(\cosh\eta-\cos\theta) - \sinh^2\eta\right]}{(\cosh\eta-\cos\theta)^{3/2}} + \frac{a\sinh^2\eta}{(\cosh\eta-\cos\theta)^{3/2}}\right\} H'\theta\psi$$

$$+ \frac{a\sinh\eta}{\sqrt{\cosh\eta-\cos\theta}}\, H''\theta\psi \qquad + \quad f(\theta\eta)\; H\theta\psi$$

$$\frac{\partial}{\partial\theta}\left(\frac{a\sinh\eta}{\cosh\eta-\cos\theta}\,\frac{\partial\phi}{\partial\theta}\right)=\frac{\partial}{\partial\theta}\left(\frac{a\sinh\eta}{\cosh\eta-\cos\theta}\,\frac{\partial}{\partial\theta}\sqrt{\cosh\eta-\cos\theta}\;H\Theta\Psi\right)$$

$$=\frac{\partial}{\partial\theta}\left\{\frac{a\sinh\eta}{\cosh\eta-\cos\theta}\left[\sqrt{\cosh\eta-\cos\theta}\;H\Theta'\Psi+\frac{1}{2}\frac{\sin\theta}{\sqrt{\cosh\eta-\cos\theta}}\,H\Theta\Psi\right]\right\}$$

$$=\frac{-\frac{1}{2}a\sinh\eta\sin\theta}{(\cosh\eta-\cos\theta)^{3/2}}\,H\Theta'\Psi+\frac{a\sinh\eta}{\sqrt{\cosh\eta-\cos\theta}}\,H\Theta''\Psi$$

$$+\frac{1}{2}\,\frac{(\cosh\eta-\cos\theta)^{3/2}a\sinh\eta\cos\theta-\frac{3}{2}a\sinh\eta\sin^2\theta\sqrt{\cosh\eta-\cos\theta}}{(\cosh\eta-\cos\theta)^{3}}\,H\Theta\Psi$$

$$+\frac{1}{2}\,\frac{a\sinh\eta\sin\theta}{(\cosh\eta-\cos\theta)^{3/2}}\,H\Theta'\Psi$$

$$\frac{\partial}{\partial\psi}=\frac{a}{\sinh\eta\sqrt{\cosh\eta-\cos\theta}}\,H\Theta\Psi''$$

$$\nabla^2\phi=\frac{a\sinh\eta}{\sqrt{\cosh\eta-\cos\theta}}\,H''\Theta\Psi+\frac{a\cosh\eta}{\sqrt{\cosh\eta-\cos\theta}}\,H'\Theta\Psi+\frac{a\sinh\eta}{\sqrt{\cosh\eta-\cos\theta}}\,H\Theta''\Psi$$

$$+\frac{a}{\sinh\eta\sqrt{\cosh\eta-\cos\theta}}\,H\Theta\Psi''$$

$$+H\Theta\Psi\left\{\frac{a}{2}\,\frac{(\cosh^2\eta-\sinh^2\eta-\cosh\eta\cos\theta)\sinh\eta}{(\cosh\eta-\cos\theta)^{5/2}}+\frac{a}{2}\,\frac{\sinh\eta\cosh\eta(\cosh\eta-\cos\theta)-\frac{1}{2}\sinh^2\eta}{(\cosh\eta-\cos\theta)^{5/2}}\right.$$

$$\left.+\frac{a}{2}\,\frac{(\cosh\eta-\cos\theta)\sinh\eta\cos\theta-\frac{3}{2}\sinh\eta\sin^2\theta}{(\cosh\eta-\cos\theta)^{5/2}}\right\}=0$$

$$\{\ \}=\frac{a}{2(\cosh\eta-\cos\theta)^{5/2}}\Big[\sinh\eta-\sinh\eta\cosh\eta\cos\theta+\sinh\eta\cosh^2\eta-\cancel{\sinh\eta\cosh\eta\cos\theta}-\frac{1}{2}\sinh^2\eta$$

$$+\sinh\eta\cosh\eta\cos\theta-\sinh\eta\cos^2\theta-\frac{3}{2}\sinh\eta\sin^2\theta\Big]$$

$$=\frac{a}{2(\cosh\eta-\cos\theta)^{5/2}}\Big[-\frac{1}{2}\sinh\eta\sin^2\theta-\sinh\eta\cosh\eta\cos\theta+\sinh\eta\cosh^2\eta-\frac{1}{2}\sinh^2\eta\Big]$$

(1)

Let $\phi = \dfrac{H(\eta)\,\Theta(\theta)\,\Psi(\varphi)}{1/\sqrt{\cosh\eta - \cos\theta}} = \sqrt{\cosh\eta - \cos\theta}\; H(\eta)\,\Theta(\theta)\,\Psi(\varphi).$

Then Laplace's equation in toroidal coordinates can be shown to reduce to

$$\frac{1}{\sinh\eta}\frac{d}{d\eta}\left(\sinh\eta\,\frac{dH}{d\eta}\right) + H\left(\frac{1}{4} - \alpha_2 - \frac{\alpha_3}{\sinh^2\eta}\right) = 0$$

$$\frac{d^2\Theta}{d\theta^2} + \alpha_2\,\Theta = 0$$

$$\frac{d^2\Psi}{d\varphi^2} + \alpha_3\,\Psi = 0 .$$

Call $\alpha_2 = p^2$, $\alpha_3 = q^2$, and $\xi = \cosh\eta$. The diff. equas. become

$$(\xi^2-1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - \left[\frac{q^2}{\xi^2-1} + (p^2 - \tfrac{1}{4})\right] H = 0$$

$$\frac{d^2\Theta}{d\theta^2} + p^2\,\Theta = 0$$

$$\frac{d^2\Psi}{d\varphi^2} + q^2\,\Psi = 0 .$$

Solutions are:

$H = P^{q}_{p-1/2}(\cosh\eta)$ and $Q^{q}_{p-1/2}(\cosh\eta)$

$\Theta = {\sin \atop \cos} p\theta$

$\Psi = {\sin \atop \cos} q\varphi .$

Thus particular solutions of Laplace's equn in toroidal coordinates are:

$$\phi = \sqrt{\cosh\eta - \cos\theta}\left.\begin{matrix} P^{q}_{p-1/2}(\cosh\eta) \\ Q^{q}_{p-1/2}(\cosh\eta)\end{matrix}\right\}\left.{\sin \atop \cos} p\theta\right\}\left.{\sin \atop \cos} q\varphi\right\}.$$

For axial symmetry,

$$\phi = \sqrt{\cosh\eta - \cos\theta}\left.\begin{matrix} P_{p-1/2}(\cosh\eta) \\ Q_{p-1/2}(\cosh\eta)\end{matrix}\right\}\left.{\sin \atop \cos} p\theta\right\}$$

Orthogonal Functions  Morse and Feshbach, chap 6

Definitions:
A set of functions $\{\phi_n(x)\}$ is orthogonal on the interval $(a,b)$ if

$$\int_a^b \phi_m(x)\cdot\phi_n(x)\,dx = 0 \quad \text{for } m\neq n,$$ i.e., for any 2 distinct values of m and n

The n'th norm for the set $\{\phi_n(x)\}$ is defined as

$$N_n = \int_a^b [\phi_n(x)]^2\,dx$$

If $\{\phi_n(x)\}$ constitutes an orthogonal set on the interval $(a,b)$, then an arbitrary function $f(x)$ may be expanded on this interval in a series of $\phi$'s.

Let
$$f(x) = \sum_{n=0}^{\infty} A_n\phi_n(x).$$

Multiply both sides by $\phi_m(x)$ and integrate.

$$\int_a^b f(x)\cdot\phi_m(x)\,dx = \sum_{n=0}^{\infty} A_n\int_a^b \phi_m(x)\,\phi_n(x)\,dx$$

$$= A_m N_m$$

because the integrals in the series all are equal to zero, except for the case when $n=m$. Thus

$$A_n = \frac{1}{N_n}\int_a^b f(x)\cdot\phi_n(x)\,dx$$

Building orthogonal sets:

Consider a set of linearly independent* functions $\{\psi_n(x)\}$, which are non-orthogonal on the interval $(a,b)$. We ask how they might be modified so that they are orthogonal on this interval. Call the orthogonal set $\{\phi_n(x)\}$

---

* The functions are linearly independent if the equation
$$a_0\psi_0(x) + a_1\psi_1(x) + a_2\psi_2(x) + \cdots + a_n\psi_n(x) = 0$$
is satisfied only when $a_0=0,\ a_1=0,\ a_2=0,\ \ldots,\ a_n=0$.

Let $\phi_0(x) = \psi_0(x)$

$$\phi_1(x) = \psi_1(x) - a_{10}\,\phi_0(x)$$

$$\phi_2(x) = \psi_2(x) - a_{20}\,\phi_0(x) - a_{21}\,\phi_1(x)$$

$$\phi_3(x) = \psi_3(x) - a_{30}\,\phi_0(x) - a_{31}\,\phi_1(x) - a_{32}\,\phi_2(x)$$

$$\cdots\cdots\cdots$$

The constants can be evaluated by the orthogonality condition. Thus

$$\int_a^b \phi_1(x)\cdot\phi_0(x)\,dx = 0 = \int_a^b [\psi_1(x) - a_{10}\phi_0(x)]\,\phi_0(x)\,dx$$

$$= \int_a^b \psi_1(x)\,\phi_0(x)\,dx - a_{10}N_0$$

and $a_{10} = \dfrac{1}{N_0}\displaystyle\int_a^b \psi_1(x)\,\phi_0(x)\,dx$.

The constants $a_{20}$ and $a_{21}$ are found in a similar manner.

$$\int_a^b \phi_2(x)\,\phi_0(x)\,dx = 0 = \int_a^b [\psi_2(x) - a_{20}\phi_0(x) - a_{21}\phi_1(x)]\,\phi_0(x)\,dx$$

$$= \int_a^b \psi_2(x)\,\phi_0(x)\,dx - a_{20}N_0 + 0$$ , where the last term is zero because $\int_a^b \phi_1(x)\,\phi_0(x)\,dx = 0$.

Thus $a_{20} = \dfrac{1}{N_0}\displaystyle\int_a^b \psi_2(x)\,\phi_0(x)\,dx$.

Using $\int_a^b \phi_2(x)\,\phi_1(x)\,dx = 0$ we get $a_{21} = \dfrac{1}{N_1}\displaystyle\int_a^b \psi_2(x)\,\phi_1(x)\,dx$.

By the same process it can be shown that

$$a_{ij} = \frac{1}{N_j}\int_a^b \psi_i(x)\,\phi_j(x)\,dx.$$

Example: From the set of functions $\psi_0(x) = 1$, $\psi_1(x) = x$, $\psi_2(x) = x^2$ build up an orthogonal set of functions on the interval $(-1, 1)$.

$$\phi_0(x) = 1$$

$$\phi_1(x) = x - a_{10}$$

$$\phi_2(x) = x^2 - a_{20} - a_{21}\,\phi_1(x)$$

$$\phi_3(x) = x^3 - a_{30} - a_{31}\,\phi_1(x) - a_{32}\,\phi_2(x)$$

Now $a_{10} = \frac{1}{N_0} \int_{-1}^{1} x \, dx = 0$.

Thus $\phi_1(x) = x$.

Also $a_{20} = \frac{1}{N_0} \int_{-1}^{1} x^2 \, dx = \frac{2}{3N_0}$ ; $N_0 = \int_{-1}^{1} [\phi_0(x)]^2 dx = 2$

$$a_{20} = \frac{1}{3}$$

$$a_{21} = \frac{1}{N_1} \int_{-1}^{1} x^2 \cdot x \, dx = 0 .$$

Therefore $\phi_2(x) = x^2 - \frac{1}{3} = \frac{1}{3}(3x^2 - 1)$.

Similarly $\phi_3(x) = \frac{1}{5}(5x^3 - 3x)$.

Eg. If $\int_a^b \lambda_0(x) \lambda_1(x) dx = 0$, then $\int_a^b a_0 \lambda_0(x) a_1 \lambda_1(x) dx = 0$.

Now, orthogonality does not depend on absolute magnitude, so any of these functions may be multiplied by any constant without destroying the orthogonality. If we compare the orthogonal functions in our example with the Legendre functions $P_n(x)$ we can see that they are identical except for numerical factors outside the parentheses.

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x) .$$

Thus we could obtain the Legendre functions by imposing orthogonality requirements on a set of power functions.

<u>Weighting functions</u>. The concept of orthogonality can be extended to include the idea of a weighting function $w(x)$. Two functions $\phi_m(x)$ and $\phi_n(x)$ are said to be orthogonal on the interval $(a,b)$ with respect to the weighting function $w(x)$ if

$$\int_a^b \phi_m(x) \phi_n(x) w(x) \, dx = 0 \quad \text{for } m \neq n .$$

If $w(x) = 1$, we call it simple orthogonality.

To expand a function $f(x)$, write

$$f(x) = \sum_{n=0}^{\infty} A_n \varphi_n(x).$$

Multiply by $w(x)\varphi_m(x)$ and integrate between a and b.

$$\int_a^b f(x)\, w(x)\, \varphi_m(x)\, dx = \sum_{n=0}^{\infty} A_n \int_a^b w(x)\, \varphi_m(x)\, \varphi_n(x)\, dx$$

$$= A_m \int_a^b w(x) [\varphi_m(x)]^2\, dx .$$

Thus
$$A_m = \frac{1}{N_m} \int_a^b f(x)\, w(x)\, \varphi_m(x)\, dx$$

where $N_m = \int_a^b w(x)\, [\varphi_m(x)]^2\, dx$.

$N_n$ is called the weighted norm.

The method of building up an orthogonal set for the case of a weighting function is the same as before. The only difference is in the expression for the coefficients $a_{ij}$.

$$a_{ij} = \frac{1}{N_j} \int_a^b w(x)\, \psi_i(x)\, \varphi_j(x)\, dx \quad , \text{ where}$$

$N_j$ is defined above.

Example: Consider the possibility of building up an orthogonal set in the interval $(0,\infty)$ from the functions $\psi_0(x)=1, \psi_1(x)=x, \psi_2(x)=x^2$

Without a weighting function all of the integrals would become infinite because of the infinite upper limit. This suggests using a weighting function of the form $e^{-x}$.

$$\varphi_0(x) = \psi_0(x) = 1$$
$$\varphi_1(x) = \psi_1(x) - a_{10}\varphi_0(x) = x - a_{10}$$

$$N_0 = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1$$

$$a_{10} = \frac{1}{N_0}\int_0^\infty e^{-x} \cdot x\, dx = \Gamma(2) = 1! = 1$$ because $\int_0^\infty x^{n-1} e^{-x}\, dx = \Gamma(n)$ and $\Gamma(n+1) = n!$ if $n$ is an integer

In this way, we would get

$$\phi_0(x) = 1$$
$$\phi_1(x) = x - 1$$
$$\phi_2(x) = x^2 - 4x + 2$$
$$\phi_3(x) = x^3 - 9x^2 + 18x - 6 .$$

These are essentially the Laguerre functions, which are ~~defined as~~ given by

$$L_0(x) = 1$$
$$L_1(x) = -(x-1)$$
$$L_2(x) = x^2 - 4x + 2$$
$$L_3(x) = -(x^3 - 9x^2 + 18x - 6)$$
$$\cdots\cdots$$

If we attempt to build an orthogonal set from the functions $\phi_0(x) = 1$, $\phi_1(x) = x$, $\phi_2(x) = x^2$, etc over the interval $(-\infty, \infty)$, we have to use a weighting function of the form $e^{-x^2}$. This leads to the set of functions called the Hermite functions, given by

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x$$
$$\cdots\cdots$$

Sturm-Liouville Systems

When the method of separation of variables can be applied to Laplace's equn., the separated ordinary differential equations will all have the form

$$\frac{d^2 Z_1}{dz^2} + P(z)\frac{dZ_1}{dz} + Q(z) Z_1 = 0$$

which is ~~the general~~ a linear, homogeneous, second-order, ordinary differential equation. Let it have solutions

$$Z_1(p, q, z) \text{ and } Z_2(p, q, z)$$

where the p and q are separation constants.

The boundary conditions will be left unspecified for the present.

We want to inquire under what conditions the solutions of the boundary value problems above constitute an orthogonal set.

~~Make a change in~~ Introduce a new variable defined by

$$u(z) = K e^{\int P(z)dz} . \quad \text{Then}$$

$$\frac{du}{dz} = K P(z) e^{\int P(z) dz}$$

and $\frac{1}{u}\frac{du}{dz} = P(z)$.

Also let $Q(z) = \frac{1}{u(z)}[v(z) + f(\lambda) w(z)]$

where $\lambda$ is the eigenvalue associated with either p or q.

The second-order diff equn. above becomes

$$\frac{d^2 Z_1}{dz^2} + \left(\frac{1}{u}\frac{du}{dz}\right)\frac{dZ_1}{dz} + \frac{1}{u}[v(z) + f(\lambda) w(z)] Z_1 = 0$$

or $$\frac{d}{dz}\left[u\frac{dZ_1}{dz}\right] + [v + f(\lambda) w] Z_1 = 0 \qquad (A)$$

Let $\lambda_m$ and $\lambda_n$ be any two separate eigenvalues with corresponding eigenfunctions $Z_m$ and $Z_n$, which are solutions of (A). Then

$$\frac{d}{dz}\left[u\frac{dZ_m}{dz}\right]+\left[v+f(\lambda_m)w\right]Z_m=0$$

and
$$\frac{d}{dz}\left[u\frac{dZ_n}{dz}\right]+\left[v+f(\lambda_n)w\right]Z_n=0.$$

Multiply the first equation by $Z_n$, the second by $Z_m$, and subtract the ~~second~~ first from the ~~first~~ second. This gives:

$$Z_m\frac{d}{dz}\left[u\frac{dZ_n}{dz}\right]-Z_n\frac{d}{dz}\left[u\frac{dZ_m}{dz}\right]$$

$$=\left[f(\lambda_m)-f(\lambda_n)\right]w\,Z_m Z_n=\frac{d}{dz}\left[uZ_m\frac{dZ_n}{dz}-uZ_n\frac{dZ_m}{dz}\right]$$

Integrate both sides of this equation with respect to $z$ over the interval $(a,b)$. This gives

$$\int_a^b\left[f(\lambda_m)-f(\lambda_n)\right]w\,Z_m Z_n\,dz$$

$$=\left[f(\lambda_m)-f(\lambda_n)\right]\int_a^b w\,Z_m Z_n\,dz=\left[uZ_m\frac{dZ_n}{dz}-uZ_n\frac{dZ_m}{dz}\right]_a^b$$

Now, the solutions of the boundary value problem will be orthogonal on the interval $(a,b)$ with respect to the weighting function $w(z)$ if $\int_a^b w(z)\,Z_m Z_n\,dz=0$,

or if
$$\left[uZ_m\frac{dZ_n}{dz}-uZ_n\frac{dZ_m}{dz}\right]_a^b=0. \qquad \text{(B)}$$

Thus we have to place conditions on our boundary values.

Boundary-value problems of the form of eq (A) and (B) are called Sturm-Liouville systems. We can see that Dirichlet, Neumann, and Churchill boundary values all are of the form of eq (B). That is, in general, they will look like

$$\cancel{k_1 Z_1 + k_2 dZ_1/dz = 0}$$

$$k_1 Z_1(a) + k_2 (dZ_1/dz)_{z=a} = 0$$

and
$$k_1 Z_1(b) + k_2 (dZ_1/dz)_{z=b} = 0 .$$

Thus, rewriting (B) as

$$u(b)\left[Z_{1m}(b)\left(\frac{dZ_n}{dz}\right)_{z=b} - Z_{1m}(b)\left(\frac{dZ_m}{dz}\right)_{z=b}\right]$$

$$- u(a)\left[Z_{1m}(a)\left(\frac{dZ_n}{dz}\right)_{z=a} - Z_{1n}(a)\left(\frac{dZ_{1m}}{dz}\right)_{z=a}\right] = 0,$$

by substitution we get

$$u(b)\left[Z_{1m}(b)\left\{-\frac{k_1}{k_2} Z_n(b)\right\} - Z_{1n}(b)\left\{-\frac{k_1}{k_2} Z_{1m}(b)\right\}\right]$$

$$- u(a)\left[Z_{1m}(a)\left\{-\frac{k_1}{k_2} Z_n(a)\right\} - Z_{1n}(a)\left\{-\frac{k_1}{k_2} Z_m(a)\right\}\right]$$

which is identically zero.

Note that for the Dirichlet problem $k_2 = 0$, and for the Neumann problem $k_1 = 0$.

Of course, all Sturm-Liouville systems need not involve Laplace's equations.

6. Cylindrical polar coordinates:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

Let $\phi = R(r)\,\Theta_1(\theta)\,Z_1(z)$. ~~Separation of variables~~ Substitution gives:

$$\Theta_1 Z_1 R'' + \frac{1}{r}\Theta_1 Z_1 R' + \frac{1}{r^2} R Z_1 \Theta_1'' + R\Theta_1 Z_1'' = 0,$$

or

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta_1''}{r^2\Theta_1} + \frac{Z_1''}{Z_1} = 0 \qquad \text{where } R'' \equiv \frac{d^2R}{dr^2}.$$

Separation of variables gives:

$$Z_1'' - p^2 Z_1 = 0$$

$$\frac{\Theta_1''}{r^2\Theta_1} + \frac{R''}{R} + \frac{R'}{rR} = -p^2.$$

Multiply the second eqn. by $r^2$. This gives:

$$\frac{\Theta_1''}{\Theta_1} + \frac{r^2R''}{R} + \frac{rR'}{R} = -p^2r^2.$$

The variables in this eqn. are separable, i.e.,

$$\Theta_1'' + q^2\Theta_1 = 0$$

and

$$r^2R'' + rR' + (p^2r^2 - q^2)R = 0.$$

Particular solutions to the last equation involve Bessel functions of order $q$ of the first and second kind. Therefore, particular solutions to Laplace's eqn can be written as:

$$\phi = \left.\begin{matrix} e^{pz} \\ e^{-pz}\end{matrix}\right\}\left.\begin{matrix}\cos \\ \sin\end{matrix}\, q\theta\right\}\left.\begin{matrix} J_q(pr) \\ Y_q(pr)\end{matrix}\right\}.$$

Other solutions can be obtained in terms of the modified Bessel's function of the first and second kind. These are:

$$I_q(pr) = i^{-q} J_q(ipr) \qquad \text{and} \qquad K_q(pr) = \frac{\pi}{2}\,\frac{I_{-q}(pr) - I_q(pr)}{\sin q\pi}$$

c. Spherical polar coordinates:

Laplace's eqn. is

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2} = 0 .$$

Consider some special cases:

i. spherical symmetry, i.e., $\phi = f(r)$ only. Then Laplace's eqn. reduces:

$$\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 0 ,$$

Integrating once,

$$r^2 \frac{d\phi}{dr} = -A_0$$

and integrating again

$$\phi = \frac{A_0}{r} + B_0 .$$ — This is a general solution.

ii. Axial symmetry, i.e., $\phi \neq f(\psi)$. Laplace's eqn. becomes:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) = 0 .$$

Let $\phi = R(r)\,\Theta(\theta)$. This gives

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \lambda .$$

If we choose $\lambda$ to be $n(n+1)$, these equations become

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)\Theta = 0 .$$

The eqn. in r can be reduced to one with constant coefficients by using $r = e^s$, as we did in the case of plane polar coordinates. This gives:

(3b)

$$\frac{d^2R}{ds^2} + \frac{dR}{ds} - n(n+1)R = 0 \qquad [D^2+D-n(n+1)]R = 0 \qquad (D-n)(D+n+1)R=0$$

which has solutions

$$R = \left.\begin{matrix} e^{ns} \\ e^{-(n+1)s} \end{matrix}\right\} \quad \text{or} \quad R = \left.\begin{matrix} r^{n} \\ r^{-(n+1)} \end{matrix}\right\}$$

The equation in $\theta$ is Legendre's equa., with $n$ being a positive integer. It can be put into the more usual form by a change in variable, namely $\mu = \cos\theta$. Then

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta\frac{d\Theta}{d\mu}$$

or $\dfrac{d}{d\theta} = -\sin\theta\dfrac{d}{d\mu}$.

Thus $\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{d\Theta}{d\theta}\right) = \dfrac{1}{\sin\theta}\left[-\sin\theta\dfrac{d}{d\mu}\left(-\sin^2\theta\dfrac{d\Theta}{d\mu}\right)\right]$

$$= \cancel{2\sin\theta\cos\theta\frac{d\Theta}{d\mu} + \sin^2\theta\frac{d^2\Theta}{d\mu^2}}$$

$$= \cancel{2\cos\theta}$$

$$= \frac{d}{d\mu}\left(\sin^2\theta\frac{d\Theta}{d\mu}\right) = \frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right].$$

The diff. equa. in $\Theta$ becomes:

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0.$$

This is Legendre's equa. It has two independent solutions, one denoted by $P_n(\mu)$ (Legendre fcn of 1st kind) and the other by $Q_n(\mu)$ (Legendre fcn of 2nd kind).

They are of degree (order) $n$, ~~and~~ (The polynomial is of degree $n$) The solutions are $P_n(\cos\theta)$

and $Q_n(\cos\theta)$ ... We shall usually consider only ... ~~non-~~ negative ~~positive~~ and integral ...

iii General case:

If $\phi = f(r, \theta, \psi)$, the separation of variables gives:

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - n(n+1) R = 0$$

$$\frac{d^2 \Psi}{d\psi^2} + m^2 \Psi = 0$$

$$\frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left\{ n(n+1) - \frac{m^2}{\sin^2\theta} \right\} \Theta = 0 .$$

The last equation is Legendre's associated equation, with $m$ and $n$ integral, $m \geq n$ $(n \geq m)$?, ~~and~~ $n$ non-negative, and $m > 0$.

The particular solutions have the form

$$\phi = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} \left.\begin{matrix} \cos \\ \sin \end{matrix}\, m\psi \right\} \left.\begin{matrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{matrix}\right\}$$

where $P_n^m$ and $Q_n^m$ are Legendre's associated functions of the first and second kind, respectively:

$$P_n^m(\mu) = (\mu^2-1)^{\frac{m}{2}} \frac{d^m P_n(\mu)}{d\mu^m}$$

$$Q_n^m(\mu) = (\mu^2-1)^{m/2} \frac{d^m Q_n(\mu)}{d\mu^m}$$

when $n$ is a ~~positive integer~~.

Note that $Q_n(\cos\theta)$ and $Q_n^m(\cos\theta) = \infty$ for $\cos\theta = \pm 1$

~~Simplification~~ ??

# Chapter II Ising & Multimers — Series Solution of Ordinary Differential Equations

First consider the ordinary diff. eqn

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 .$$

The points at which $P(x)$ and $Q(x)$ are analytic functions are called ordinary points. [A function is analytic at a point if it can be expanded in a Taylor series valid in some neighborhood of the point.]

Assume a power series solution of the form

$$y = \sum_{r=0}^{\infty} a_r (x - x_0)^r$$ where $x_0$ is an ordinary point of the diff. eqn.

Then substitute the series into the diff. eqn. and evaluate the $a_r$'s, using boundary conditions and recurrence relations.

Example:

Consider $\frac{d^2y}{dx^2} + y = 0$ .

Comparing with the more general eqn, $P(x) = 0$ and $Q(x) = 1$ .

Let $x_0 = 0$. Then

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} r a_r x^{r-1}$$ and

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} = \sum_{r=-2}^{\infty} (r+2)(r+1) a_{r+2} x^r .$$

Substitute into the diff. eqn. This gives

$$\sum_{r=-2}^{\infty} (r+1)(r+2) a_{r+2} x^r + \sum_{r=0}^{\infty} a_r x^r = 0 .$$

Equate coefficients of $x^r$. This gives

$$(r+1)(r+2) a_{r+2} + a_r = 0 .$$

This is called a recurrence relation. It gives $a_{r+2}$ if $a_r$ is known. We find $a_0$ and $a_1$ from the boundary conditions (2nd order diff eqn — three 2 arbitrary constants).

From the recurrence relation we find,

for $r=0$: $\quad 2a_2 + a_0 = 0 \quad$ or $\quad a_2 = \frac{-a_0}{2!}$

for $r=1$: $\quad 2\cdot3a_3 + a_1 = 0 \quad$ or $\quad a_3 = \frac{-a_1}{3!}$

for $r=2$: $\quad 3\cdot4a_4 + a_2 = 0 \quad$ or $\quad 3\cdot4a_4 = -\frac{a_0}{2!}, \quad a_4 = \frac{a_0}{4!}$.

for $r=3$: $\quad 4\cdot5a_5 + a_3 = 0 \quad$ or $\quad 4\cdot5a_5 = -\frac{a_1}{3!}, \quad a_5 = \frac{a_1}{5!}$, etc.

Thus $y = \sum_{r=0}^{\infty} a_r x^r = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$.

We can recognize the first series as ~~sin~~ cos $x$ and the second as ~~cos~~ sin $x$. Thus

$$y = a_0 \sin x + a_1 \cos x$$

which we could have found by more conventional methods.

---

The point $x = x_0$ is a singular point of the diff. eqn if $P(x)$ and/or $Q(x)$ are not analytic at $x = x_0$. If, when we rewrite the diff. eqn as

$$(x-x_0)^2 \frac{d^2y}{dx^2} + p(x)(x-x_0)\frac{dy}{dx} + q(x)\, y = 0,$$

the $p(x_0)$ and $q(x_0)$ are finite, then $x_0$ is called a regular singular point. If $p(x_0)$ and/or $q(x_0)$ ~~are~~ not finite, then $x_0$ is called an irregular singular point.

Theorem of Fuchs: The diff. eqn. above with $p(x)$ and $q(x)$ has a general solution in the form of a linear combination of convergent series of the type

$$y = (x-x_0)^s \sum_{r=0}^{\infty} a_r (x-x_0)^r$$

provided that $p(x)$ and $q(x)$ are expressible in Taylor series form about the point $x = x_0$.

$$\text{Put } p(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n$$

$$q(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n$$

into the diff. eqn., with $y = (x-x_0)^s \sum_{r=0}^{\infty} a_r (x-x_0)^r$. This gives

$$(x-x_0)^2 \sum_{r=0}^{\infty} (s+r)(s+r-1)\, a_r\, (x-x_0)^{s+r-2} + (x-x_0)\sum_{m=0}^{\infty} p_m (x-x_0)^m \sum_{r=0}^{\infty} (s+r)\, a_r\, (x-x_0)^{s+r}$$

$$+ \sum_{m=0}^{\infty} q_m (x-x_0)^m \sum_{r=0}^{\infty} a_r (x-x_0)^{s+r} = 0$$

or

$$\sum_{r=0}^{\infty} (s+r)(s+r-1)\, a_r (x-x_0)^{s+r} + \sum_{m=0}^{\infty} p_m (x-x_0)^m \sum_{r=0}^{\infty} (s+r) a_r (x-x_0)^{s+r}$$

$$+ \sum_{m=0}^{\infty} q_m (x-x_0)^m \sum_{r=0}^{\infty} a_r (x-x_0)^{s+r} = 0 .$$

Equate the coefficient of $(x-x_0)^s$ to zero ($r=0$, $m=0$). This gives

$$a_0 s(s-1) + p_0 a_0 s + q_0 a_0 = 0.$$

Divide by $a_0$. This gives

$$s(s-1) + p_0 s + q_0 = 0$$

which is called the <u>indicial equation</u>.

Next equate the coefficient of $(x-x_0)^{r+s}$ to zero. But first for convenience let us rewrite the equation as

$$\sum_{r=0}^{\infty} (s+r)(s+r-1) a_r (x-x_0)^{s+r} + \sum_{m=0}^{\infty} p_m \Big\{ s a_0 (x-x_0)^{s+m} + (s+1) a_1 (x-x_0)^{1+s+m}$$

$$+ (s+2) a_2 (x-x_0)^{2+s+m} + \dots + (s+r-1)\, a_{r-1} (x-x_0)^{r-1+s+m}$$

$$+ (s+r)\, a_r (x-x_0)^{r+s+m} + \dots \Big\}$$

$$+ \sum_{m=0}^{\infty} q_m \Big\{ a_0 (x-x_0)^{s+m} + a_1 (x-x_0)^{1+s+m} + a_2 (x-x_0)^{2+s+m} + \dots$$

$$+ a_{r-1} (x-x_0)^{r-1+s+m} + a_r (x-x_0)^{r+s+m} + \dots \Big\} = 0$$

Now equate the coefficients of $(x-x_0)^{r+s}$ to zero. This gives

$$(s+r)(s+r-1) a_r + p_0 (s+r) a_r + p_1 (s+r-1) a_{r-1} + \dots + p_r s a_0$$

$$+ q_0 a_r + q_1 a_{r-1} + \dots + q_r a_0 ,$$

or

$$(s+r)(s+r-1) a_r + \sum_{m=0}^{r} \big[ p_m (s+r-m) + q_m \big] a_{r-m} = 0 .$$

(41) Separate the first term in the series and put it outside the series. This gives

$$[(s+r)(s+r-1)+p_0(s+r)+q_0]a_r+\sum_{m=1}^{r}[p_m(s+r-m)+q_m]a_{r-m}=0.$$

This is a recurrence relation. It gives $a_r$ in terms of $a_{r-m}$.

Call the two roots of the indicial equation $s_1$ and $s_2$. Then the solutions of the diff. eqn. are

$$y_1(x)=\sum_{r=0}^{\infty}a_r(x-x_0)^{r+s_1}$$

$$y_2(x)=\sum_{r=0}^{\infty}a'_r(x-x_0)^{r+s_2}$$

where the $a_r$ and $a'_r$ satisfy the recurrence relation above.

Three different cases can arise, depending on the value of $s_1$ and $s_2$.

1) The general case is when $s_1-s_2$ is neither zero nor an integer. Then the general solution of the diff. eqn is

$$y=y_1(x)+y_2(x)=\sum_{r=0}^{\infty}a_r(x-x_0)^{r+s_1}+\sum_{r=0}^{\infty}a'_r(x-x_0)^{r+s_2}.$$

2) the special case $s_1=s_2$ or $s_1-s_2=0$. Then the general solution is $y=y_1(x)+y_2(x)$ where [" that $\frac{dc^m}{dm}=c^m \ln c$

$$y_1(x)=(x-x_0)^{s_1}\sum_{r=0}^{\infty}a_r(x-x_0)^r$$

and

$$y_2(x)=\left(\frac{\partial y_1}{\partial s}\right)_{s=s_1}=y_1(x)\ln(x-x_0)+(x-x_0)^{s_1}\sum_{r=0}^{\infty}\left(\frac{\partial a_r}{\partial s}\right)_{s=s_1}(x-x_0)^r$$

3) The special case $s_1-s_2$ equals an integer. In this case all the coefficients $a_r$ become either infinite or indeterminate for all $r$ greater than a certain number, say $r=\lambda$. If some of the coefficients $a_r$ become infinite when $s=s_2$, we modify the form of $y$ by replacing $a_0$ by $(s-s_2)$ in the series $y=x^s\sum_{r=0}^{\infty}a_r x^r$ and then evaluate the resultant series $(y)_{s=s_1}$ and $\left(\frac{\partial y}{\partial s}\right)_{s=s_1}$. The general solution is a linear combination of these independent solutions.

If the root $s_1$ makes a coefficient $a_r$ of the series indeterminate, that root determines the general solution of the diff. eqn. The remaining root $s_2$ gives a multiple of one of the series already determined in the general solution.

This method of solving diff. eqns. by integration in series is a powerful one, which can be used in most cases. Certain diff. eqns., when solved by this method, have series solutions which define the special functions. Some of these diff. eqns. are:

Bessel's $\quad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0$

Legendre's $\quad (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\, y = 0$

Laguerre's $\quad x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0$

Hermite's $\quad \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$

Hypergeometric or Gauss's $\quad x(1-x)\frac{d^2y}{dx^2} + \left\{ c - (a+b+1)x \right\}\frac{dy}{dx} - aby = 0$

In this class we'll just consider Bessel's & Legendre's equations.

(43) <u>Bessel's Equation and Bessel Functions</u>

The diff. eqn. is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0 \quad \text{where } n \ge 0.$$

Note that n need not be an integer

Divide through by $\frac{1}{x^2}$ to make the coefficient of $\frac{d^2y}{dx^2} = 1$. Then $P(x)$, the coefficient of $\frac{dy}{dx}$, equals $\frac{1}{x}$ which is infinite at $x=0$. The ~~origin~~ point $x=0$ is a regular singular point of the diff. eqn. It is a <u>regular</u> singular point because neither $p(x) = 1$ or $q(x) = x^2-n^2$ is infinite at $x=0$.

From before, the indicial equation is

$s^2 + (p_0-1)s + q_0 = 0$.

We also had that

$p(x) = \sum_{n=0}^{\infty} p_n(x-x_0)^n$ and $q(x) = \sum_{n=0}^{\infty} q_n(x-x_0)^n$ where now $x_0=0$.

Thus $1 = \sum_{n=0}^{\infty} p_n x^n$ and $x^2-n^2 = \sum_{n=0}^{\infty} q_n x^n$.

Equating coefficients of $x^n$, from the first eqn. we get

$p_0=1$, $p_n=0$ for $n\ge 1$

and from the second equation

$q_0=-n^2$, $q_1=0$, $q_2=1$, $q_n=0$ for $n\ge 3$.

Thus the indicial equation becomes

$s^2+(1-1)s-n^2=0$ or $s^2-n^2=0$

which has roots $s_1=+n$, $s_2=-n$ or $s_1-s_2=2n$.

The first ~~root~~ will give us a solution

$$y_1(x) = \sum_{r=0}^{\infty} a_r x^{r+n}.$$

Now the general recurrence relation we developed earlier was

$$[(s+r)(s+r-1)+p_0(s+r)+q_0]a_r + \sum_{m=1}^{r}[p_m(s+r-m)+q_m]a_{r-m} = 0.$$

For Bessel's eqn this becomes $p_0=1$ all the other $p_m$, $q_1=0$

$$[(n+r)(n+r-1)+n+r-n^2]a_r + 1\,a_{r-2} = 0$$

or $[(n+r)^2-n^2]a_r = -a_{r-2}$. ~~for r≥2~~ (Note that $a_{-k}$ is zero where k is a ~~undefined~~ positive integer)

In particular, with $r=1$,

$[(m+1)^2 - m^2]a_1 = 0$ or $(2m+1)a_1 = 0$.

If $m$ is positive, as we required in Bessel's diff. eqns., ~~then~~ this equation can be satisfied only if $a_1 = 0$.

But we can see from the recurrence relation at the bottom of pg. 43 that if $a_1$ is zero, then all the $a_r$ with odd $r$ are zero. For example, let $r = 3$. Then

$[(m+3)^2 - m^2]a_3 = -a_1 = 0$ or $(6m+9)a_3 = 0$. Therefore $a_3 = 0$.

If we let $r = 0$, we get from the recurrence relation

$(m^2 - m^2)a_0 = 0$ which does not require $a_0$ to equal zero.

Putting $r = 2$, we get

$$[(m+2)^2 - m^2]a_2 = -a_0 \quad \text{or} \quad a_2 = \frac{-a_0}{2(2m+2)}.$$

For $r = 4$, we get

$$a_4 = -\frac{a_2}{4(2m+4)} = \frac{+a_0}{2\cdot 4(2m+2)(2m+4)}.$$

Thus the one particular solution $y_1(x)$ of Bessel's eqn. is

$$y_1(x) = a_0 x^m \left[1 - \frac{x^2}{2(2m+2)} + \frac{x^4}{2\cdot 4(2m+2)(2m+4)} - \cdots\right].$$

When $a_0$ is placed equal to $\frac{1}{2^m m!}$ the solution $y_1(x)$ is called Bessel's function of the first kind of order $m$, and is written $J_m(x)$.

It is

$$J_m(x) = \cancel{\frac{x^m}{2^m m!}} \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{m+2r}}{r!\,\Gamma(m+r+1)}$$

where $\Gamma(m+r+1) = (m+r)!$

(45) The sum of two linearly independent particular solutions will be the general solution of the diff. eqn. We get the second particular solution from the root $s_2 = -n$ of the indicial equation, with $n \geq 0$.

Now, if $n = 0$, then the two roots $s_1$ and $s_2$ will give the same particular solutions, which are not independent. This is a special case.

Also, if $2n$ is an integer we have a special case. To see this, suppose $n = \frac{1}{2}$ and look at the recurrence relation for $a_1$. It is

$$\left[\left(-\tfrac{1}{2}+1\right)^2 - \left(\tfrac{1}{2}\right)^2\right] a_1 = 0$$

which does not require $a_1 = 0$ as for the ~~first~~ root $s = +n$. It is easy to show that, for this case,

$$y_2(x) = a_0 x^{-1/2}\left[1 - \frac{x^2}{2(-1+2)} + \frac{x^4}{2\cdot 4(-1+2)(-1+4)} - \cdots\right]$$
$$+ a_1 x^{-1/2}\left[x - \frac{x^3}{3(-1+3)} + \frac{x^5}{3\cdot 5(-1+3)(-1+5)} - \cdots\right]$$

where $a_0$ and $a_1$ are arbitrary constants. This is a general solution to the Bessel's diff. eqn. In this case $y_1(x)$ is just a simple multiple of the second series and adds nothing to the general solution. Thus,

$$y(x) = A\,J_{1/2}(x) + B\,J_{-1/2}(x)$$ is a general solution.

For the general case, $n \neq 0$ nor $2n$ an integer nor $n$ an integer, the solution

$$y_2(x) = J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$$

and $$y(x) = A\,J_n(x) + B\,J_{-n}(x)$$ is the general solution.

The last special case is for $n$ equal to an integer.

Now $\frac{1}{\Gamma(m)} = 0$ for $m = 0$ or a negative integer. Thus $\frac{1}{\Gamma(-n+r+1)} = 0$ for $r = 0, 1, 2, \ldots (n-1)$

and, for $n$ an integer,

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{-n+2r}}{r!\,\Gamma(-n+r+1)} = \sum_{r=n}^{\infty} (-1)^r \frac{(x/2)^{-n+2r}}{r!\,(r-n)!}$$

Define $r' = r - n$ so that $r = r' + n$. Then

$$J_{-n}(x) = \sum_{r'=0}^{\infty} (-1)^{r'+n} \frac{(x/2)^{n+2r'}}{r'!\,(r'+n)!} = (-1)^n \sum_{r'=0}^{\infty} (-1)^{r'} \frac{(x/2)^{n+2r'}}{r'!\,\Gamma(n+r'+1)}$$

$$= (-1)^n J_n(x).$$

Thus, for $n$ an integer, $J_{-n}(x)$ is not linearly independent of $J_n(x)$. We need a second independent solution for the general solution.

Let us define a function, called Bessel's function of the second kind and written as $Y_n(x)$, by the equation

$$Y_n(x) = \frac{\cos n\pi\, J_n(x) - J_{-n}(x)}{\sin n\pi}.$$

For $n$ not zero or an integer, $Y_n(x)$ is a linear combination of $J_n(x)$ and $J_{-n}(x)$, which are independent solutions of Bessel's eqn. so $Y_n(x)$ is also a solution. Thus, for $n$ not zero or an integer, we can write another general solution as

$$y(x) = A J_n(x) + B\, Y_n(x).$$

For $n$ zero or an integer, the expression for $Y_n(x)$ becomes indeterminate and must be evaluated by L'Hospital's rule. If this is done, there results

$$Y_n(x) = \left(\frac{2}{\pi}\right)\Bigg[\{\ln x - \ln 2 + \gamma\} J_n(x) - \frac{1}{2}\sum_{s=0}^{n-1} \frac{(n-s-1)!}{s!}\left(\frac{x}{2}\right)^{-n+2s}$$

$$- \frac{1}{2}\sum_{s=0}^{\infty} (-1)^s \frac{(x/2)^{n+2s}}{s!\,(s+n)!}\{\phi(s) + \phi(s+n)\}\Bigg]$$

where $\gamma = 0.577216$ (Euler's constant) and $\phi(0) = 0$, $\phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p}$

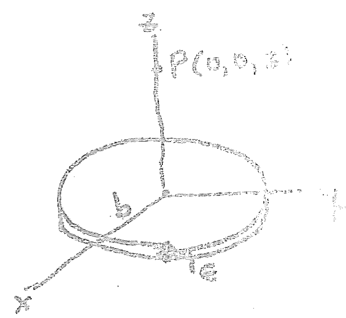
POTENTIAL THEORY (GPH 204)

PROBLEM SET #1

ASSIGNED: SEPT 10, 1971 DUE: SEPT 17,

1. CONSIDER A THIN CIRCULAR CYLINDER (DISC) OF THICKNESS $\epsilon$ AND RADIUS $b$. THE DENSITY $\rho = \epsilon\sigma$ ($\sigma$ IS CALLED THE SURFACE DENSITY AND HAS DIMENSIONS $ML^{-2}$.)

   a. DERIVE EXPRESSIONS FOR THE NEWTONIAN POTENTIAL AND FIELD INTENSITY AT A POINT $P(0,0,z)$ ON THE AXIS OF THE DISC, ASSUMING $\sigma$ IS A CONSTANT.




   b. SET UP EXPRESSIONS, IN THE FORM OF DEFINITE INTEGRALS, FOR THE NEWTONIAN POTENTIAL AND FIELD INTENSITY OF THE DISC AT SOME GENERAL POINT $P(x,y,z)$. WHAT DIFFICULTIES WILL YOU ENCOUNTER IN ATTEMPTING TO CARRY OUT THE INTEGRATION?

2. CONSIDER A SPHERE IN WHICH THE DENSITY $\rho = \rho_0(1-kr)$, WHERE $\rho_0$ AND $k$ ARE CONSTANTS AND $r$ IS DISTANCE FROM THE CENTER OF THE SPHERE.

   a. DERIVE EXPRESSIONS FOR THE NEWTONIAN POTENTIAL AND FIELD INTENSITY AT POINTS OUTSIDE THE SPHERE

   b. DERIVE EXPRESSIONS FOR THE NEWTONIAN POTENTIAL AND FIELD INTENSITY AT POINTS WITHIN THE SPHERE.

   c. DISCUSS THE CONTINUITY OF $U$, $\frac{\partial U}{\partial x}$ AND $\frac{\partial^2 U}{\partial x^2}$ AT THE BOUNDARY OF THE SPHERE.

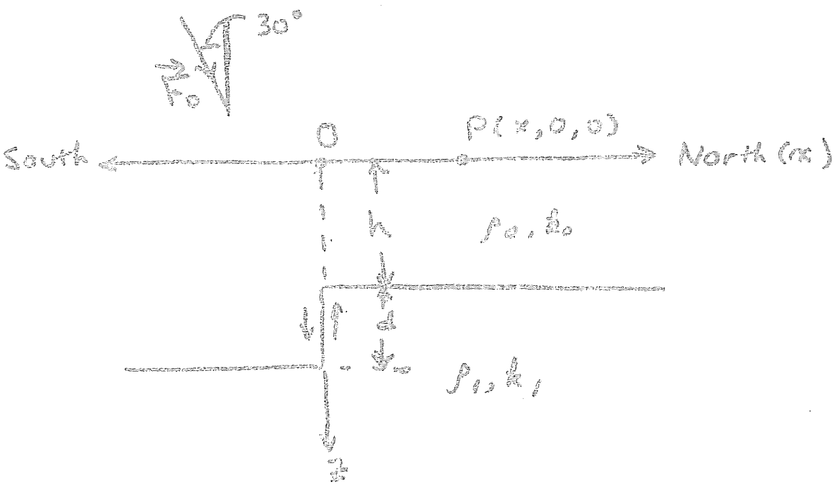
3. a. SHOW THAT $\lim_{R \to \infty} U(P) = 0$, WHERE $R$ IS THE DISTANCE FROM A POINT IN THE BODY TO THE POINT P.

b. SHOW THAT $\lim_{R \to \infty} R \cdot U(P) = GM$, WHERE $G$ IS THE CONSTANT OF GRAVITATION AND $M$ IS THE MASS OF THE BODY.

c. SHOW THAT $\lim_{R \to \infty} \vec{F}(P) = 0$.

IN PROVING THESE RESULTS, USE $U = G\int_V \frac{\rho \, dv}{r}$ AND $\vec{F} = -G \int_V \frac{\rho \, \vec{e}_r \, dv}{r^2}$.

4. A FAULT, WITH VERTICAL DISPLACEMENT $d$ ON A VERTICAL FAULT SURFACE, IS SHOWN SCHEMATICALLY IN THE FIGURE. $\rho_0, k_0$ ARE THE DENSITY AND MAGNETIC SUSCEPTIBILITY IN THE UPPER LAYER. SUBSCRIPTS "1" REFER TO THE LOWER SPACE. THE EARTH'S MAGNETIC FIELD INTENSITY IS INCLINED AT 30° TO THE VERTICAL. $\rho_1 > \rho_0$ AND $k_1 > k_0$.




a. DERIVE AN EXPRESSION FOR THE GRAVITY ANOMALY, $\Delta g$, AT THE POINT $P(x,0,0)$.

b. DERIVE AN EXPRESSION FOR THE VERTICAL COMPONENT MAGNETIC FIELD INTENSITY ANOMALY $\Delta Z$ AT THE POINT $P(x,0,0)$.

POTENTIAL THEORY (GPH 204)

PROBLEMS

ASSIGNED: NOV. 5, 1971

DUE : NOV. 15, 1971

1. SHOW THAT THE 3 POSSIBLE SOLUTIONS OF THE FORM $\phi(r,z,\theta) = R(r)\,Z(z)\,\Theta(\theta)$, WHICH ARE <u>EVERY WHERE</u> FINITE AND SINGLE-VALUED FUNCTIONS OF POSITION, ARE GIVEN BY

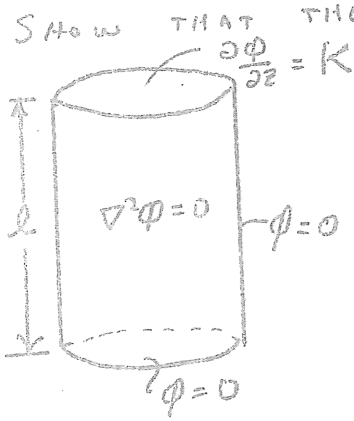
1) $\phi(r,z,\theta) = (Ae^{-pz} + Be^{pz})(C\sin n\theta + D\cos n\theta)\,J_n(pr)$

2) $\phi(r,z,\theta) = (A\sin pz + B\cos pz)(C\sin n\theta + D\cos n\theta)\,I_n(pr)$

3) $\phi(r,z,\theta) = (Az + B)(C\sin n\theta + D\cos n\theta)\,r^n$,

where $n$ is zero or a positive integer.

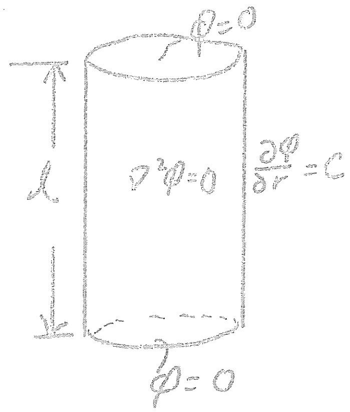
2. SHOW THAT THE SOLUTION TO THE BOUNDARY VALUE PROBLEM IS




$$\phi(r,z) = \frac{2K}{\ell a}\sum_{i=1}^{\infty}\frac{\sinh(p_i z)\,J_0(p_i r)}{p_i^2\cosh(p_i \ell)\,J_1(p_i a)}$$

where the $p_i$ are the roots of $J_0(pa) = 0$ and $a$ is the radius of the cylinder.

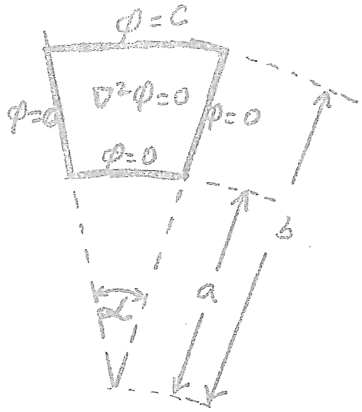
3. SHOW THAT THE SOLUTION TO THE BOUNDARY VALUE PROBLEM IS




$$\phi(r,z) = \frac{2Cl}{\pi^2}\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n^2 I_1\left(\frac{n\pi a}{l}\right)}\sin\left(\frac{n\pi}{l}z\right) I_0\left(\frac{n\pi}{l}r\right)$$

where $a$ is the radius of the cylinder.

4. LET $r=a$ BE THE INNER RADIUS AND $r=b$ BE THE OUTER RADIUS OF A PLANE AREA IN WHICH $\nabla^2\phi = 0$. THE POTENTIAL $\phi(r,\theta)$ IS ZERO ON THE 3 SIDES OF THE PLANE AREA, AND EQUALS THE CONSTANT $C$ ON $r=b$. SHOW THAT



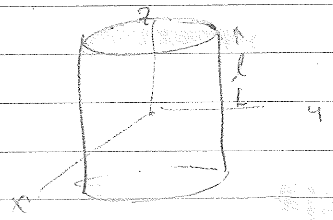

$$\phi(r,\theta) = \sum_{n=1}^{\infty} A_n\left[-a^{-2n\pi/\alpha}\, r^{n\pi/\alpha} + r^{-n\pi/\alpha}\right]\sin\left(\frac{n\pi\theta}{\alpha}\right)$$

where

$$A_n = \frac{2C}{n\pi}\,\frac{(1-\cos n\pi)}{\left[b^{-n\pi/\alpha} - a^{-2n\pi/\alpha}\, b^{n\pi/\alpha}\right]}.$$

Problem — Exterior Dirichlet

and $\varphi \neq \varphi(\theta)$




$\nabla^2 \varphi = 0$ ~~for $r > a$, $|z| < l$~~ outside the cylinder

$\varphi(a, z) = f(r)$ for $|z| < l$

$\varphi(r, \pm l) = 0$ for $r < a$

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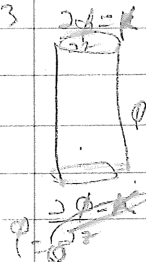


$$\varphi^{(r,z)} = \sum_{i=1}^{\infty} A_i \sinh p_i z \, J_0(p_i r)$$

where $p_i$ are the roots of $J_0(p a) = 0$

$$\frac{\partial \varphi}{\partial z} = \sum p_i A_i \cosh p_i z \, J_0(p_i r)$$

$$\left(\frac{\partial \varphi}{\partial z}\right)_{z=l} = K = \sum p_i A_i \cosh p_i l \, J_0(p_i r)$$

$$\int_0^a K r J_0(p_i r) \, dr = p_i A_i \cosh p_i l \left\{\frac{a^2}{2}[J_1(p_i a)]^2\right\}$$

and $$A_i = \frac{\frac{2}{p_i^2} K \int_0^a p_i r \, J_0(p_i r) \, d(p_i r)}{l a^2 p_i \cosh p_i l \, \{J_1(p_i a)\}^2}$$

$\int (p_i r) J_0(p_i r) \, d(p_i r) = (p_i r) J_1(p_i r)$

$$= \frac{2K}{p_i^3} \frac{J_1(p_i a) \cdot (p_i a)}{l a^2 \cosh p_i l \, \{J_1(p_i a)\}^2}$$

$$= \frac{2 K a J_1(p_i a)}{p_i^2 l a^2 \cosh p_i l \, \{J_1(p_i a)\}^2}$$

$$= \frac{2K}{l a p_i^2 \cosh p_i l \, J_1(p_i a)}$$

and $$\varphi(r, z) = \frac{2K}{l a} \sum_{i=1}^{\infty} \frac{\sinh(p_i z) J_0(p_i r)}{p_i^2 \cosh(p_i l) J_1(p_i a)}$$

(1) Problem Show that the three possible solutions of the form $\phi = Z^{(z)} R^{(r)} \Theta^{(\theta)}$ which are everywhere finite and single-valued functions of position, are given by

1) $\phi(r,z,\theta) = (Ae^{-pz} + Be^{pz})(C\sin n\theta + D\cos n\theta)\, J_n(pr)$

2) $\phi(r,z,\theta) = (A\sin pz + B\cos pz)(C\sin n\theta + D\cos n\theta)\, I_n(pr)$

3) $\phi(r,z,\theta) = (Az + B)(C\sin n\theta + D\cos n\theta)\, r^n$

where $n$ is a positive integer or zero.

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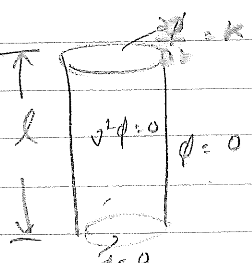
See also Bowman, pg 46

(2) Show that the solution to the boundary value problem is

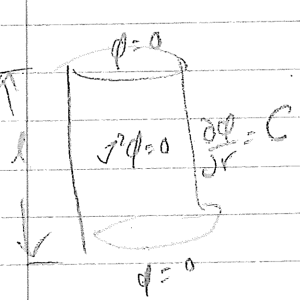



$$\phi(r,z) = \frac{2K}{la} \sum_{i=1}^{\infty} \frac{\sinh(p_i z)\, J_0(p_i r)}{p_i^2 \cosh(p_i l)\, J_1(p_i a)}$$

where the $p_i$ are the roots of $J_0(pa) = 0$.

and $a$ is the radius of the cylinder.

(3) Show that the solution to the boundary value problem is




$$\phi(r,z) = \frac{2Cl}{\pi^2} \sum_{n=1}^{\infty} \frac{(1-\cos n\pi)}{n^2 I_1\left(\frac{n\pi a}{l}\right)} \sin\left(\frac{n\pi}{l} z\right) I_0\left(\frac{n\pi}{l} r\right)$$

where $a$ is the radius of the cylinder.

1. Show that solutions to Laplace's equation in plane polar coordinates are $\varphi = \left.\begin{matrix}1\\ \theta\end{matrix}\right\} \left.\begin{matrix}1\\ \ln r\end{matrix}\right\} + \left.\begin{matrix}\sin n\theta\\ \cos n\theta\end{matrix}\right\} \left\{\begin{matrix}r^{n}\\ r^{-n}\end{matrix}\right.$

2. Given $\nabla^2\varphi = 0$ in the region outside the circle $r = b$.

1) $\varphi = 0$ on $r = b$ for $0 \le \theta < 2\pi$

2) $\lim_{r\to\infty} \nabla\varphi = \cancel{B}\, X_0 \vec{e}_x$ or $\lim_{r\to\infty}\varphi = X_0 r\cos\theta$

Show that, for $r \geq b$, $\varphi = X_0\left(1 - \frac{b^2}{r^2}\right) r\cos\theta$

3. Given $\nabla^2\varphi = 0$ in the region outside the circle $r = b$.

1) $\frac{\partial\varphi}{\partial r} = 0$ on $r = b$ for $0 \le \theta < 2\pi$

2) $\lim_{r\to\infty} \nabla\varphi = X_0 \vec{e}_x$

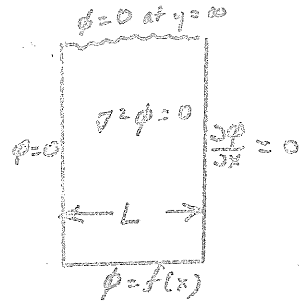
Show that, for $r \geq b$, $\varphi = X_0\left(1 + \frac{b^2}{r^2}\right) r\cos\theta$ .

POTENTIAL THEORY (GPH 204)
MID-SEMESTER EXAM

OCT. 22, 1971 B. NUTTLI

1. CONSIDER PLANE POLAR COORDINATES $x = r\cos\theta$, $y = r\sin\theta$.
   a. DERIVE AN EXPRESSION FOR LAPLACE'S EQUATION IN THIS COORDINATE SYSTEM.
   b. SEPARATE LAPLACE'S EQUATION INTO ORDINARY DIFFERENTIAL EQUATIONS.

2. SOLVE THE BOUNDARY VALUE PROBLEM:




$\nabla^2\phi = 0$ FOR $0<x<L$, $0<y<\infty$

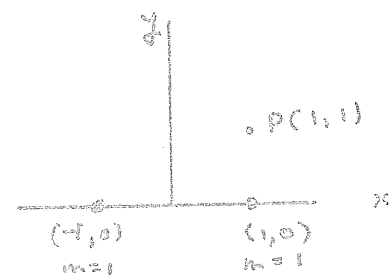
$\phi(0,y) = 0$ FOR $0<y<\infty$,

$\frac{\partial\phi}{\partial x}(L,y) = 0$ FOR $0<y<\infty$,

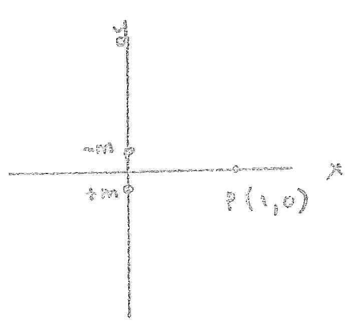
$\phi(x,0) = f(x)$ FOR $0<x<L$,

$\lim_{y\to\infty} \phi(x,y) = 0$ FOR $0<x<L$.

3. a. UNIT POSITIVE MASSES ARE LOCATED AT THE POINTS (1,0) AND (−1,0).
   DETERMINE THE VALUES OF THE NEWTONIAN POTENTIAL AND FIELD INTENSITY AT THE POINT P(1,1.)

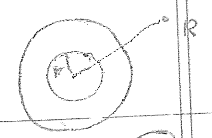



   b. AN INFINITESIMAL DIPOLE HAS ITS AXIS ON THE Y-AXIS, AS SHOWN IN THE FIGURE.




   DETERMINE THE VALUES OF THE MAGNETIC POTENTIAL AND FIELD INTENSITY AT THE POINT P(1,0).

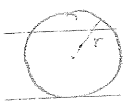
$\rho = \rho_0 (1 - Kr)$



$U_{shell}(\text{outside}) = \frac{4\pi G \sigma r^2}{R}$ where $\sigma = \rho\, dr$

(a) $U(\text{outside}) = \frac{4\pi G}{R} \int_0^b \rho_0 (1 - Kr)\, r^2 dr = \frac{4\pi G \rho_0}{R} \left[ \frac{b^3}{3} - \frac{K b^4}{4} \right]$

$\vec{F}(\text{outside}) = \nabla U = -\frac{4\pi G \rho_0\, b^3}{R^2} \left( \frac{1}{3} - \frac{Kb}{4} \right) \vec{e_r}$



(b) $U_{shell}(\text{inside}) = 4\pi G \sigma r$

$U(\text{inside}) = \frac{4\pi G \rho_0}{\cancel{R}} \left( \frac{r^3}{3} - \frac{K r^4}{4} \right) + \int_r^b 4\pi G r \rho_0 (1 - Kr)\, dr$

$= \cancel{\frac{4\pi G \rho_0}{R}}$

$= 4\pi G \rho_0 r^2 \left( \frac{1}{3} - \frac{Kr}{4} \right) + 4\pi G \rho_0 \left[ \left( \frac{b^2 - r^2}{2} \right) - K \left( \frac{b^3 - r^3}{3} \right) \right]$

$\vec{F}(\text{inside}) = 4\pi G \rho_0 \left[ \frac{2}{3} r \vec{e_r} - \frac{3}{4} K r^2 \vec{e_r} \right] + 4\pi G \rho_0 \left[ -r \vec{e_r} + K r^2 \vec{e_r} \right]$

$= 4\pi G \rho_0 \left[ -\frac{1}{3} r \vec{e_r} + \frac{1}{4} K r^2 \vec{e_r} \right]$

(c) $\left[ U(\text{outside}) \right]_{r=b} = 4\pi G \rho_0 b^2 \left[ \frac{1}{3} - \frac{Kb}{4} \right]$ ✓

$\left[ U(\text{inside}) \right]_{r=b} = 4\pi G \rho_0 b^2 \left[ \frac{1}{3} - \frac{Kb}{4} \right]$

$\left[ \frac{\partial U}{\partial x}(\text{outside}) \right]_{r=b} = -4\pi G \rho_0 x \left( \frac{1}{3} - \frac{Kb}{4} \right)$ ✓

$\left[ \frac{\partial U}{\partial x}(\text{inside}) \right]_{r=b} = 4\pi G \rho_0 \left[ \frac{2}{3} x - \frac{3}{4} K b x - x + K b x \right]$

$\left[ \frac{\partial^2 U}{\partial x^2}(\text{outside}) \right]_{r=b} = \left( -\frac{1}{b^3} + \frac{3x^2}{b^5} \right) 4\pi G \rho_0 \left[ \frac{b^3}{3} - \frac{K b^4}{4} \right]$

$= 4\pi G \rho_0 \left( -\frac{1}{3} + \frac{Kb}{4} + \frac{x^2}{b} - \frac{3}{4} \frac{x^2 K}{b} \right)$

$\left[ \frac{\partial^2 U}{\partial x^2}(\text{inside}) \right]_{r=b} = 4\pi G \rho_0 \left( -\frac{1}{3} + Kb + \frac{K x^2}{b} \right)$

(3) $U_{inside} = 4\pi G \rho_0 \left[ \frac{r^2}{3} - \frac{r^2}{2} + \frac{b^2}{2} + K \left( \frac{-r^3}{4} + \frac{r^3}{3} - \frac{b^3}{3} \right) \right]$

$= 4\pi G \rho_0 \left[ -\frac{r^2}{6} + \frac{b^2}{2} + K \left( \frac{r^3}{12} - \frac{b^3}{3} \right) \right]$

Pg 130, Prob. 3

Show that the solution of the equation $\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial y^2}$

which (3) tends to zero as $y \to \infty$ and which satisfies the conditions

(1) $z = f(x)$ when $y = 0$, $x > 0$

(2) $z = 0$ when $y > 0$, $x = 0$

may be written in the form $\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(\xi) e^{\xi x - y\sqrt{\xi}}\, d\xi$,

Evaluate this integral when $f(x)$ is a constant $k$.

---

Solution: Use the Laplace transform, giving

$$\bar{z}(x,y) = \int_0^\infty e^{-\xi x} z(x,y)\, dx .$$

Then

$$\int_0^\infty e^{-\xi x} \frac{\partial z}{\partial x}\, d\xi = \left[z e^{-\xi x}\right]_0^\infty + \xi \int_0^\infty z e^{-\xi x}\, dx = 0 - \overset{\overset{0 \text{ from condition (2)}}{=}}{z} + \xi\int_0^\infty z e^{-\xi x}\, dx = \xi \bar{z}(x,y)$$

Also

$$\int_0^\infty \frac{\partial^2 z}{\partial y^2} e^{-\xi x}\, dx = \frac{\partial^2 \bar{z}}{\partial y^2} .$$

Thus

$$\int_0^\infty \left[\frac{\partial z}{\partial x} - \frac{\partial^2 z}{\partial y^2}\right] e^{-\xi x}\, dx = \xi \bar{z} - \frac{d^2 \bar{z}}{dy^2} = 0 .$$

This can be written as the diff. equn. $(D^2 - \xi)\bar{z} = 0$

which has solutions $D = \pm\sqrt{\xi}$ or $\bar{z} = c_1 e^{\sqrt{\xi} y} + c_2 e^{-\sqrt{\xi} y} = c_2 e^{-\sqrt{\xi} y}$ by bound. cond. (3).

At $y = 0$, $\bar{z}(0) = c_2$. But $\bar{z}(0) = \int_0^\infty e^{-\xi x} f(x)\, dx = \bar{f}(\xi)$ ~~. Also, at $y = 0$, we know that $\frac{\partial z}{\partial y} = 0 = \sqrt{\xi} c_1 - \sqrt{\xi} c_2$, or $c_1 = c_2$~~.

Therefore $\bar{z} = \bar{f}(\xi) e^{-\sqrt{\xi} y}$. Therefore,

by the inversion theorem for Laplace transforms,

$$z = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{(\xi x - \sqrt{\xi} y)} \bar{f}(\xi)\, d\xi \qquad \text{Q.E.D.}$$

Next let $f(x) = k$. Then

$$\bar{f}(\xi) = \int_0^\infty k e^{-\xi x}\, dx = \left[\frac{-k e^{-\xi x}}{\xi}\right]_{x=0}^\infty = -\frac{k}{\xi}(0 - 1) = \frac{k}{\xi} .$$

Therefore

$$z = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(\xi) e^{\xi x - y\sqrt{\xi}}\, d\xi = \frac{k}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{1}{\xi} e^{\xi x - y\sqrt{\xi}}\, d\xi$$

Evaluate this last integral by contour integration. Let $\xi = \beta + i\eta$

(over)

Prob. 3, pg 130 (cont'd)

For the integral around L, let $r_0$ be the radius of the circular arc, and let $r_0 \to 0$

On the circle of radius $r_0$,

$$\zeta = r_0 e^{i\theta} \text{ and } \sqrt{\zeta} = \sqrt{r_0}\, e^{i\theta/2}, \quad d\zeta = i r_0 e^{i\theta} d\theta$$

$\therefore$ Thus

$$\int_L = \int_{\pi}^{-\pi} \frac{1}{r_0} e^{-i\theta} e^{x\zeta - y\sqrt{\zeta}}\, i r_0 e^{i\theta} d\theta$$

$$= \int_{\pi}^{-\pi} \frac{i}{r_0} e^{-i\theta} e^{x r_0(\cos\theta + i\sin\theta) - y\sqrt{r_0}\,(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})}\, r_0 e^{i\theta} d\theta$$

$$\lim_{r_0 \to 0} \int_L = \int_{\pi}^{-\pi} i\, d\theta = -2\pi i$$

Thus $\displaystyle\int_{C'} \frac{1}{\zeta} e^{x\zeta - y\sqrt{\zeta}} d\zeta = 0 = -2\pi i + 4i \int_0^{\infty} e^{-\left(\frac{x u^2}{y^2}\right)} \cdot \frac{\sin u}{u} \cdot du$

and

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{1}{\zeta} e^{x\zeta - y\sqrt{\zeta}} d\zeta = 2\pi i - 4i \int_0^{\infty} e^{-\left(\frac{x u^2}{y^2}\right)} \cdot \frac{\sin u}{u} du$$

$$= 2\pi i \left(1 - \frac{2}{\pi} \int_0^{\infty} e^{-\left(\frac{x u^2}{y^2}\right)} \cdot \frac{\sin u}{u} du\right)$$

Now from tables of integrals:

$$\int_0^{\infty} e^{-\left(\frac{x u^2}{y^2}\right)} \cos \alpha u \, du = \frac{1}{2}\sqrt{\frac{\pi y^2}{x}}\, e^{-\frac{\alpha^2 y^2}{4x}}$$

Integrate both sides of this equation with respect to $\alpha$, from 0 to 1. Then

$$\int_0^{\infty}\int_0^1 e^{-\left(\frac{x u^2}{y^2}\right)} \cos \alpha u \, d\alpha\, du = \int_0^{\infty} \frac{e^{-\frac{x u^2}{y^2}}}{u} \sin \alpha u\, du$$

$$\int_0^1 \frac{1}{2}\sqrt{\frac{\pi y^2}{x}}\, e^{-\frac{\alpha^2 y^2}{4x}} d\alpha = \int_0^{y/2\sqrt{x}} \frac{1}{2}\sqrt{\frac{\pi y^2}{x}}\, e^{-\lambda^2} \frac{2\sqrt{x}}{y} d\lambda$$

where
$\frac{\alpha^2 y^2}{4x} = \lambda^2$
$\alpha = \frac{2\lambda\sqrt{x}}{y}$
$d\alpha = \frac{2\sqrt{x}}{y} d\lambda$

$$= \sqrt{\pi} \int_0^{y/2\sqrt{x}} e^{-\lambda^2} d\lambda$$

Therefore $\displaystyle \frac{2}{\pi} \int_0^{\infty} e^{-\left(\frac{x u^2}{y^2}\right)} \frac{\sin u}{u} du = \frac{2}{\sqrt{\pi}} \int_0^{\frac{y}{2\sqrt{x}}} e^{-\lambda^2} d\lambda = \operatorname{erf}\left(\frac{y}{2\sqrt{x}}\right)$

Finally,

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{1}{\zeta} e^{x\zeta - y\sqrt{\zeta}} d\zeta = 2\pi i \left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{x}}\right)\right].$$

Prob 5 Pg 120

The variation of the function $z$ over the $xy$ plane and for $t \geq 0$ is determined by the equation

$$\nabla_1^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}.$$

If, when $t = 0$, $z = f(x,y)$ and $\frac{\partial z}{\partial t} = 0$, show that, at any subsequent time,

$$z(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi,\eta)\cos(ct\sqrt{\xi^2+\eta^2})\, e^{-i(\xi x+\eta y)}\, d\xi\, d\eta$$

where $F(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i(\xi x+\eta y)}\, dx\, dy$.

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Solution:

Use the Fourier transform. Let

$$\bar{Z}(\xi,\eta,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z(x,y,t)\, e^{i\xi x}\, e^{i\eta y}\, dx\, dy = \frac{1}{2\pi}\iint_{-\infty}^{\infty} z(x,y,t)\, e^{i(\xi x+\eta y)}\, dx\, dy$$

Then,

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{\partial^2 z}{\partial t^2}\, e^{i(\xi x+\eta y)}\, dx\, dy = \frac{d^2\bar{Z}}{dt^2}$$

and

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty}\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}\right] e^{i(\xi x+\eta y)}\, dx\, dy = \frac{1}{c^2}\frac{d^2\bar{Z}}{dt^2} + (\xi^2+\eta^2)\bar{Z} = 0.$$

since

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty} z(x,y,t)\, e^{i(\xi x+\eta y)}\, dx\, dy = \frac{-1}{2\pi}(\xi^2+\eta^2)\bar{Z} \quad \text{[see attached sheet]}$$

Thus we must solve the ordinary diff. equa.

$$[D^2 + c^2(\xi^2+\eta^2)]\bar{Z} = 0.$$

The solutions are:

$$D = \pm ic\sqrt{\xi^2+\eta^2}$$

or

$$\bar{Z} = K_1 e^{ic\sqrt{\xi^2+\eta^2}\,t} + K_2 e^{-ic\sqrt{\xi^2+\eta^2}\,t}$$

to here

From the boundary conditions,

$$\bar{z}(\xi,\eta,0) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} f(x,y)\, e^{i(\xi x+\eta y)}\, dx\, dy = K_1 + K_2.$$

$$\frac{\partial\bar{z}}{\partial t}(\xi,\eta,0) = (K_1 - K_2)\, ic\sqrt{\xi^2+\eta^2} = \frac{1}{2\pi}\iint_{-\infty}^{\infty} 0\cdot e^{i(\xi x+\eta y)}\, dx\, dy = 0 \quad \text{or} \quad K_1 = K_2.$$

Call $K_1 = K_2 = K$. Then

$$\bar{z}(\xi,\eta,0) = 2K = F(\xi,\eta) \quad \text{where} \quad F(\xi,\eta) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} f(x,y)\, e^{i(\xi x+\eta y)}\, d\xi\, d\eta.$$

and

$$\bar{z}(\xi,\eta,t) = \frac{1}{2}\cdot F(\xi,\eta)\left[e^{ic\sqrt{\xi^2+\eta^2}\,t} + e^{-ic\sqrt{\xi^2+\eta^2}\,t}\right] = F(\xi,\eta)\cos(ct\sqrt{\xi^2+\eta^2})$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2 z}{\partial x^2} e^{i(\xi x+\eta y)}\,dx\,dy = \int_{-\infty}^{\infty} e^{i\eta y}\left\{\left[e^{i\xi x}\frac{\partial z}{\partial x}\right]_{-\infty}^{\infty} - i\xi\int_{-\infty}^{\infty}\frac{\partial z}{\partial x} e^{i\xi x}\,dx\right\}dy$$

$$= \int_{-\infty}^{\infty} e^{i\eta y}\left\{\left[e^{i\xi x}\frac{\partial z}{\partial x} - i\xi z e^{i\xi x}\right]_{-\infty}^{\infty} - \xi^2\int_{-\infty}^{\infty} z e^{i\xi x}\,dx\right\}dy$$

$$= -\xi^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\, e^{i(\xi x+\eta y)}\,dx\,dy + \int_{-\infty}^{\infty}\left[e^{i\xi x}\frac{\partial z}{\partial x} - i\xi z e^{i\xi x}\right]_{-\infty}^{\infty} e^{i\eta y}\,dy$$

Similarly,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2 z}{\partial y^2} e^{i(\xi x+\eta y)}\,dx\,dy = -\eta^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\, e^{i(\xi x+\eta y)}\,dx\,dy + \int_{-\infty}^{\infty}\left[e^{i\eta y}\frac{\partial z}{\partial y} - i\eta z e^{i\eta y}\right]_{-\infty}^{\infty} e^{i\xi x}\,dx$$

Thus,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) e^{i(\xi x+\eta y)}\,dx\,dy = -(\xi^2+\eta^2)\,\bar{Z}(\xi,\eta)$$

if we assume that $z$, $\frac{\partial z}{\partial x}$, and $\frac{\partial z}{\partial y}$ are zero at $\pm\infty$.

Prob. 1, Pg. 130

The temperature $\theta$ in the semi-infinite solid $0 \le x < \infty$ is determined by the diff. eqn.

$$\frac{\partial \theta}{\partial t} = K \frac{\partial^2 \theta}{\partial x^2}$$

and the conditions

(1) $\theta = 0$ when $t = 0$, $x \geq 0$

(2) $\theta = \theta_0$ = const. when $x = 0$ and $t > 0$.

Making use of the sine transform, show that

$$\theta(x,t) = \frac{2}{\pi} \int_0^\infty \frac{\sin(\xi x)}{\xi} \left(1 - e^{-K\xi^2 t}\right) d\xi .$$

Problem for class

Now the transform

$$\bar{\theta}_1(\xi, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \theta(x,t) \sin(\xi x)\, dx$$

and

$$K \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 \theta}{\partial x^2} \sin(\xi x)\, dx = K \left\{ \left[ \frac{\partial \theta}{\partial x} \sqrt{\frac{2}{\pi}} \sin \xi x - \theta \sqrt{\frac{2}{\pi}}\, \xi \cos \xi x \right]_0^\infty \right.$$

$$\left. + \int_0^\infty \theta \left\{ -\sqrt{\frac{2}{\pi}}\, \xi^2 \sin \xi x \right\} dx \right\}$$

$$= +K \left\{ \sqrt{\frac{2}{\pi}}\, \xi\, \theta_0 - \xi^2 \bar{\theta} \right\}$$

Now

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \frac{\partial \theta}{\partial t} \sin(\xi x)\, dx = \frac{d\bar{\theta}_1}{dt} .$$

Thus

$$\int_0^\infty \left[ K \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial \theta}{\partial t} \right] \sqrt{\frac{2}{\pi}} \sin(\xi x)\, dx = +K \xi^2 \bar{\theta} + K \sqrt{\frac{2}{\pi}}\, \xi\, \theta_0 - \frac{d\bar{\theta}}{dt} = 0$$

$$\left(D + K\xi^2\right) \bar{\theta} = +K \sqrt{\frac{2}{\pi}}\, \xi\, \theta_0$$

$$\bar{\theta} = Ce^{-K\xi^2 t} + \text{a particular integral}$$

assume $\bar{\theta} = A$ (constant) Then $K\xi^2 A = -K\sqrt{\frac{2}{\pi}}\, \xi\, \theta_0$ or $A = -\sqrt{\frac{2}{\pi}} \frac{\theta_0}{\xi}$

Thus

$$\bar{\theta} = +\sqrt{\frac{2}{\pi}} \frac{\theta_0}{\xi} + Ce^{-K\xi^2 t}$$

Now when $t = 0$, $\theta = 0$ and $\bar{\theta}(0) = 0$ and $C = -\sqrt{\frac{2}{\pi}} \frac{\theta_0}{\xi}$.

therefore

$$\bar{\theta} = \sqrt{\frac{2}{\pi}} \frac{\theta_0}{\xi} - \sqrt{\frac{2}{\pi}} \frac{\theta_0}{\xi} e^{-K\xi^2 t} = \sqrt{\frac{2}{\pi}} \frac{\theta_0}{\xi} \left(1 - e^{-K\xi^2 t}\right)$$

and

$$\theta = \theta_0 \cdot \frac{2}{\pi} \int_0^\infty \frac{\sin \xi x}{\xi} \left(1 - e^{-K\xi^2 t}\right) d\xi$$

#3-1-55 (b)

| | |
|---|---|
| Cape Girardeau, Saint Louis University | $\Delta = 38.1$ |
| Cincinnati, Xavier University | $\Delta = 38.5$ |
| Weston, Weston College | $\Delta = 41.0$ |
| Georgetown, Georgetown University | $\Delta = 41.6$ |
| New Orleans, Loyola University | $\Delta = 44.5$ |
| Spring Hill, Spring Hill College | $\Delta = 44.5$ |


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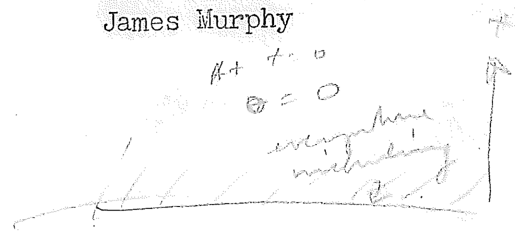



Let $\theta = X(x)\,T(t)$

$$XT' = KTX''$$

$$\frac{T'}{T} = \frac{K X''}{X} = -c^2$$

$D^2 + $ [illegible]

$(D + 2i)(D - 2i)$

$D^2 + 0$

$$\left(D^2 + \frac{c^2}{K}\right)X = 0, \quad D = \pm i\frac{c}{\sqrt{K}}$$

$$X(x) = \left[A e^{i\frac{c}{\sqrt{K}}x} + B e^{-i\frac{c}{\sqrt{K}}x}\right] \text{ or } = A\cos\left(\frac{c}{\sqrt{K}}x + \epsilon\right)$$

$$\frac{T'}{T} = -c^2, \quad T' + c^2 T = 0, \quad (D + c^2)T = 0, \quad D = -c^2$$

$$T(t) = a e^{-c^2 t} \cancel{+ \text{c particular soln}}$$

Initial conditions:

$\theta = 0$ for $t = 0,\ x > 0$

$\theta = \theta_0$ for $x = 0,\ t > 0$

$$\theta = F e^{-c^2 t}\cos\left(\frac{c}{\sqrt{K}}x + \epsilon\right)$$

$$\theta(x,0) = 0 = F\cos\left(\frac{c}{\sqrt{K}}x + \epsilon\right)$$

$$\theta(x,0) = F\left[A\left\{\cos\frac{c}{\sqrt{K}}x + i\sin\frac{c}{\sqrt{K}}x\right\} + B\left\{\cos\frac{c}{\sqrt{K}}x - i\sin\frac{c}{\sqrt{K}}x\right\}\right]$$

Example 11, pg 128

Derive the solution of the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{for } r \geq 0,\ z \geq 0$$

satisfying the conditions (i) $V \to 0$ as $z \to \infty$ and as $r \to \infty$
(ii) $V = f(r)$ on $z = 0$, $r \geq 0$

Solution:
Introduce the Hankel transform $\bar{V} = \int_0^\infty r\,V(r,z)\,J_0(\xi r)\,dr$

In eq (1) pg 126, we have for our problem: $a = 1$, $b = \frac{1}{r}$, $c = 0$, $L = \frac{\partial^2}{\partial z^2}$, $f = 0$.

Therefore, from the equation between (2) and (3) on pg 126, (see red check)

$$\int_0^\infty \left\{\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}\right\} r\,J_0(\xi r)\,dr = g(\xi,z) + \int_0^\infty V\left\{\frac{\partial^2}{\partial r^2}\left(r J_0(\xi r)\right) - \frac{\partial}{\partial r}\left(J_0(\xi r)\right)\right\} dr$$

where $g(\xi,z) = \left[\frac{\partial V}{\partial r} r J_0(\xi r) + V\left\{J_0(\xi r) - \frac{\partial}{\partial r}\left(r J_0(\xi r)\right)\right\}\right]_0^\infty$

$\frac{\partial}{\partial r} J_0(\xi r) = -\xi J_1(\xi r)$

Thus

$$g(\xi,z) = \left[r\frac{\partial V}{\partial r} J_0(\xi r) + V\left\{J_0(\xi r) - J_0(\xi r) + \xi r J_1(\xi r)\right\}\right]_0^\infty$$

$$= \left[r\frac{\partial V}{\partial r} J_0(\xi r) + \xi r V J_1(\xi r)\right]_0^\infty$$

$= 0$ if we can show that $\lim_{r\to\infty}\left[r\frac{\partial V}{\partial r} J_0(\xi r)\right] = 0$ and $\lim_{r\to 0}\left[\xi r V J_1(\xi r)\right] = 0$

$(L + \lambda)\,\bar{V}(\xi, z) = 0$

also

$$\int_0^\infty V\left\{\frac{\partial^2}{\partial r^2}\left(r J_0(\xi r)\right) - \frac{\partial}{\partial r}\left(\overbrace{r J_0(\xi r)}^{K}\frac{1}{r}\right)\right\} dr = \int_0^\infty V \frac{\partial}{\partial r}\left\{J_0(\xi r) - r\xi J_1(\xi r)\right\} + \xi J_1(\xi r)\Big\} dr$$

The term in brackets is $\Delta K$.

$$= \int_0^\infty V\left\{-\xi J_1(\xi r) + \xi J_1(\xi r) - \frac{\partial}{\partial r}\left[r\xi J_1(\xi r)\right]\right\} dr$$

$$= \int_0^\infty -V r \xi^2 J_0(\xi r)\,dr = -\xi^2 \bar{V}.$$

Thus

$$0 = \int_0^\infty \left[\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2}\right] r J_0(\xi r)\,dr = -\xi^2 \bar{V} + \int_0^\infty \frac{\partial^2}{\partial z^2}\left[V r J_0(\xi r)\right] dr$$

$$= -\xi^2 \bar{V} + \frac{d^2 \bar{V}}{dz^2} \quad \text{since } \bar{V} \equiv \int_0^\infty V r J_0(\xi r)\,dr.$$

Solutions of $\frac{d^2\bar{V}}{dz^2} - \xi^2 \bar{V} = 0$ are

$$\bar{V} = C_1 e^{\xi z} + C_2 e^{-\xi z}$$

From the boundary conditions, $V$ and consequently $\bar{V} \to 0$ as $z \to \infty$. Therefore $C_1 = 0$. The second boundary condition gives:

Example 12, Pg 129

Determine the solution of $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial y^2} = 0$, $-\infty < x < \infty$, $y \geq 0$

satisfying the conditions

(1) $z$ and its partial derivatives tend to zero as $x \to \pm\infty$

(2) $z = f(x)$, $\partial z/\partial y = 0$ on $y = 0$.

---

Take $Z(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z(x,y)\, e^{i\xi x}\, dx$ [ $Z$ is the Fourier transform of $z$ ]

This diff equn is of higher order than the second. It is a fourth order equn. Therefore consider

$$\int_{-\infty}^{\infty} \frac{\partial^4 z}{\partial x^4} \cdot \frac{1}{\sqrt{2\pi}} e^{i\xi x}\, dx = I.$$

and integrate by parts.

$$I = \frac{1}{\sqrt{2\pi}} \left\{ \left[ e^{i\xi x} \frac{\partial^3 z}{\partial x^3} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} i\xi\, e^{i\xi x} \frac{\partial^3 z}{\partial x^3}\, dx \right\} = -\frac{i\xi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \frac{\partial^3 z}{\partial x^3}\, dx$$

Integrating by parts 2 more times,

$$I = \frac{(-i\xi)^3}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \frac{\partial z}{\partial x}\, dx = \frac{(-i\xi)^3}{\sqrt{2\pi}} \left\{ \left[ e^{i\xi x} z \right]_{-\infty}^{\infty} - i\xi \int_{-\infty}^{\infty} z\, e^{i\xi x}\, dx \right\}$$

$$= \frac{(-i\xi)^4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{i\xi x}\, dx = \xi^4 Z.$$

Also

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 z}{\partial y^2} e^{i\xi x}\, dx = \frac{d^2 Z}{dy^2}.$$

Therefore

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \left[ \frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial y^2} \right] dx = \frac{d^2 Z}{dy^2} + \xi^4 Z = 0.$$

$$(D^2 + \xi^4) Z = 0, \quad D = \pm i \xi^2$$

and $Z = c_1 e^{i\xi^2 y} + c_2 e^{-i\xi^2 y}$.

$Z(y=0) = c_1 + c_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\xi x}\, dx = F(\xi)$ from boundary condition (2) where $F(\xi)$ is the Fourier transform of $f(x)$.

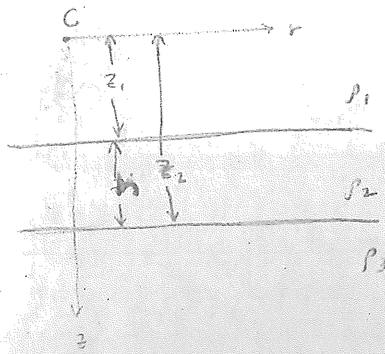
$\frac{\partial Z}{\partial y}(y=0) = i\xi^2 c_1 - i\xi^2 c_2 = 0$, from bdy. condition (2). Then $c_1 = c_2$.

and $F(\xi) = 2c_1$.

Therefore

$$Z = \tfrac{1}{2} F(\xi) e^{i\xi^2 y} + \tfrac{1}{2} F(\xi) e^{-i\xi^2 y} = F(\xi) \cos(\xi^2 y)$$

7.




$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0 \text{ except at } C(0,0)$$

Boundary conditions:

1) $\phi_1 = \phi_2$ at $z = z_1$
2) $\phi_2 = \phi_3$ at $z = z_2$
3) $\frac{1}{\rho_1}\frac{\partial\phi_1}{\partial z} = \frac{1}{\rho_2}\frac{\partial\phi_2}{\partial z}$ at $z = z_1$
4) $\frac{1}{\rho_2}\frac{\partial\phi_2}{\partial z} = \frac{1}{\rho_3}\frac{\partial\phi_3}{\partial z}$ at $z = z_2$
5) $\phi_1 = \phi_2 = \phi_3 = 0$ at $r = \infty$
6) $\phi_1 = 0$ at $z = -\infty$
7) $\phi_3 = 0$ at $z = \infty$
8) $\phi$ is finite everywhere except at $C(0,0)$.

Particular solutions are of form $\phi = \left.\begin{matrix} e^{\lambda z} \\ e^{-\lambda z}\end{matrix}\right\} \left.\begin{matrix} J_0(\lambda r) \\ Y_0(\lambda r)\end{matrix}\right\}$, but $Y_0(0) = -\infty$, so coefficients of $Y_0(r)$ must be zero. Thus solutions are of form

$$\phi = \left.\begin{matrix} e^{\lambda z} \\ e^{-\lambda z}\end{matrix}\right\} J_0(\lambda r).$$

The potential produced by a point source in a homogeneous infinite space of resistivity $\rho_1$ is

$$\phi = \frac{\rho_1 I}{4\pi R} = \frac{\rho_1 I}{4\pi\sqrt{r^2+z^2}} = \frac{\rho_1 I}{4\pi}\int_0^\infty e^{-\lambda z} J_0(\lambda r)\,d\lambda .$$

Absolute value signs because $z$ can be negative in this half-space; and $\int_0^\infty e^{-\lambda|z|} J_0(\lambda r)\,d\lambda$ is the expression for $1/R$, the inverse distance.

Thus we may write

$$\phi_1 = \frac{\rho_1 I}{4\pi}\int_0^\infty \left\{ e^{-\lambda|z|} + f_1(\lambda)e^{-\lambda z} + g_1(\lambda)e^{\lambda z}\right\} J_0(\lambda r)\,d\lambda$$

$$\phi_2 = \frac{\rho_1 I}{4\pi}\int_0^\infty \left\{ e^{-\lambda z} + f_2(\lambda)e^{-\lambda z} + g_2(\lambda)e^{\lambda z}\right\} J_0(\lambda r)\,d\lambda$$

$$\phi_3 = \frac{\rho_1 I}{4\pi}\int_0^\infty \left\{ e^{-\lambda z} + f_3(\lambda)e^{-\lambda z} + g_3(\lambda)e^{\lambda z}\right\} J_0(\lambda r)\,d\lambda .$$

From 6), $f_1(\lambda) = 0$. From 7), $g_3(\lambda) = 0$.

From 1),

$$e^{-\lambda z_1} + g_1(\lambda)e^{\lambda z_1} = e^{-\lambda z_1} + f_2(\lambda)e^{-\lambda z_1} + g_2(\lambda)e^{\lambda z_1}$$

or

$$g_1(\lambda) = f_2(\lambda)e^{-2\lambda z_1} + g_2(\lambda). \qquad (A)$$

From 3),

$$-\lambda e^{-\lambda z_1} + \lambda g_1(\lambda)e^{\lambda z_1} = \frac{\rho_1}{\rho_2}\left\{-\lambda e^{-\lambda z_1} - \lambda f_2(\lambda)e^{-\lambda z_1} + \lambda g_2(\lambda)e^{\lambda z_1}\right\}$$

or

$$g_1(\lambda) = \frac{\rho_1}{\rho_2}\left\{-e^{-2\lambda z_1} - f_2(\lambda)e^{-2\lambda z_1} + g_2(\lambda)\right\} + e^{-2\lambda z_1} \qquad (B)$$

Equating (A) and (B),

$$f_2(\lambda)e^{-2\lambda z_1} + g_2(\lambda) = \frac{\rho_1}{\rho_2}\left\{-e^{-2\lambda z_1} - f_2(\lambda)e^{-2\lambda z_1} + g_2(\lambda)\right\} + e^{-2\lambda z_1}$$

or

$$f_2(\lambda) = K_{21}\left\{1 - g_2(\lambda)e^{2\lambda z_1}\right\} \quad (C) \text{ where } K_{21} = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}.$$

From B) and C)

$$g_1(\lambda) = K_{21}\{1-g_2(\lambda)\}e^{2\lambda z_1} + g_2(\lambda) = K_{21}e^{-2\lambda z_1} + (1-K_{21})\, g_2(\lambda). \qquad (D)$$

From 2),

$$e^{-\lambda z_2} + f_2(\lambda)\, e^{-\lambda z_2} + g_2(\lambda)\, e^{\lambda z_2} = e^{-\lambda z_2} + f_3(\lambda)\, e^{-\lambda z_2}$$

or

$$f_3(\lambda) = f_2(\lambda)\, \cancel{e^{-\lambda z_2}} + g_2(\lambda)\, e^{2\lambda z_2} \cancel{= e^{-\lambda z_2} + f_3(\lambda) e^{-\lambda z_2}}$$

$$= K_{21}\{1 - g_2(\lambda) e^{2\lambda z_1}\} + g_2(\lambda) e^{2\lambda z_2}$$

Thus

$$f_3(\lambda) = K_{21} + g_2(\lambda)\, e^{2\lambda z_1}\{e^{2\lambda h} - K_{21}\} \qquad (E)$$

where $z_2 = z_1 + h$

From 4),

$$\{-\lambda e^{-\lambda z_2} - \lambda f_2(\lambda) e^{-\lambda z_2} + \lambda g_2(\lambda) e^{\lambda z_2}\} = \frac{\rho_2}{\rho_3}\{-\lambda e^{-\lambda z_2} - \lambda f_3(\lambda) e^{-\lambda z_2}\}$$

or

$$\{-1 - f_2(\lambda) + g_2(\lambda) e^{2\lambda z_2}\} = \frac{\rho_2}{\rho_3}\{-1 - f_3(\lambda)\}.$$

Thus

$$f_2(\lambda) = g_2(\lambda)\, e^{2\lambda z_2} - \left(\frac{\rho_3 - \rho_2}{\rho_3}\right) + \frac{\rho_2}{\rho_3} f_3(\lambda)$$

$$= g_2(\lambda)\, e^{2\lambda z_2} - \left(\frac{\rho_3 - \rho_2}{\rho_3}\right) + \frac{\rho_2}{\rho_3}\{K_{21} + g_2(\lambda)\, e^{2\lambda z_1}(e^{2\lambda h} - K_{21})\}$$

$$= g_2(\lambda) e^{2\lambda z_1}\left\{e^{2\lambda h}\left(\frac{\rho_3 + \rho_2}{\rho_3}\right) - \frac{\rho_2}{\rho_3} K_{21}\right\} + \frac{\rho_2 K_{21} - \rho_3 + \rho_2}{\rho_3}. \qquad (F)$$

Equating (C) and (F),

$$K_{21}\{1 - g_2(\lambda) e^{2\lambda z_1}\} = g_2(\lambda)\, e^{2\lambda z_1}\left\{e^{2\lambda h}\left(\frac{\rho_3+\rho_2}{\rho_3}\right) - \frac{\rho_2}{\rho_3} K_{21}\right\} + \frac{\rho_2 K_{21} - \rho_3 + \rho_2}{\rho_3}$$

Solving for $g_2(\lambda)$,

$$g_2(\lambda) = \frac{e^{-2\lambda z_2} K_{32}(K_{21}+1)}{1 + K_{21} K_{32}\, e^{-2\lambda h}} \qquad (G)$$

where $K_{32} = \dfrac{\rho_3 - \rho_2}{\rho_3 + \rho_2}$.

From (C) and (G),

$$f_2(\lambda) = K_{21}\left\{1 - \frac{e^{2\lambda z_1} e^{-2\lambda z_2} K_{32}(K_{21}+1)}{1 + K_{21}K_{32}\, e^{-2\lambda h}}\right\} = \frac{K_{21}(1 - K_{32} e^{-2\lambda h})}{1 + K_{21}K_{32}\, e^{-2\lambda h}} \qquad (H)$$

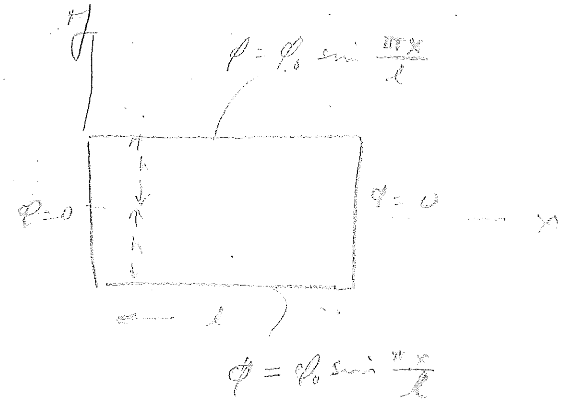
From (A), (G), and (H),

$$g_1(\lambda) = \frac{K_{21} e^{-2\lambda z_1} + K_{32}\, e^{-2\lambda z_2}}{1 + K_{21}K_{32}\, e^{-2\lambda h}} \qquad (I)$$

From (E) and (G),

$$f_3(\lambda) = \frac{K_{21} + K_{32} + K_{21}K_{32}(1 - e^{-2\lambda h})}{1 + K_{21}K_{32}\, e^{-2\lambda h}} \qquad (J)$$

Substitution into the expressions for $\phi_1$, $\phi_2$, and $\phi_3$ gives, after simplifying, the




$$\phi = \left.\begin{matrix}\sin px \\ \cos px\end{matrix}\right\} \left.\begin{matrix}\sinh py \\ \cosh py\end{matrix}\right\}$$

$X = A \sin px + B \cos px$

$X(0) = 0 = B$

$X(l) = 0 = A \sin pl$

$\therefore p = \frac{n\pi}{l}$

$$\phi = A \sin \frac{n\pi}{l} x \sinh \frac{n\pi}{l} y + B \sin \frac{n\pi}{l} x \cosh \frac{n\pi}{l} y$$

$$\phi(x,h) = \phi_0 \sin \frac{\pi x}{l} = A \sin \frac{n\pi x}{l} \sinh \frac{n\pi h}{l} + B \sin \frac{n\pi x}{l} \cosh \frac{n\pi h}{l}$$

$$\phi(x,-h) = \phi_0 \sin \frac{\pi x}{l} = -A \sin \frac{n\pi x}{l} \sinh \frac{n\pi h}{l} + B \sin \frac{n\pi x}{l} \cosh \frac{n\pi h}{l}$$

Add (1) and (2).

$\phi_0 = B \cosh \frac{n\pi h}{l}$ or $B = \frac{\phi_0}{\cosh \frac{n\pi h}{l}}$ when $n = 1$

Subtract (2) from (1). $A = 0$.

Thus the solution is

$$\phi = \frac{\phi_0 \sin \frac{\pi x}{l} \cosh \frac{\pi y}{l}}{\cosh \frac{\pi h}{l}}$$

POTENTIAL THEORY (GPH 204)

DUE: JAN. 26, 1971

1. SHOW THAT THE SOLUTION TO THE BOUNDARY VALUE PROBLEM

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{FOR} \quad -\infty < x < \infty \; , \; 0 < y < 1$$

AND

$$\phi(x,0) = e^{-2|x|}$$

$$\phi(x,1) = 0$$

$$\phi(x,y) \to 0 \quad \text{AS} \quad x \to \pm\infty$$

IS

$$\phi(x,y) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sinh[\xi(1-y)]\, e^{-i\xi x}\, d\xi}{(4+\xi^2)\sinh \xi} .$$

HINT: USE $\int_0^\infty e^{-2x} \cos \xi x \, dx = \frac{2}{4+\xi^2}$.

2. GIVEN $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ FOR $-\infty < x < \infty$, $y > 0$

AND $\phi(x,0) = f(x)$ FOR $-\infty < x < \infty$.

USING THE FOURIER TRANSFORM, SHOW THAT

$$\bar{\phi}(\xi,y) = \bar{\phi}(\xi,0)\, e^{-|\xi| y} = e^{-|\xi| y} \int_{-\infty}^{\infty} f(x')\, e^{-i\xi x'} dx'$$

AND

$$\phi(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{i\xi(x-x') - |\xi| y} \, d\xi \right\} f(x')\, dx'$$

$$= \frac{2y}{2\pi} \int_{-\infty}^{\infty} \frac{f(x')\, dx'}{y^2 + (x-x')^2} .$$

Gph 204 January 23, 1968
Potential Theory O. Nuttli

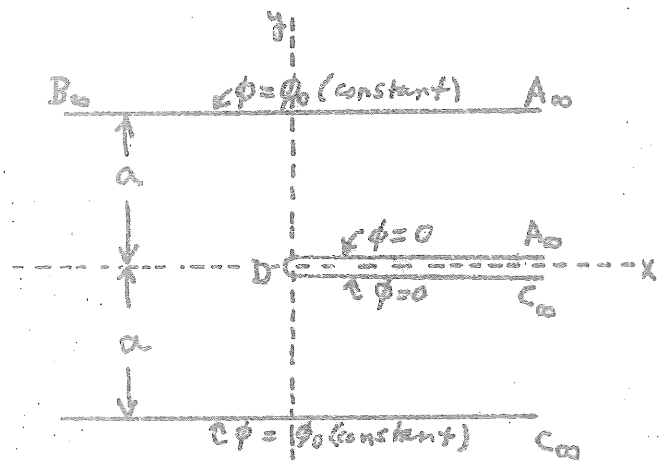
Semester Examination

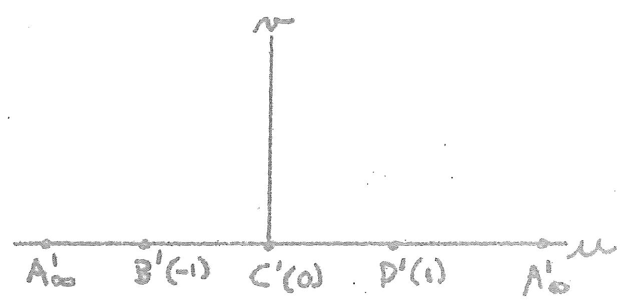
1. Given $\nabla^2\phi = o$ in the interior of a spherical region of radius b, and $\frac{\partial\phi}{\partial n} = f(\theta)$ on $r = b$.

   a. Show that
   $$\phi = \sum_{n=o}^{\infty} A_n r^n P_n(\cos\theta)$$
   where
   $$A_n = \frac{2n+1}{2nb^{n-1}} \int_0^{\pi} f(\theta) P_n(\cos\theta)\sin\theta\, d\theta.$$

   b. If $f(\theta) = A\cos\theta$, find $\phi$.

2. a. Show by using the Schwarz-Christoffel transformation that the figure in the z-plane is transformed to that in the w-plane by the transformation
   $$z = \frac{a}{\pi}\left[\ln\frac{(w+1)^2}{w} - 2\ln 2\right].$$




   b. Find the potential $\phi$ in the region bounded by the surfaces $y = \pm a$, given that $\nabla^2\phi = o$ in the interior of that region.

3. Using the Fourier transform, show that the solution to the boundary value problem
   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for } -\infty<x<\infty, \quad 0<y<1$$

with

$$u(x,o) = e^{-2|x|}$$

$$u(x,1) = o$$

$u(x,y) \to o$ uniformly in $y$ as $x \to \pm \infty$

is

$$u = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sinh [\xi(1-y)]\, e^{-i\xi x} d\xi}{(4 + \xi^2) \sinh \xi}$$

Hint: you may use the relation

$$\int_0^\infty e^{-2x} \cos \xi x \, dx = \frac{2}{4 + \xi^2} .$$

# POTENTIAL THEORY (PROBLEMS)

1. GIVEN $\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{\partial^2\phi}{\partial z^2} = 0$

AND $\phi$ AND ITS FIRST DERIVATIVES VANISH AT INFINITY.

ALSO, LET $\bar{F}(\xi,0) = \dfrac{2V_0 \sin b\xi}{\pi\xi^2}$ WHERE $\bar{F}$ IS THE HANKEL TRANSFORM OF $\phi(r,0)$ WITH RESPECT TO $r$.

SHOW THAT

$$\phi(r,z) = V_0 \quad \text{FOR} \quad 0 \le r \le b$$

$$\phi(r,z) = \frac{2V_0}{\pi}\sin^{-1}\left(\frac{b}{r}\right) \quad \text{FOR} \quad r > b.$$

2. GIVEN $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$ FOR $-\infty < x < \infty$, $0 < y < 1$

AND $\phi(x,0) = e^{-2|x|}$, $\phi(x,1) = 0$, $\phi(x,y) \to 0$ IN $y$ AS $x \to \pm\infty$

SHOW THAT

$$\phi(x,y) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sinh[\xi(1-y)]\, e^{-i\xi x}}{(4+\xi^2)\sinh\xi}\, d\xi$$

3. GIVEN $\dfrac{\partial^2\phi}{\partial x^2} = K\dfrac{\partial\phi}{\partial t}$ WITH $\phi = \phi_0(1 - x/\ell)$ AT $t=0$ ($\phi_0 = $ const)

$\phi = 0$ AT $x=0$ AND $x=\ell$ for $t>0$.

SHOW THAT

$$\phi(x,t) = 2\sum_{m=1}^{\infty} \frac{\phi_0}{m\pi}\, e^{-\frac{m^2\pi^2 t}{\ell^2 K}} \sin\frac{m\pi x}{\ell}.$$

# POTENTIAL THEORY (PROBLEMS)

1. GIVEN $\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{\partial^2\phi}{\partial z^2} = 0$

AND $\phi$ AND ITS FIRST DERIVATIVES VANISH AT INFINITY.
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2. GIVEN $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$ FOR $-\infty < x < \infty$, $0 < y < 1$

AND $\phi(x,0) = e^{-2|x|}$, $\phi(x,1) = 0$, $\phi(x,y) \to 0$ IN $y$ AS $x \to \pm\infty$

SHOW THAT

$$\phi(x,y) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sinh[\xi(1-y)]\, e^{-i\xi x}}{(4+\xi^2)\sinh\xi}\, d\xi$$

3. GIVEN $\dfrac{\partial^2\phi}{\partial x^2} = K\dfrac{\partial\phi}{\partial t}$ WITH $\phi = \phi_0(1 - x/l)$ AT $t=0$ ($\phi_0 = \text{const.}$)

$\phi = 0$ AT $x=0$ AND $x=l$ for $t > 0$.

SHOW THAT

$$\phi(x,t) = 2\sum_{n=1}^{\infty} \frac{\phi_0}{n\pi}\, e^{-\frac{n^2\pi^2}{l^2 K}}\, \sin\frac{n\pi x}{l}.$$

(1) Problem: Assume a model of the earth which is spherical and in which the density $\rho$ is a function of distance $r$ from the center only. By solving Poisson's equation, find the potential $\varphi$ at points inside the ~~center~~ Earth. You may use the fact that at the surface of the Earth, of radius $r_0$, that $\varphi = \frac{GM}{r_0}$ where $M$ is the total mass.

(2) Problem: Consider Poisson's equation in elliptical cylindrical coordinates, where $Q$ is a constant. Show that a solution is

$$\varphi = \Phi - Q\left[\frac{z^2}{2} + \frac{a^2}{8}(\cosh 2\eta + \cos 2\psi)\right]$$

where $\nabla^2\varphi = \frac{1}{a^2(\cosh^2\eta - \cos^2\psi)}\left[\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\psi^2}\right] + \frac{\partial^2\varphi}{\partial z^2} = -Q$

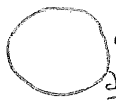
and $\nabla^2\Phi = 0$.

(3) Problem: Given $\frac{1}{a^2(\cosh^2\eta - \cos^2\psi)}\left(\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\psi^2}\right) = -Q$ (constant)

and
$$\left.\begin{aligned}\varphi(\eta_0,\psi) &= \varphi_0\\ \varphi(0,\psi) &= 0\end{aligned}\right\}\text{ boundary conditions.}$$

Show that
$$\varphi(\eta,\psi) = \frac{Qa^2}{8}\left[\cosh 2\eta + \frac{(1-\cosh 2\eta_0)\sinh 2\eta}{\sinh 2\eta_0}\right]\cos 2\psi + \frac{Qa^2}{8}\left[1 - \cosh 2\eta - \cos 2\psi\right].$$

1.



$\nabla^2\varphi = 0$

$\varphi = f(\theta)$ (a)

$\frac{\partial\varphi}{\partial r} = f(\theta)$ (b)

a) $$\varphi(r,\theta) = \sum_{n=0}^{\infty} A_n r^{-(n+1)} P_n(\cos\theta)$$

$$f(\theta) = \varphi(b,\theta) = \sum_{n=0}^{\infty} A_n b^{-(n+1)} P_n(\cos\theta)$$

$$A_n b^{-(n+1)} = \frac{2n+1}{2}\int_0^{2\pi} f(\theta)\sin n\theta\, P_n(\cos\theta)\,d\theta$$

$$\varphi(r,\theta) = \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\right)\left[\int_0^{\pi} f(\theta)\sin n\theta\, P_n(\cos\theta)\,d\theta\right]\left(\frac{b}{r}\right)^{n+1} P_n(\cos\theta)$$

b) $$\frac{\partial\varphi}{\partial r} = \sum -(n+1) A_n r^{-(n+2)} P_n(\cos\theta)$$

$$f(\theta) = \left(\frac{\partial\varphi}{\partial r}\right)_{r=b} = \sum -(n+1) A_n b^{-(n+2)} P_n(\cos\theta)$$

$$-(n+1)b^{-(n+2)} A_n = \frac{2n+1}{2}\int_0^{\pi} f(\theta)\sin\theta\, P_n(\cos\theta)\,d\theta$$

$$\varphi(r,\theta) = \sum_{n=0}^{\infty}\left(-\frac{(2n+1)}{2(n+1)}\, b^{n+2}\left\{\int_0^{\pi} f(\theta)\sin\theta\, P_n(\cos\theta)\,d\theta\right\}\frac{1}{r^{n+1}} P_n(\cos\theta)\right)$$

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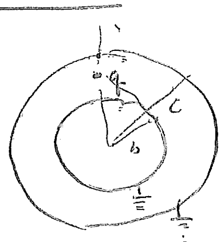
Point charge located between 2 grounded, concentric spherical conducting surfaces. Find electric potential for $b \le r \le c$

2.



$$\varphi(r,\theta) = \frac{q}{R} + \sum\left(A_n r^n + B_n r^{-(n+1)}\right)P_n(\cos\theta)$$

Let charge be at $(r_0, 0)$ and $R$ the distance from $(r_0, 0)$ to $(r,\theta)$.

B. conditions:

1) $\varphi(b,\theta) =$ 2) $\varphi(c,\theta) = 0$.

By cond. 1)

$$\frac{q}{\sqrt{r_0^2 + b^2 - 2br_0\cos\theta}} + \sum\left(A_n b^n + B_n b^{-(n+1)}\right)P_n(\cos\theta) = 0$$

$$\frac{q}{r_0\sqrt{1+\left(\frac{b}{r_0}\right)^2 - 2\left(\frac{b}{r_0}\right)\cos\theta}} + \sum(\quad)P_n(\cos\theta) = 0$$

$$\frac{q}{r_0}\sum\left(\frac{b}{r_0}\right)^n P_n(\cos\theta) + \sum(\quad)P_n(\cos\theta) = 0$$

$$\frac{q\, b^n}{r_0^{n+1}} + A_n b^n + B_n b^{-(n+1)} = 0$$

Bdy cond. 2)

$$\frac{q}{\sqrt{r_0^2+c^2-2cr_0\cos\theta}} + \sum \left(A_n c^n + B_n c^{-(n+1)}\right) P_n(\cos\theta) = 0$$

$$\frac{q}{c\sqrt{1+\left(\frac{r_0}{c}\right)^2 - 2\left(\frac{r_0}{c}\right)\cos\theta}} + \text{''} = 0$$

$$\frac{q}{c}\sum\left(\frac{r_0}{c}\right)^n P_n(\cos\theta) + \text{''} = 0$$

$$\frac{q}{c^{n+1}} r_0^n + A_n c^n + B_n c^{-(n+1)} = 0$$

$$B_n = -\left[A_n b^{2n+1} + \frac{q}{r_0^{n+1}} b^{2n+1}\right]$$

$$\frac{q}{c^{n+1}} r_0^n + A_n c^n - \left[A_n b^{2n+1} + \frac{q}{r_0^{n+1}} b^{2n+1}\right] c^{-(n+1)}$$

$$A_n = \frac{\begin{vmatrix} -\frac{q b^n}{r_0^{n+1}} & b^{-(n+1)} \\ -\frac{q r_0^n}{c^{n+1}} & c^{-(n+1)} \end{vmatrix}}{\begin{vmatrix} b^n & b^{-(n+1)} \\ c^n & c^{-(n+1)} \end{vmatrix}} \qquad B_n = \frac{\begin{vmatrix} b^n & -\frac{q b^n}{r_0^{n+1}} \\ c^n & -\frac{q r_0^n}{c^{n+1}} \end{vmatrix}}{D}$$

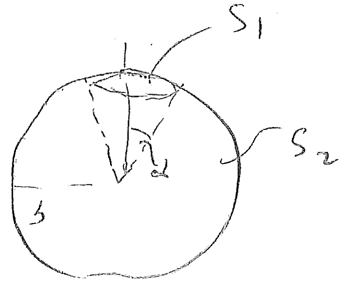
$$A_n = \frac{\frac{q b^n}{(r_0 c)^{n+1}} - \frac{q r_0^n}{(bc)^{n+1}}}{\frac{b^n}{c^{n+1}} - \frac{c^n}{b^{n+1}}} = -\frac{\frac{q\left[b^{2n+1} - r_0^{2n+1}\right]}{(bcr_0)^{n+1}}}{\frac{b^{2n+1}-c^{2n+1}}{(bc)^{n+1}}} = -\frac{\left(b^{2n+1} - r_0^{2n+1}\right)q}{r_0^{n+1}\left(b^{2n+1}-c^{2n+1}\right)}$$

$$B_n = \frac{\frac{q(br_0)^n}{c^{n+1}} - \frac{q(bc)^n}{r_0^{n+1}}}{\frac{b^n}{c^{n+1}} - \frac{c^n}{b^{n+1}}} = q\frac{\left[\frac{b^n r_0^{2n+1} - b^n c^{2n+1}}{(cr_0)^{n+1}}\right]}{\frac{b^{2n+1}-c^{2n+1}}{(bc)^{n+1}}} = \frac{-q b^{2n+1}}{r_0^{n+1}}\left[\frac{r_0^{2n+1} - c^{2n+1}}{b^{2n+1}-c^{2n+1}}\right]$$

$$\varphi(r,\theta) = \frac{q}{R} + q\sum\left\{\left(\frac{b^{2n+1}-r_0^{2n+1}}{b^{2n+1}-c^{2n+1}}\right)\frac{r^n}{r_0^{n+1}} + \left(\frac{b^{2n+1}}{r_0^{n+1}}\right)\left(\frac{r_0^{2n+1}-c^{2n+1}}{b^{2n+1}-c^{2n+1}}\right)\frac{1}{r^{n+1}}\right\}P_n(\cos\theta)$$

Answer

3.




$$\varphi(S_1) = \Phi_0 \qquad \nabla^2\varphi = 0 \text{ for } r<b$$

$$\varphi(S_2) = 0$$

$$\varphi(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

$$\int_0^\pi \phi(s)\sin\theta\, P_m(\cos\theta)\,d\theta = \sum_{n=0}^{\infty}\int_0^\pi A_n b^n\, P_n \sin\theta\, \cancel{P_m} \cdot P_n(\cos\theta)\cdot P_m(\cos\theta)\,d\theta$$

$$= \left(\frac{2}{2m+1}\right) A_m b^m$$

$$\int_0^\alpha \varphi_0 \sin\theta\, P_m(\cos\theta)\,d\theta$$

$$= \varphi_0 \int_1^{\cos^{-1}(\alpha)} -P_m(x)\,dx = \varphi_0 \int_{\cos^{-1}(\alpha)}^{1} P_m(x)\,dx = \frac{\Phi_0}{2m+1}\left(P_{m-1}(\cos\alpha) - P_{m+1}(\cos\alpha)\right.$$

for $n \geq 1$

$$A_n = \frac{\Phi_0}{2b^n}\left[P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)\right]$$

$$\varphi(r,\theta) = \cancel{\sum \frac{\Phi_0}{2b^n}}$$

$$\frac{\Phi_0}{2}\sum\left[A_0 + \left\{P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)\right\}\frac{r^n}{b^n} P_n(\cos\theta)\right]$$

---

For $A_0$ : $\int_0^\alpha \phi_0 \sin\theta\, d\theta = \left[-\varphi_0\cos\theta\right]_0^\alpha = -\varphi_0(\cos\alpha - 1)$

Therefore

$$\varphi(r,\theta) = \frac{\Phi_0}{2}\left\{1-\cos\alpha + \sum_{n=1}^{\infty}\left[P_{n-1}(\cos\alpha) - P_{n+1}(\cos\alpha)\right]\left(\frac{r}{b}\right)^n P_n(\cos\theta)\right.$$