Also

$$\frac{1}{2R^3}\int_B \sigma(3r^2\cos^2\alpha - r^2)\,dV = \frac{1}{2R^3}\int_B \sigma(3\cdot r^2 - 3r^2\sin^2\alpha - r^2)\,dV$$

$$= \frac{1}{R^3}\int_B \sigma r^2\,dV - \frac{3}{2R^3}\int_B \sigma\, r^2\sin^2\alpha\,dV$$

The first of the integrals on the right is the moment of inertia of the body B with respect to the origin O. The second integral is the moment of inertia of B with respect to the line (about the axis) $\overline{OP} \equiv R$. Call these moments of inertia I_0 and I_R respectively. Then,

$$\varphi = \frac{M}{R} + \frac{Mq}{R^2} + \frac{2I_0 - 3I_R}{R^3} + \cdots$$

and $\varphi = \dfrac{M}{R} + \dfrac{2I_0 - 3I_R}{R^3} + \cdots$ if the origin of coordinates is taken at the mass center of the body.

The expression for the potential can be written in terms of the coordinates (x, y, z) of P and (ξ, η, ζ) of the element of volume. Now

$$r = (\xi^2 + \eta^2 + \zeta^2)^{1/2}, \quad R = (x^2 + y^2 + z^2)^{1/2},$$

and $r\cos\alpha = \dfrac{\xi x + \eta y + \zeta z}{R}$. By simple substitution and algebraic reduction it can be shown that

$$\varphi = \frac{M}{R} + \frac{x}{R^3}\int_B \xi\,dm + \frac{y}{R^3}\int_B \eta\,dm + \frac{z}{R^3}\int_B \zeta\,dm + \frac{(2x^2 - y^2 - z^2)}{2R^5}\int_B \xi^2\,dm + \frac{(-x^2 + 2y^2 - z^2)}{2R^5}\int_B \eta^2\,dm$$

$$+ \frac{(-x^2 - y^2 + 2z^2)}{2R^5}\int_B \zeta^2\,dm + \frac{3xy}{R^5}\int_B \xi\eta\,dm + \frac{3yz}{R^5}\int_B \eta\zeta\,dm + \frac{3xz}{R^5}\int_B \zeta\xi\,dm$$

$$+ \frac{(2x^3 - 3xy^2 - 3xz^2)}{2R^7}\int_B \xi^3\,dm + \frac{(-3x^2y + 2y^3 - 3yz^2)}{2R^7}\int_B \eta^3\,dm + \frac{(-3x^2z - 3y^2z + 2z^3)}{2R^7}\int_B \zeta^3\,dm$$

$$+ \frac{(12x^2y - 3y^3 - 3yz^2)}{2R^7}\int_B \xi^2\eta\,dm + \frac{(-3x^2z + 12y^2z - 3z^3)}{2R^7}\int_B \eta^2\zeta\,dm + \frac{(-3x^3 - 3xy^2 + 12xz^2)}{2R^7}\int_B \zeta^2\xi\,dm$$

$$+ \frac{(12x^2z - 3y^2z - 3z^3)}{2R^7}\int_B \xi^2\zeta\,dm + \frac{(-3x^3 + 12xy^2 - 3xz^2)}{2R^7}\int_B \xi\eta^2\,dm + \frac{(-3x^2y - 3y^3 + 12yz^2)}{2R^7}\int_B \eta\zeta^2\,dm$$

Associated Legendre Functions (Irving and Mullineux, pg 186 ff)

Legendre's associated eqn., which arises in the solution of problems in spherical coordinates in which there is no axial symmetry, is

$$(1-x^2)v'' - 2xv' + \left\{n(n+1) - \frac{m^2}{(1-x^2)}\right\}v = 0 \qquad (11.3.1)$$

In (11.3.1), let $v = (1-x^2)^{m/2}u$, where m is a positive integer. This gives

$$(1-x^2)u'' - 2(m+1)xu' + (n-m)(n+m+1)u = 0 \qquad (11.3.2)$$

Differentiate Legendre's eqn. (11.1) m times, using the notation $y^{(m)} = \frac{d^m y}{dx^m}$

This gives, after rearranging,

$$(1-x^2)\frac{d^2y^{(m)}}{dx^2} - 2(m+1)x\frac{dy^{(m)}}{dx} + (n-m)(n+m+1)y^{(m)} = 0 \qquad (11.3.3)$$

Comparison of (11.3.2) and (11.3.3) gives

$$u = y^{(m)} = \frac{d^m y}{dx^m} = \frac{d^m}{dx^m}\left\{AP_n(x) + BQ_n(x)\right\}$$

where $y = AP_n(x) + BQ_n(x)$ is the general solution of (11.1).

Thus the general solution of (11.3.1) is

$$v = (1-x^2)^{m/2}u = (1-x^2)^{m/2}\frac{d^m}{dx^m}\left\{AP_n(x) + BQ_n(x)\right\}$$

which may be written as

$$v = AP_n^m(x) + BQ_n^m(x) \qquad (11.3.4)$$

where

$$P_n^m(x) \equiv (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m} \qquad (11.3.5)$$

and

$$Q_n^m(x) \equiv (1-x^2)^{\frac{m}{2}}\frac{d^m Q_n(x)}{dx^m} \qquad (11.3.6)$$

$P_n^m(x)$ and $Q_n^m(x)$ are called Legendre's associated functions of the first and second kind, respectively. $Q_n^m(x)$ has infinite singularities at $x = \pm 1$.

Now $P_n^m(x)$ for $m > n$ equals zero, which can be seen from eq. (11.3.5) ~~and~~ if we remember that $P_n(x)$ is a polynomial of degree n in x. Also, from (11.3.5), $P_n^0(x) = P_n(x)$.

Rodrigues' formula (11.9) gives

$$P_n^m(x) = (1-x^2)^{m/2} \frac{d^{(m+n)}}{dx^{(m+n)}} \left\{ \frac{(x^2-1)^n}{2^n n!} \right\} \qquad (11.3.7)$$

Recurrence relations are given as equations (11.3.8).

The associated Legendre functions are orthogonal, with weight factor 1 in the interval -1 to $+1$. That is, if n_1, n_2, & m are positive integers such that m is not greater than n_1 · n_2 (11.3.9)

$$\int_{-1}^{1} P_{n_1}^m(x)\, P_{n_2}^m(x)\, dx = 0 \quad \text{for } n_1 \neq n_2.$$

For $n_1 = n_2$,

$$\int_{-1}^{1} \left\{ P_n^m(x) \right\}^2 dx = \frac{2(n+m)!}{(2n+1)(n-m)!} \qquad (11.3.12)$$

Also $\int_{-1}^{1} P_n^{m_1}(x)\, P_n^{m_2}(x) \left(\frac{1}{1-x^2}\right) dx = 0$ for $m_1 \neq m_2$ and See pp 56 f-1, 2

$$\int_{-1}^{1} \left(\frac{1}{1-x^2}\right) \left[P_n^m(x)\right]^2 dx = \frac{(n+m)!}{m(n-m)!}$$

Spherical Harmonics

We have already discussed the solution of Laplace's eqn. in spherical coordinates by the method of separation of variables. It is

$$\phi = \left. \begin{matrix} r^n \\ r^{-(n+1)} \end{matrix} \right\} \left. \begin{matrix} \cos m\psi \\ \sin m\psi \end{matrix} \right\} \begin{matrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{matrix}.$$

The functions

$$r^n \cos m\psi \; P_n^m(\cos\theta)$$
$$r^n \sin m\psi \; P_n^m(\cos\theta)$$

are called solid spherical harmonics.

Example of Solution of 3-D Laplace Equation in Spherical Coordinates (Morse & Feshbach pg 231)

Consider the interior Dirichlet problem for a sphere of radius a. On $r=a$, let $\phi = F(\theta,\psi)$.

We have shown that solutions to $\nabla^2\phi = 0$ in spherical coordinates are

$$\phi = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} \left.\begin{matrix} \cos \\ \sin \end{matrix}\, m\psi \right\} \left.\begin{matrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{matrix}\right\}.$$

We discard solutions involving $Q_n^m(\cos\theta)$, because it is infinite on the axis $\theta=0$ or $\theta=\pi$. For the interior problem we also discard solutions involving $r^{-(n+1)}$, which would become infinite at $r=0$. Therefore the solution has the form

$$\phi = \sum_{n=0}^{\infty} B_{0n}(r/a)^n P_n(\cos\theta) + \sum_{n=1}^{\infty}\sum_{m=1}^{n}\{B_{mn}(r/a)^n P_n^m(\cos\theta)\cdot\cos m\psi$$

$P_n^0(\mu)=P_n(\mu)$

$$+ A_{mn}(r/a)^n P_n^m(\cos\theta)\cdot\sin m\psi\}.$$

We must evaluate the coefficients B_{0n}, B_{mn}, and A_{mn}.

At $r=a$,

$$F(\theta,\psi) = \sum_{n=0}^{\infty} B_{0n} P_n(\mu) + \sum_{n=1}^{\infty}\sum_{m=1}^{n}\{B_{mn} P_n^m(\mu)\cdot\cos m\psi + A_{mn} P_n^m(\mu)\cdot\sin m\psi\}.$$

To find B_{0n}, multiply both sides by $d\psi$ and integrate. This gives

$$\int_0^{2\pi} F(\theta,\psi)\,d\psi = \sum_{n=0}^{\infty} B_{0n}P_n(\mu)\int_0^{2\pi}d\psi = 2\pi\sum_{n=0}^{\infty} B_{0n}P_n(\mu),$$ where the terms involving the double summations are zero because

$$\int_0^{2\pi}\cos m\psi\,d\psi = \int_0^{2\pi}\sin m\psi\,d\psi = 0.$$

Next multiply by $P_k(\mu)$ and integrate with respect to μ from -1 to $+1$ (k is an integer). This gives

$$\int_{-1}^{1}\int_0^{2\pi} F(\theta,\phi)\, P_k(\mu)\, d\phi\, d\mu = 2\pi \sum_{n=0}^{\infty} B_{0n} \int_{-1}^{1} P_n(\mu)\, P_k(\mu)\, d\mu$$

From the orthogonality of Legendre functions in the interval $(-1,1)$,

$$\int_{-1}^{1} P_n(\mu)\, P_k(\mu)\, d\mu = 0 \quad \text{if } n \neq k$$
$$= \frac{2}{2n+1} \quad \text{if } n = k.$$

Thus
$$\int_{-1}^{1}\int_0^{2\pi} F(\theta,\phi)\, P_n(\mu)\, d\phi\, d\mu = 2\pi\, B_{0k} \cdot \frac{2}{2k+1} = \frac{4\pi}{2k+1} B_{0k}$$

where the only term in the summation which gives an integral other than zero is $n = k$. (We then immediately switch back and call $k = n$.)

Thus
$$B_{0n} = \frac{2n+1}{4\pi} \int_{-1}^{1}\int_0^{2\pi} F(\theta,\phi)\, P_n(\mu)\, d\phi\, d\mu .$$

To find the B_{mn}, we make use of the orthogonality of the cosine function in the interval 0 to 2π. Multiply the expression for $F(\theta,\phi)$ by $\cos k\phi$ and integrate with respect to ϕ from 0 to 2π. This gives

$$\int_0^{2\pi} F(\theta,\phi) \cos k\phi\, d\phi = \sum_{n=0}^{\infty}\sum_{m=1}^{n} B_{mn}\, P_n^m(\mu) \int_0^{2\pi} \cos m\phi \cos k\phi\, d\phi$$
$$= \pi \sum_{n=0}^{\infty} B_{mn}\, P_n^m(\mu)$$

where $\int_0^{2\pi} \cos k\phi\, d\phi = 0$, $\int_0^{2\pi} \cos k\phi \sin m\phi\, d\phi = 0$

and $\int_0^{2\pi} \cos k\phi \cos m\phi\, d\phi = \begin{matrix} 0 & \text{if } k \neq m \\ \pi & \text{if } k = m \end{matrix}$.

Next multiply by $P_s^m(\mu)\, d\mu$ and integrate with respect to μ from -1 to $+1$.

$$\int_{-1}^{1}\int_{0}^{2\pi} F(\theta,\psi)\cos k\psi \, P_s^{m}(\mu)\, d\psi\, d\mu = \pi \sum_{n=0}^{\infty} B_{mn} \int_{-1}^{1} P_n^{m}(\mu)\, P_s^{m}(\mu)\, d\mu .$$

Now $\displaystyle\int_{-1}^{1} P_n^{m}(\mu)\, P_s^{m}(\mu)\, d\mu = \begin{cases} 0 & \text{if } n \neq s \\ \dfrac{2}{(2n+1)} \dfrac{(n+m)!}{(n-m)!} & \text{if } n = s . \end{cases}$

Thus

$$\pi \sum_{n=0}^{\infty} B_{mn} \int_{-1}^{1} P_n^{m}(\mu)\, P_s^{m}(\mu)\, d\mu = \frac{2\pi}{(2s+1)} \frac{(s+m)!}{(s-m)!} B_{ms}$$

and

$$B_{mn} = \frac{(2n+1)}{2\pi} \frac{(n-m)!}{(n+m)!} \int_{-1}^{1}\int_{0}^{2\pi} F(\theta,\psi)\, P_n^{m}(\mu) \cos m\psi \, d\psi\, d\mu .$$

Similarly,

$$A_{mn} = \frac{(2n+1)}{2\pi} \frac{(n-m)!}{(n+m)!} \int_{-1}^{1}\int_{0}^{2\pi} F(\theta,\psi)\, P_n^{m}(\mu) \sin m\psi \, d\psi\, d\mu .$$

Poisson's equation

Given $\nabla^2 \phi = -Q(u_1, u_2, u_3)$ where u_1, u_2, u_3 are generalized coordinates.

To find a solution, assume $\phi = \Phi + f$ where $\nabla^2 \Phi = 0$. Then $\nabla^2 f = -Q$.

Any f that satisfies $\nabla^2 f = -Q$ will, when added to Φ, give a solution to Poisson's equation.

In physical problems, we will have boundary conditions, similar to the Dirichlet, Neumann or ~~...~~ Churchill problems for Laplace's equation.

It can be shown that solutions to Poisson's eqn. that satisfies the boundary conditions are unique, similar to those for Laplace's eqn.

So, if we can find a particular solution to Poisson's eqn and make it fit the bdy conditions, it will be the solution to the problem.

Example: Poisson's eqn. in rectangular coordinates, with Q a constant

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -Q \quad \text{or} \quad \nabla^2 \Phi = 0 \text{ and } \nabla^2 f = -Q, \text{ where } \phi = \Phi + f$$

Assume f is a function of x only. Then

$\nabla^2 f = -Q$ reduces to $\frac{d^2 f}{dx^2} = -Q$ or $f = -\frac{Qx^2}{2}$.

Then

$$\phi(x,y,z) = \left.\begin{matrix} \sin px \\ \cos px \end{matrix}\right\} \left.\begin{matrix} \sin qy \\ \cos qy \end{matrix}\right\} \left.\begin{matrix} e^{-\sqrt{p^2+q^2}\, z} \\ e^{\sqrt{p^2+q^2}\, z} \end{matrix}\right\} - \frac{Qx^2}{2}$$

is one solution to Poisson's equation.

We could just as well have used $-\frac{Qy^2}{2}$, $-\frac{Qz^2}{2}$, $-\frac{Q}{2}(x^2+y^2)$, etc., $-\frac{Q}{2}(x^2+y^2+z^2)$ as particular solutions to be added to Φ, to give $\phi = \Phi + f$.

The boundary conditions may require that part of the terms in $-\frac{Q}{2}(x^2+y^2+z^2)$ drop out, e.g. leaving only $-\frac{Q}{2}(x^2+z^2)$

Example: Poisson's equation in cylindrical polar coordinates

Let $\varphi = \Phi + f$ and $\nabla^2 \Phi = 0$

Then $$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} = -Q$$

Suppose Q = a constant.

1) Assume $f = f(z)$ only. Then $\frac{d^2 f}{dz^2} = -Q$ and $f = -\frac{Qz^2}{2}$.

2) Assume $f = f(r)$ only. Then
$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} = -Q \quad \text{or} \quad r^2\frac{d^2 f}{dr^2} + r\frac{df}{dr} + Qr^2 = 0.$$
A solution is $f = -\frac{Qr^2}{4}$.

3) Assume $f = f(\theta)$ only. This leads to
$$\frac{1}{r^2}\frac{d^2 f}{d\theta^2} = -Q \quad \text{or} \quad \frac{d^2 f}{d\theta^2} = -Qr^2$$
. But this leads to a contradiction, because the f would be a function of both θ and r, and r only. Therefore, there is no solution for $f = f(\theta)$ only.

Thus, solutions to $\nabla^2 \varphi = -Q$ (constant) in cylindrical coordinates are
$$\varphi(r,\theta,z) = \left.\begin{matrix} e^{pz} \\ e^{-pz} \end{matrix}\right\} \left.\begin{matrix} \sin q\theta \\ \cos q\theta \end{matrix}\right\} \left.\begin{matrix} J_q(pr) \\ Y_q(pr) \end{matrix}\right\} - Q\left(\frac{z^2}{2} + \frac{r^2}{4}\right)$$
where either of the two terms multiplying Q could be zero. The boundary conditions will determine which, if either, is zero.

Further example: Suppose in the equation above Q is not constant, but a function of r, say $g(r)$. Then the only value of f would be ~~the~~ a solution of the ordinary diff. eq.
$$r^2\frac{d^2 f}{dr^2} + r\frac{df}{dr} + r^2 g(r) = 0$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = -Q$$

$\varphi = \Phi + f_1$ where $\nabla^2 \Phi = 0$

where f is any particular solution

$$\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -Q$$

Assume $f = f(x)$ only. Then

$$\frac{d^2 f}{dx^2} = -Q \quad \text{and} \quad f = -\frac{Q x^2}{2}.$$

Solutions are:

$$\varphi = \left.\begin{matrix}\sin px \\ \cos px\end{matrix}\right\} \left.\begin{matrix}e^{\cancel{-qy}} \\ e^{\cancel{+qy}}\end{matrix}\right\} \left.\begin{matrix}\sin qy \\ \cos qy\end{matrix}\right\} \begin{matrix}e^{-\sqrt{p^2+q^2}\,z} \\ e^{+\sqrt{p^2+q^2}\,z}\end{matrix} \quad - \frac{Q x^2}{2}$$

$\varphi =$

Cylindrical polar coord

Assume $f = f(z)$ only. Then $\frac{d^2 f}{dz^2} = -Q$ and $f = -\frac{Q z^2}{2}$

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} = -Q$$

If $f = f(r)$ only, reduces to

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} = -Q$$

Solution is $f = -\frac{Q r^2}{4}$

$$\frac{d^2 f}{d\theta^2} = -Q r^2$$

$f \neq f(\theta)$ only

Solutions of Poisson's Equation

Poisson's equation is

$\nabla^2\phi = -Q(u_1, u_2, u_3)$ where Q is either a known function of the coordinates (u_1, u_2, u_3) or is constant.

The equation is valid only if $\nabla^2\phi$ is continuous in the neighborhood of the point (u_1, u_2, u_3). [Elaborate.]

In order to obtain a solution to Poisson's equation, we shall make a change in variable by defining

$$\phi = \Phi + f(u_1, u_2, u_3)$$

where $f(u_1, u_2, u_3)$ is chosen so that $\nabla^2\Phi = 0$. At present we have one method (separation of variables) to find Φ. Any problem [f is a particular solution – not unique] thus reduces to finding f, when the separation of variables is applicable to $\nabla^2\Phi = 0$.

First consider Q = constant. Then

$$\nabla^2\phi = \nabla^2\Phi + \nabla^2 f = \nabla^2 f = -Q .$$

If our coordinate system is such that $\nabla^2\phi$ contains only the term $\frac{\partial^2\phi}{\partial u_1^2}$ and no other term relating ϕ and u_1 (e.g. $\frac{\partial\phi}{\partial u_1}$ or $\frac{1}{u_1}\frac{\partial\phi}{\partial u_2}$), then we may write $\phi = \Phi - Q\frac{u_1^2}{2}$ ~~Taking~~ or $f = -Q\frac{u_1^2}{2}$. This value of f is independent of u_2 and u_3. Therefore $\nabla^2 f = \frac{\partial^2 f}{\partial u_1^2} = -Q$, which establishes that $\Phi - Q\frac{u_1^2}{2}$ is a solution of $\nabla^2\phi = -Q$.

We obtain Φ by finding the solution of $\nabla^2\phi = 0$.

Example: Poisson's equation in rectangular coordinates.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -Q .$$

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We may select u, as x (y or z would have been equally valid).

Then $\phi = \Phi - Q\frac{x^2}{2}$, where one form of Φ would be

$$\Phi = \left.\begin{matrix}\sin\\ \cos\end{matrix}\, px\right\} \left.\begin{matrix}\sin\\ \cos\end{matrix}\, qy\right\} \left.\begin{matrix}e^{\sqrt{p^2+q^2}\,z}\\ e^{-\sqrt{p^2+q^2}\,z}\end{matrix}\right\}.$$

Example: <u>Poisson's ~~equa. in~~ polar cylindrical coordinates</u>.

a.) $$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2} + \frac{\partial^2\phi}{\partial z^2} = -Q.$$

In this equation the only term relating ϕ to z is $\frac{\partial^2\phi}{\partial z^2}$. Therefore

$f = -Q\frac{z^2}{2}$ and $\phi = \Phi - \frac{Qz^2}{2}$ where, e.g.,

$$\Phi = \left.\begin{matrix}e^{pz}\\ e^{-pz}\end{matrix}\right\} \left.\begin{matrix}\sin\\ \cos\end{matrix}\, q\theta\right\} \left.\begin{matrix}J_q(pr)\\ Y_q(pr)\end{matrix}\right\}.$$

b.) Suppose that ϕ is a function of r and θ only. Then

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2} = -Q.$$

$\frac{\partial^2\phi}{\partial r^2}$ or $\frac{\partial^2\phi}{\partial \theta^2}$ by itself on

There is no way of getting only the left-hand side of the equation. However, multiply through by r^2. This gives

$$r^2\frac{\partial^2\phi}{\partial r^2} + r\frac{\partial \phi}{\partial r} + \frac{\partial^2\phi}{\partial \theta^2} + Qr^2 = 0$$

We cannot make use of $\frac{\partial^2\phi}{\partial \theta^2}$ being alone on the left-hand side, ~~is~~ multiplied by r^2. ~~on the right-hand side~~. However, a because Q is now ~~not on the right-hand side~~ $r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + Qr^2 = 0$ will ~~be~~ give a particular solution of ~~in terms of~~ θ. A solution integral of the ~~original~~ differential equation. of this equation is $\phi = -\frac{Qr^2}{4}$, which can be verified by substitution. Then $\phi = \Phi - Q\frac{r^2}{4}$ is the solution to the Poisson equation, where $\nabla^2\Phi = 0$.

$$\phi = \left.\begin{matrix}\sin\\ \cos\end{matrix}\, p\theta\right\} \left.\begin{matrix}J_0(pr)\\ Y_0(pr)\end{matrix}\right\} - \frac{Qr^2}{4}$$

Check by substitution: $\left[\frac{\partial^2\Phi}{\partial r^2} - \frac{Q}{2}\right] + \frac{1}{r}\left[\frac{\partial\Phi}{\partial r} - \frac{Qr}{2}\right] + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}$

$\nabla^2\Phi \quad Q \quad Q \quad = Q$ Q.E.D.

c.) Suppose that ϕ is a function of r and z only. Then

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2} = -Q .$$

One particular solution is $\phi = \Phi + f = \Phi - \frac{Qz^2}{2}$.

Another particular solution can be obtained from

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr} = -Q$$

which is $f = -\frac{Q}{4}r^2$. Therefore we can write as another particular solution

$$\phi = \Phi + f = \Phi - \frac{Qr^2}{4} = \left.\begin{matrix}\sinh\\ \cosh\end{matrix}\, pz\right\} \left.\begin{matrix}J_0(pr)\\ Y_0(pr)\end{matrix}\right\} - \frac{Qr^2}{4} .$$

The boundary conditions, which determine the coefficients of the ~~constants~~ terms appearing in the solution of $\nabla^2\Phi=0$, provide a unique solution for ϕ.

-48- Example: Poisson's eqn. in polar spherical coordinates

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \psi^2} = -Q$$

[Note: Be careful that the form of Laplace's eqn. used to write Poisson's equation has not been multiplied or divided by r^2!]

Multiply by r^2. This gives

$$r^2 \frac{\partial^2 \phi}{\partial r^2} + 2r\frac{\partial \phi}{\partial r} + Qr^2 + \frac{\partial^2 \phi}{\partial \theta^2} + \cot\theta \frac{\partial \phi}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 \phi}{\partial \psi^2} = 0.$$

Separating variables, ~~we have~~

$$r\left[r\frac{\partial^2 f}{\partial r^2} + 2\frac{\partial f}{\partial r} + Qr\right] = 0, \quad \text{where } \phi = \Phi + f$$

A solution is $f = -\frac{Qr^2}{6}$, as may be verified by substituting into the ~~previous~~ equation. Therefore, the solution to Poisson's eqn. in polar spherical coordinates is

$$\phi = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} \left.\begin{matrix} \sin \\ \cos \end{matrix}\ m\psi \right\} \left.\begin{matrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{matrix}\right\} - Q\frac{r^2}{6}$$

It can be shown in a similar manner that if $\phi = \phi(r, \theta)$,

$$\phi = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} \left.\begin{matrix} P_n(\cos\theta) \\ Q_n(\cos\theta) \end{matrix}\right\} - Q\frac{r^2}{6}.$$

Also, if $\phi = \phi(r)$ only

$$\phi = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} - Q\frac{r^2}{6}.$$

There are no solutions to Poisson's equation if ϕ is a function of θ only or if ϕ is a function of ψ only. or θ and ψ only

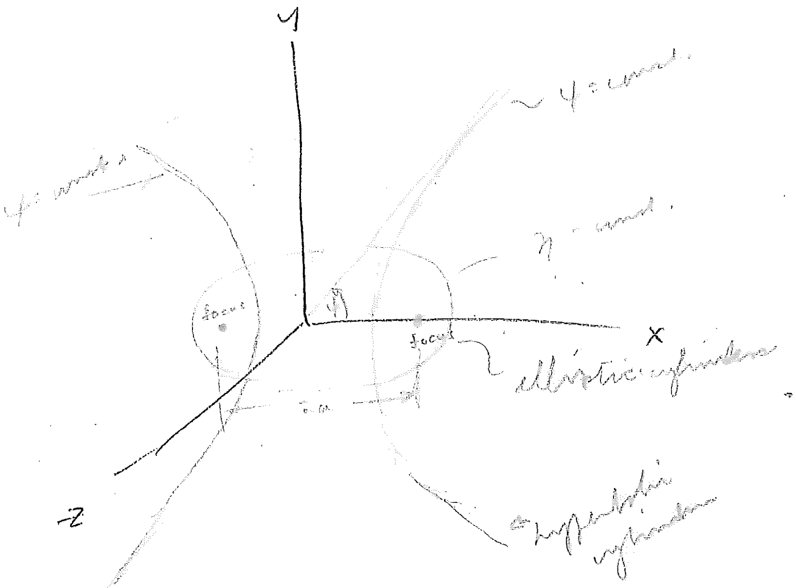
Suppose $\phi = \phi(r, \psi)$, then $\phi = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} \left.\begin{matrix} \sin \\ \cos \end{matrix}\ m\psi \right\} - Q\frac{r^2}{6}$

Consider Poisson's eqn. in elliptic cylindrical coordinates

$$\cancel{\nabla^2\phi = \frac{1}{a^2(\cosh^2\eta\,\sin^2\psi + \sinh^2\eta\,\cos^2\psi)}\left\{\frac{\partial}{\partial\eta}\right.}$$

$$\nabla^2\phi = \frac{1}{a^2(\cosh^2\eta - \cos^2\psi)}\left\{\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\psi^2}\right\} + \frac{\partial^2\phi}{\partial z^2} = -Q$$

where Q is constant



$\psi = 0$ on the right hand side of x-z plane beyond focus
$\psi = \pi$ " " left " beyond focus
$\psi = \pi/2$ ~~on the upper~~ half of y-z plane
$\psi = \frac{3\pi}{2}$ " "lower" " "
$\eta = 0$ in x-z plane between foci

ψ is measured from x axis in the x-z plane — it varies from 0 to 2π.

~~If~~ $\phi = \phi(\eta, \psi, z)$, or $\phi = \phi(\eta, z)$, or $\phi = \phi(\psi, z)$,

$\phi = \Phi - \frac{Qz^2}{2}$, where Φ is the solution of $\nabla^2\phi = 0$.

However, a ~~more~~ interesting case arises when $\phi = \phi(\psi, \eta)$

Then Poisson's eqn. becomes

$$\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\psi^2} = -Qa^2(\cosh^2\eta - \cos^2\psi)$$

$$= -Qa^2\left[\tfrac{1}{2}(\cosh 2\eta + 1) - \tfrac{1}{2}(1 + \cos 2\psi)\right] = -\frac{Qa^2}{2}(\cosh 2\eta - \cos$$

In this case the Q appears in terms multiplying η and in those multiplying ψ. Thus, we cannot use the previous methods of solution. However, we can write

$$\phi = \Phi + f_1(\eta) + f_2(\psi)$$

Then

$$\frac{\partial^2 \varphi}{\partial \eta^2} = \frac{\partial^2 \Phi}{\partial \eta^2} + \frac{\partial^2 f_1}{\partial \eta^2}$$

$$\frac{\partial^2 \varphi}{\partial \psi^2} = \frac{\partial^2 \Phi}{\partial \psi^2} + \frac{\partial^2 f_2}{\partial \psi^2} \quad ,$$

and Poisson's equa. becomes

$$\nabla^2 \Phi + \frac{\partial^2 f_1}{\partial \eta^2} + \frac{Qa^2}{2}\cosh 2\eta + \frac{\partial^2 f_2}{\partial \psi^2} - \frac{Qa^2}{2}\cos 2\psi = 0$$

This equation can be satisfied if

$$\cancel{d^2 f_1(\eta) = -\frac{Ka^2}{2}\cosh 2\eta}$$

$$\cancel{f_2(\psi) = -Ka^2}$$

$$\frac{d^2 f_1}{d\eta^2} = -\frac{Qa^2}{2}\cosh 2\eta$$

$$\frac{d^2 f_2}{d\psi^2} = \frac{Qa^2}{2}\cos 2\psi \quad .$$

Integration gives particular solutions such as

$$f_1(\eta) = \frac{-Qa^2}{8}\cosh 2\eta \qquad \left\{ \begin{array}{l} \int \cosh 2\eta \, d\eta = \frac{1}{2}\sinh 2\eta \\ \frac{1}{2}\int \sinh 2\eta \, d\eta = \frac{1}{4}\cosh 2\eta \end{array} \right.$$

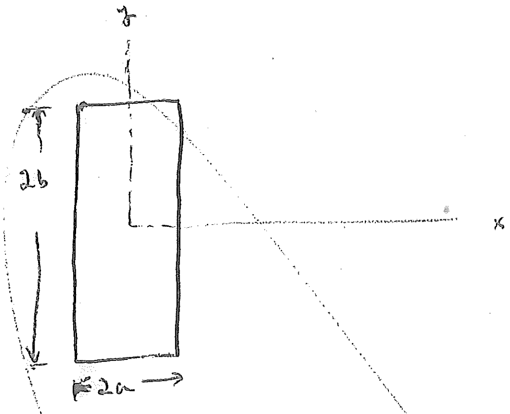
$$f_2(\psi) = \frac{-Qa^2}{8}\cos 2\psi \quad .$$

Thus

$$\varphi = \left.\begin{array}{l} e^{p\eta} \\ e^{-p\eta} \end{array}\right\} \left.\begin{array}{l} \sin \\ \cos \end{array} p\psi \right\} - \frac{Qa^2}{8}(\cosh 2\eta + \cos 2\psi)$$

Problem: Find gravitational potential inside a homogeneous rotating spheroid, where (see bottom of pg. 48.

"Steady" State
Temperature within a 2-D Rectangular Bar



The temperature inside the bar is φ.

It satisfies

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = -\frac{Q}{K} - Q$$

where K is the thermal conductivity.

Drop terms in $\sin p\xi$ because of symmetry

$$\Phi = A \cosh p\eta \cos p\xi + B \sinh p\eta \cos p\xi + C + D\eta$$

$$\varphi = (A \cosh p\eta + B \sinh p\eta) \cos p\xi + C + D\eta - \frac{Q a^2}{8}(\cosh 2\eta + \cos 2\xi)$$

$$\varphi = \varphi_0 \frac{\eta}{\eta_0} + \frac{Q a^2}{8} \left\{ \left[\frac{(\cosh 2\eta - 1)\sinh 2\eta_0 - (\cosh 2\eta_0 - 1)\sinh 2\eta}{\sinh 2\eta_0} \right. \right.$$

$\cos 2\xi$

This is the main part

$$- (\cosh 2\eta_0 - 1)\left(1 - \frac{\eta}{\eta_0}\right) \Big\}$$

$m = 2 \quad n = 1$

$\sin \frac{\pi}{2}$

$$\frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} \Bigg]_{-\pi}^{\pi}$$

$$\int_{-\pi}^{\pi} A \cos p\xi \cos m\xi \, d\xi = \int_{-\pi}^{\pi} \frac{Q a^2}{8}(1 + \cos 2\xi) \cos m\xi \, d\xi$$

Let $m = p$

$$\text{L.H.S} = \frac{A}{2p}(p\xi + \sin p\xi \cos p\xi)\Big]_{-\pi}^{\pi} = \frac{A}{2p}(2p\pi) = A\pi$$

$$A = \frac{Q a^2}{8}$$

$$\text{R.H.S} = \frac{Q a^2}{8}\left\{ \left[\frac{1}{p}\sin p\xi\right]_{-\pi}^{\pi} + \pi \right\} = \frac{Q a^2 \pi}{8}$$

, p must $= 2$

Example:

Consider the boundary value problem

$$\frac{1}{a^2(\cosh^2\eta - \cos^2\psi)}\left(\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\psi^2}\right) = -Q \qquad \text{where } Q \text{ is constant}$$

$$\left.\begin{aligned}\varphi(\eta_0, \psi) &= \Phi_0 \\ \varphi(0, \psi) &= 0\end{aligned}\right\} \text{ boundary conditions.}$$

We have particular solutions of the differential equation of the kind

$$\varphi = \left.\begin{matrix} e^{p\eta} \\ e^{-p\eta}\end{matrix}\right\} \left.\begin{matrix}\sin \\ \cos\end{matrix}\, p\psi\right\} - \frac{Qa^2}{8}(\cosh 2\eta + \cos 2\psi)$$

Because of symmetry with respect to the yz plane, we can see that φ will be an even-valued function of ψ. Therefore the coefficients of $\sin p\psi$ must be zero.

Use the boundary condition $\varphi(0,\psi) = 0$. Because the non-homogeneous part ~~boundary~~ term in the solution arising from the non-homogeneous part involves $\cosh 2\eta$, it appears to be advisable to use hyperbolic, rather than exponential, terms in the particular solution involving $p\eta$. This will give

$$\varphi = (A\cosh p\eta + B\sinh p\eta)\cos p\psi + C - \frac{Qa^2}{8}(\cosh 2\eta + \cos 2\psi)$$

and

$$\varphi(0,\psi) = A\cos p\psi - \frac{Qa^2}{8}(1 + \cos 2\psi) + C = 0$$

$\left\{ \int_{-\pi}^{\pi} \cos px \cos mx \, dx = 0 \text{ unless } p = m \right.$ see over for details.

From the orthogonality properties of the cosine functions $\cos p\psi$ ~~term~~ in the interval π to $(-\pi)$, we can equate the coefficients of ~~cos~~ 2ψ $n\psi$. This gives

for $n = 2$, $p = 2$ ~~and~~, $A = \dfrac{Qa^2}{8}$,

~~and~~ for $n = 0$, $C = \dfrac{Qa^2}{8}$.

-52- $\phi = \left(\frac{Qa^2}{8}\cosh 2\eta + \sum_{p=0}^{\infty} B \sinh p\eta\right)\cos p\psi + \frac{Qa^2}{8} - \frac{Qa^2}{8}(\cosh 2\eta + \cos 2\psi)$

The second boundary condition gives:

$$\phi(\eta_0, \psi) = \phi_0 = \frac{Qa^2}{8}\cosh 2\eta_0 [\cos 2\psi - 1] + B \sinh 2\eta_0 \cos 2\psi - \frac{Qa^2}{8}\cos 2\psi + \frac{Qa^2}{8}$$

Equate coefficients of $\cos n\psi$. For $n = 0$,

$$\phi_0 = -\frac{Qa^2}{8}\cosh 2\eta_0 + \frac{Qa^2}{8}, \quad Q = \frac{8\phi_0}{a^2(1-\cosh 2\eta_0)}$$

For $n = 2$,

$$\frac{Qa^2}{8}\cosh 2\eta_0 + B \sinh 2\eta_0 - \frac{Qa^2}{8} = 0$$

$$B = \frac{Qa^2}{8}(1 - \cosh 2\eta_0)/\sinh 2\eta_0 .$$

Thus

$$\phi = \left[\frac{Qa^2}{8}\cosh 2\eta + \frac{Qa^2}{8}\frac{(1-\cosh 2\eta_0)\sinh 2\eta}{\sinh 2\eta_0}\right]\cos 2\psi + \frac{Qa^2}{8} - \frac{Qa^2}{8}(\cosh 2\eta + \cos 2\psi)$$

$$\cancel{= \frac{Qa^2}{8}\cos 2\psi\left[\cosh 2\eta + \frac{(1-\cosh 2\eta_0)\sinh 2\eta}{\sinh 2\eta_0} - 1\right]}$$

$$\cancel{+ \frac{Qa^2}{8}\cosh 2\eta}$$

$$= \frac{Qa^2}{8}\left\{1 + \cosh 2\eta(\cos 2\psi - 1) + \cos 2\psi\left[\frac{(1-\cosh 2\eta_0)\sinh 2\eta}{\sinh 2\eta_0} - 1\right]\right\}$$

$$\phi = \phi_0 \left\{\frac{1 + \cosh 2\eta(\cos 2\psi - 1) + \cos 2\psi\left[\frac{(1-\cosh 2\eta_0)\sinh 2\eta}{\sinh 2\eta_0} - 1\right]}{1-\cosh 2\eta_0}\right\}$$

Check:

$\phi(0, \psi) = 0$

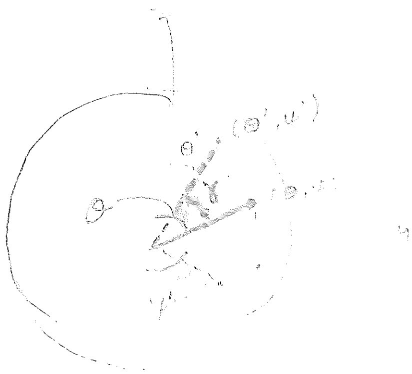
$$\phi(\eta_0, \psi) = \phi_0 = \frac{Qa^2}{8}\left\{\left[\cosh 2\eta_0 + \frac{(1-\cosh 2\eta_0)\sinh 2\eta_0}{\sinh 2\eta_0}\right]\cos 2\psi - \cosh 2\eta_0 - \cos 2\psi + 1\right\}$$

$$= \frac{Qa^2}{8}\left[\cos 2\psi - \cos 2\psi - \cosh 2\eta_0 + 1\right]$$

Addition Theorem of Spherical Harmonics (stated without proof)

See [illegible], Found. of Math Physics, pg [illegible], # 50.

Consider a pair of points on the surface of a unit sphere. Let the



first point have spherical coordinates

$(x,y,z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and

the second point the coordinates

$(x',y',z') = (\sin\theta'\cos\varphi', \sin\theta'\sin\varphi', \cos\theta')$.

The angle between the position vectors of the two points is

$$\cos\gamma = xx' + yy' + zz' = \sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta'.$$

The addition formula for Legendre polynomials is

$$P_n(\cos\gamma) = P_n(\cos\theta)\,P_n(\cos\theta') + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\,P_n^m(\cos\theta)\,P_n^m(\cos\theta')\cos m(\varphi-\varphi')$$

$$r^2 \frac{\partial^2 f}{\partial r^2} + r \frac{\partial f}{\partial r} + Q r^2 + \underbrace{\boxed{r^2 \frac{\partial^2 \Phi}{\partial r^2} + r \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial \theta^2}}}_{= 0 \text{ if } \nabla^2 \Phi = 0} + \frac{\partial^2 f}{\partial \theta^2} = 0$$

The first 3 terms involve only f, r, and Q. We wish to determine f as a function of Q and r. Then set

$$r^2 \frac{d^2 f}{d r^2} + r \frac{d f}{d r} + Q r^2 = 0$$

$$\frac{d^2 f}{d r^2} + \frac{1}{r} \frac{d f}{d r} + Q = 0 \qquad \text{which has solution } f = -\frac{Q r^2}{4}$$

$$\frac{d f}{d r} = -\frac{Q r}{2}, \quad \frac{d^2 f}{d r^2} = -\frac{Q}{2}.$$

Bessel's Equation

Standard form: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$ where n is positive ($n \geq 0$)

For real values of n and finite values (positive or negative) of x, a solution is:

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(\frac{x}{2})^{n+2r}}{r!\,\Gamma(n+r+1)}.$$

This series converges for any value of n and any finite value of x. Both n and x may be complex. (See Stratton, Electromagnetic Theory, pg 357.)

If n does not equal zero or an integer, a second solution is

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(\frac{x}{2})^{-n+2r}}{r!\,\Gamma(-n+r-1)}.$$

For n an integer, $J_{-n}(x) = (-1)^n J_n(x)$ [not linearly independent]

The functions $J_n(x)$ and $J_{-n}(x)$ are linearly independent, provided n does not equal zero or an integer. For n not an integer, a general solution to Bessel's eqn. is

$$y = A J_n(x) + B J_{-n}(x)$$

It can be shown that $J_n(0)$ is finite for $n \geq 0$. Also $J_{-n}(0)$ is finite for n an integer but infinite for n not an integer. (Look at figures on pg. 130, Irving and Mullineux.)

$J_n(x)$ is oscillatory and has an infinite number of zeros. It tends to zero as x tends to infinity. This is also true for $J_{-n}(x)$, for all values of n.

The series expression for $J_n(x)$ above is, considering practical reasons, useless for large x because it converges so slowly. However, it is satisfied formally by

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\left[\left\{1 - \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8x)^2} + \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)(4n^2-7^2)}{4!\,(8x)^4} - \cdots\cdots\right\}\cos\phi\right.$$

$$\left. - \left\{\frac{4n^2-1^2}{8x} - \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{3!\,(8x)^3} + \cdots\cdots\right\}\sin\phi\right]$$

where $\phi = x - (n+\frac{1}{2})\frac{\pi}{2}$ and $|x| \gg 1$, $|x| \gg |n|$.

Irving and Mullineux (pg 131) use the ~ rather than the = sign to indicate that it is an asymptotic expansion. Actually (Stratton, pp 358-359) the series diverges for all values of x. However, if x is very large, the first few terms diminish rapidly in magnitude and in this sense the series is semi-convergent. If the expansions are broken off near or

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before the point at which the successive terms begin to grow larger, they lead to an approximate value of the function and the error incurred can be estimated.

If m is zero or a positive integer, $J_{-m}(x) = (-1)^m J_m(x)$, and $J_{-m}(x)$ is not linearly independent of $J_m(x)$. The additional solution of Bessel's equation is required. It is the Bessel function of the second kind of order m, $Y_m(x)$.

$y = AJ_m(x) + BJ_{-m}(x)$ is no longer a general solution.

$$Y_m(x) = \frac{\cos m\pi \, J_m(x) - J_{-m}(x)}{\sin m\pi}.$$

For any value of m (integral + or −, zero, or non-integral), $Y_m(x)$ is an independent (of $J_m(x)$) solution of Bessel's equation.

In this formula $Y_m(x)$ is indeterminate $\left(\frac{0}{0}\right)$ for $m = 0$ or an integer. For these cases it can be written as (see eq 3.4, pg 134)

Don't write out

$$Y_m(x) = \frac{2}{\pi}\left[\{\ln x - \ln 2 + \gamma\} J_m(x) - \frac{1}{2}\sum_{s=0}^{m-1} \frac{(m-s-1)!}{s!}\left(\frac{x}{2}\right)^{-m+2s} - \frac{1}{2}\sum_{s=0}^{\infty} (-1)^s \frac{(x/2)^{m+2s}}{s!\,(s+m)!}\{\phi(s) + \phi(s+m)\}\right]$$

where $\gamma = \lim_{m\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} - \ln m\right) \simeq 0.577216$

and $\phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p}$ with $\phi(0) = 0$.

Look at fig. on pg 136 (Irv. & Mull.) for $Y_m(x)$. Note that

$Y_m(0) = -\infty$.

The asymptotic form of $Y_m(x)$, which is also "semi-convergent" as the similar one for $J_m(x)$, is given by

$$Y_m(x) \sim \sqrt{\frac{2}{\pi x}}\left[\left\{1 - \frac{(4m^2-1^2)(4m^2-3^2)}{2!\,(8x)^2} + \frac{(4m^2-1^2)(4m^2-3^2)(4m^2-5^2)(4m^2-7^2)}{4!\,(8x)^4} - \cdots\right\}\sin\phi + \left\{\frac{4m^2-1^2}{8x} - \frac{(4m^2-1^2)(4m^2-3^2)(4m^2-5^2)}{3!\,(8x)^3} + \cdots\right\}\cos\phi\right]$$ where $|x| \gg 1$, $|x| \gg |m|$

where $\phi = x - (m + \frac{1}{2})\frac{\pi}{2}$

For $|x| \ggg |m|$,

$$J_m(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{2m+1}{4}\pi\right)$$

$$Y_m(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{2m+1}{4}\pi\right).$$ These are the proper functions for the representation of cylindrical standing waves.

Orthogonality of Bessel's Functions

Bessel's diff. eqn is

$$x^2 y'' + x y' + (k^2 x^2 - m^2) y = 0$$ (2-3) of Irv. & Mull, pg 128

Comparing this with the Sturm-Liouville diff. equation

$$\frac{d}{dz}\left[u(z) \frac{dZ_1}{dz} \right] + \left[v(z) + f(\lambda)\, w(z) \right] Z_1 = 0 .$$

Divide Bessel's eqn by x. Then

$$x y'' + y' + \left(k^2 x - \frac{m^2}{x} \right) y = 0$$

and

$$x y'' + y' = \frac{d}{dx}(x y') .$$ Thus, by comparison,

$$z = x, \quad Z_1 = y, \quad u = x, \quad v = \frac{-m^2}{x}, \quad f(\lambda) = k^2, \quad w = x .$$

(We could also select $v = k^2 x$, $w = -\frac{1}{x}$ and $f(\lambda) = m^2$. This does ~~not~~ not have many application to physical problems, however.)

The solutions to Bessel's eqn will be orthogonal with respect to the weighting function x if

$$\left[x\, y_i \frac{dy_j}{dx} - x\, y_j \frac{dy_i}{dx} \right]_a^b = 0$$

where $y = J_m(kx)$ and $y_i = J_m(\xi_i x)$, $y_j = J_m(\xi_j x)$ ($k = \xi_i, \xi_j$ are eigenvalues)

where the ξ_i, ξ_j are eigenvalues and the y_i, y_j are eigenfunctions.

For Bessel functions the interval is customarily taken as $(0, a)$.

Thus $\left[x \left(y_i \frac{dy_j}{dx} - y_j \frac{dy_i}{dx} \right) \right]_0^a = 0$. [The $k = \xi_i$ are roots of this equation.]

which ~~can be written as~~ is equivalent to

$$y(a) + h \left(\frac{dy}{dx} \right)_{x=a} = 0$$

or $J_m(ka) + hk J_m'(ka) = 0$, h being a constant

(Irv. & Mull. eq (8-6), pg 155)

Norm, $N_{n,i}$.

The norm is defined as

$$N_{n,i} = \int_0^a x\left[J_n(\xi_i x)\right]^2 dx .$$ To evaluate it, consider the indefinite integral

$$\int x\left[J_n(kx)\right]^2 dx$$ where k is an arbitrary constant.

Now $J_n(kx)$ is one of the solutions of Bessel's diff. eqn. Multiply this diff. eqn. by $2y'$, to obtain

$$2x^2 y' y'' + 2xy'^2 + 2(k^2x^2 - n^2)yy' = 0$$

which can be written as

$$\frac{d}{dx}\left(x^2y'^2 - n^2y^2 + k^2x^2y^2\right) - 2k^2xy^2 = 0 .$$

Integrating with respect to x,

$$2k^2\int xy^2\,dx = x^2y'^2 - n^2y^2 + k^2x^2y^2 .$$

But $y = J_n(x)$. Then the integral above can be written as

$$\int x\left[J_n(kx)\right]^2 dx = \frac{x^2}{2}\left[\left(1 - \frac{n^2}{k^2x^2}\right)\{J_n(kx)\}^2 + \{J_n'(kx)\}^2\right]$$

where $y' \equiv \frac{dJ_n(kx)}{dx}$ and $J_n'(kx) \equiv \frac{dJ_n(kx)}{d(kx)} = \frac{y'}{k}$.

Using recurrence relations, one can show

$$\int x\left[J_n(kx)\right]^2 dx = \frac{x^2}{2}\left[J_n^2(kx) - J_{n-1}(kx)J_{n+1}(kx)\right].$$

Going to the definite integral,

$$\int_0^a x\{J_n^2(kx)\}\,dx = \frac{a^2}{2}\left[J_n^2(ka) - J_{n-1}(ka)J_{n+1}(ka)\right]$$

and also $$= \frac{a^2}{2}\left[\left(1 - \frac{n^2}{k^2a^2}\right)J_n^2(ka) + \{J_n'(ka)\}^2\right]$$

a) If $k = \xi_i$ is a root of $h J_n(ak) + k J_n'(ak) = 0$,

then $$\int_0^a x J_n^2(\xi_i x)\,dx = \frac{a^2}{2}\left[\left(1 - \frac{n^2}{a^2\xi_i^2}\right)J_n^2(\xi_i a) + \frac{h^2}{\xi_i^2}J_n^2(\xi_i a)\right]$$

~~Sign differs from Carr + Miller eq (8.12) pg 136~~

$$\int_0^a x J_n^2(\xi_i x)\,dx = \frac{a^2}{2\xi_i^2} J_n^2(\xi_i a)\left[\xi_i^2 - \frac{n^2}{a^2} + k^2\right)$$

b) If $k = \xi_i$ is a root of $J_n(ak) = 0$, then

$$\int_0^a x J_n^2(\xi_i x)\,dx = -\frac{a^2}{2} J_{n-1}(\xi_i a) J_{n+1}(\xi_i a) = \frac{a^2}{2}\{J_n'(\xi_i a)\}^2$$

$$= +\frac{a^2}{2} J_{n-1}^2(\xi_i a)$$

or

$$= +\frac{a^2}{2} J_{n+1}^2(\xi_i a)$$

from recurrence relations

<u>Expansion of $f(x)$ in terms of $J_n(\xi_i x)$</u>

Let $f(x) = \sum_{i=1}^{\infty} A_i J_n(\xi_i x)$.

Multiply both sides by $x J_n(\xi_j x)$ and integrate with respect to x between 0 and a. This gives

$$\int_0^a x f(x) J_n(\xi_j x)\,dx = \int_0^a \sum_{i=1}^{\infty} A_i x J_n(\xi_i x) J_n(\xi_j x)\,dx.$$

Integrate the series on the right term by term. By the orthogonality properties of the Bessel function, all ~~terms~~ integrals except

$$\int_0^a A_j x J_n^2(\xi_j x)\,dx$$

are zero.

c) If the ξ_j are roots of $J_n(a\xi) = 0$, then the integral equals

$$\frac{A_j a^2}{2} J_{n-1}^2(\xi_j a),$$

and

$$A_i = \frac{2}{a^2}\frac{\int_0^a x f(x) J_n(\xi_i x)\,dx}{\{J_n'(\xi_i a)\}^2}.$$

Thus

$$f(x) = \frac{2}{a^2}\sum_{i=1}^{\infty}\left[\left\{\int_0^a x f(x) J_n(\xi_i x)\,dx\right\} \Big/ \left\{J_n'(\xi_i a)\right\}^2\right] J_n(\xi_i x) \qquad (\mathrm{I})$$

57 If the ξ_j are roots of $h J_n(ak) + k J_n'(ak) = 0$, then the integral equals

$$\int_0^a A_j x J_n^2(\xi_j x)\,dx = \frac{A_j a^2}{2\xi_j^2} J_n^2(\xi_j a)\left[\xi_i^2 + h^2 - \frac{n^2}{a^2}\right]$$

and

$$A_i = \frac{2\xi_i^2 \int_0^a x f(x) J_n(\xi_i x)\,dx}{a^2 J_n^2(\xi_i a)\left[\xi_i^2 + h^2 - \frac{n^2}{a^2}\right]} .$$

Thus

$$f(x) = \frac{2}{a^2}\sum_{i=1}^{\infty} \frac{\xi_i^2 \left\{\int_0^a x f(x) J_n(\xi_i x)\,dx\right\}}{\left\{h^2 + \left(\xi_i^2 - \frac{n^2}{a^2}\right)\right\}\left\{J_n^2(\xi_i a)\right\}} J_n(\xi_i x) \qquad \text{II}.$$

The function $\int_0^a x f(x) J_n(\xi_i x)\,dx \equiv g(\xi_i)$ is called the finite Hankel transform of $f(x)$. Eqs. (I) and (II) are called the inversion theorems for the finite Hankel transform. The ~~valuation~~ evaluation of the integrals in (I) and (II) requires integrating expressions of the form

$$\int_0^a x f(x) J_n(\xi_i x)\,dx .$$

These can be integrated in closed form for only a few functions $f(x)$. In the majority of cases one must resort to numerical integration.

Integrals involving Bessel functions —

see Ince + Abell, pp 157-173.

Modified Bessel Functions

The equation $$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+n^2)y = 0$$

can be put in the form of Bessel's diff. eqn $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2-n^2)y = 0$ by letting $x = -it$. The independent solutions of the diff. eqn. are

$$J_n(t) \text{ and } Y_n(t)$$

where $$J_n(t) = J_n(ix) \quad\text{and}\quad Y_n(t) = Y_n(ix).$$

$J_n(ix)$ and $Y_n(ix)$ are not always real solutions. Therefore it is convenient to define real solutions, which are called modified Bessel functions of the first and second kind, as

$$I_n(x) = i^{-n} J_n(ix) = \sum_{r=0}^{\infty} \frac{(x/2)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

$$K_n(x) = \frac{\pi}{2}\left(\frac{I_{-n}(x) - I_n(x)}{\sin n\pi}\right).$$

$I_n(x)$ and $K_n(x)$ are not oscillating (see pg 3-11, fig 1&3 I&M, and $K_n(x) \to 0$ as $x \to \infty$. Asymptotic $K_n(x) \to \infty$ as $x \to 0$

expansions are:

$$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left[1 - \frac{4n^2-1}{1!\,(8x)} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8x)^2} - \cdots\right] \quad |x| \gg 1,\ |x| \gg n$$

$$K_n(x) \sim \sqrt{\frac{\pi}{2x}}\,e^{-x}\left[1 + \frac{4n^2-1^2}{1!\,(8x)} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8x)^2} + \cdots\right]$$

Because the modified Bessel functions have no real roots except the origin, there is no general orthogonality property.
(Collins, Math. Meth. for Phys. & Eng., pg. 82)

Hankel Functions (Bessel Functions of Third Kind)

Definitions:

$$H_n^{(1)}(x) = J_n(x) + i\,Y_n(x)$$

$$H_n^{(2)}(x) = J_n(x) - i\,Y_n(x).$$

These functions are complex for a real argument. The Hankel functions are important in physical problems owing to the fact that among all the Bessel functions they are the only which vanish for an infinite complex argument of the Bessel or cylindrical fn, that is,

$\lim_{z\to\infty} H_n^{(1)}(z) = 0$

$$\lim_{r\to\infty} H_n^{(1)}(re^{i\theta}) = 0 \qquad \text{if } 0 \le \theta \le \pi.$$

$$\lim_{r\to\infty} H_n^{(2)}(re^{-i\theta}) = 0$$

Their asymptotic forms are:

$$H_n^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)}\left[1 - \frac{4n^2-1^2}{1!\,(8ix)} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8ix)^2} - \cdots\right]$$

$$H_n^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{-i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)}\left[1 + \frac{4n^2-1^2}{1!\,(8ix)} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8ix)^2} + \cdots\right]$$

$$|x| \gg |n|$$

Multiplication of the asymptotic expansions for $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ by $e^{\pm i\omega t}$ gives <u>traveling waves</u>. Oftentimes, only the first term in the expansion is retained.

Consider for lecture material pp 158–174 of Irv. & Mull.

Complex Potential — 2 Dimensional Problems

We shall denote Ω, a complex function of the complex variable z, by

$$\Omega \equiv f(z) \equiv \phi + i\psi ,$$

where ϕ and ψ are real functions of x, y. We shall assume that $\Omega = f(z)$ is an analytic function.

Because Ω is assumed analytic, the Cauchy-Riemann conditions are satisfied. This gives

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} .$$

Thus

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial x \partial y} = 0 .$$

We conclude that both the real, ϕ, and imaginary part, ψ, of Ω satisfy Laplace's equation in two dimensions.

The functions $\phi(x,y)$ and $\psi(x,y)$ are conjugate harmonic functions i.e., the curves ϕ = constant always intersect the curves ψ = constant orthogonally. [A conjugate harmonic function is not the same as the complex conjugate $\phi - i\psi$.] For proof, see pg 62a

Consider the curves $\varphi = \text{constant} = C$ and $\psi = \text{constant} = K$.

On the curve $\varphi = C$, $d\varphi = 0 = \dfrac{\partial \varphi}{\partial x}dx + \dfrac{\partial \varphi}{\partial y}dy$

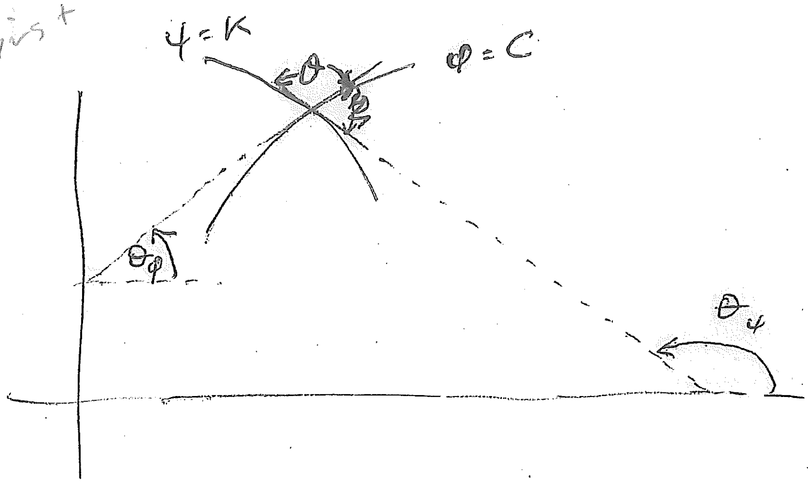
and on the curve $\psi = K$, $d\psi = 0 = \dfrac{\partial \psi}{\partial x}dx + \dfrac{\partial \psi}{\partial y}dy$.

do this second

Then

$$\left(\frac{dy}{dx}\right)_{\varphi = c} = \frac{-\dfrac{\partial \varphi}{\partial x}}{\dfrac{\partial \varphi}{\partial y}} \quad \text{and} \quad \left(\frac{dy}{dx}\right)_{\psi = K} = \frac{-\dfrac{\partial \psi}{\partial x}}{\dfrac{\partial \psi}{\partial y}} \qquad (3)$$

Do this first



$$\left.\begin{aligned} \left(\frac{dy}{dx}\right)_{\varphi = C} &= \tan \theta_\varphi \\ \left(\frac{dy}{dx}\right)_{\psi = K} &= \tan \theta_\psi \end{aligned}\right\} \quad (1)$$

Let θ be the angle between the tangents to the two curves at their point of intersection. Then

$$\tan \theta = \tan(\theta_\psi - \theta_\varphi) = \frac{\tan \theta_\psi - \tan \theta_\varphi}{1 + \tan \theta_\psi \tan \theta_\varphi}$$

From (1)

$$\tan \theta = \frac{\left(\frac{dy}{dx}\right)_{\psi = K} - \left(\frac{dy}{dx}\right)_{\varphi = C}}{1 + \left(\frac{dy}{dx}\right)_{\psi = K}\left(\frac{dy}{dx}\right)_{\varphi = C}} \qquad (2)$$

(see above)

Substituting (3) into (2),

$$\cancel{\tan\theta =} \qquad \tan \theta = \frac{-\dfrac{\partial \psi / \partial x}{\partial \psi / \partial y} + \dfrac{\partial \varphi / \partial x}{\partial \varphi / \partial y}}{1 + \left(\dfrac{\partial \psi / \partial x}{\partial \psi / \partial y} \cdot \dfrac{\partial \varphi / \partial x}{\partial \varphi / \partial y}\right)} = \frac{-\dfrac{\partial \psi/\partial x}{\partial \psi/\partial y} + \dfrac{\partial \varphi/\partial x}{\partial \varphi/\partial y}}{1 + \left(\dfrac{-\partial \varphi/\partial y}{\partial \varphi/\partial x} \cdot \dfrac{\partial \varphi/\partial x}{\partial \varphi/\partial y}\right)}$$

by Cauchy-Riemann conditions

Thus the denominator equals zero, and $\tan \theta = \infty$, or $\theta = \frac{\pi}{2}$.

Q.E.D.

The complex potential arises in many branches of physics. For examples, discuss table I, pg 443 of Irving & Mullineux. Most of the examples in the book are from hydrodynamics (ideal fluid), but they can be ~~the~~ interpreted directly in terms of other fields.

~~In hydrodynamics, not every~~

We might note that every analytic complex function is not of interest in hydrodynamics. ~~The~~ The only values of Ω which are applicable are those for which the flow at large distances from the origin approximate to a uniform stream (i.e. the velocity $\vec{V}$ is constant at large distances from the origin). ϕ, the velocity potential, and ψ, the stream function, satisfy the equations

$$V_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{(the second equality comes from the Cauchy-Riemann)}$$

$$V_y = \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x}.$$

(V), V_x, V_y constant)

If the stream is uniform,

$$\frac{d\Omega}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = V_x - iV_y$$

$= \text{constant}$ because both V_x and V_y are assumed constant.

In this case, Ω is a linear function of z, as can be seen by integrating $\frac{d\Omega}{dz} = \text{constant}$.

We can ~~there~~ see that for $\Omega = az$ the stream is uniform everywhere, and in particular at large z. We shall further see that if Ω, in the general case, is to approximate to a uniform stream at infinity, then

$$\Omega = Az + B\ln z + \sum_{m=1}^{\infty} a_m/z^m \quad \text{where } A, B, a_m \text{ are complex numbers.}$$

This is the most general expression for the complex potential of a velocity field in which the velocity is uniform at infinity.

5.1 Uniform Stream

Let $\Omega = V_0 e^{-i\alpha} z$ where V_0 and α are real constants.

Then $\Omega = V_0 (\cos\alpha - i \sin\alpha)(x + iy)$

and $\phi = V_0 (x \cos\alpha + y \sin\alpha)$

$\psi = V_0 (y \cos\alpha - x \sin\alpha)$.

Therefore $V_x = \dfrac{\partial\phi}{\partial x} = V_0 \cos\alpha$

$V_y = \dfrac{\partial\phi}{\partial y} = V_0 \sin\alpha$, and V_x, V_y are constant.

Thus $\Omega = V_0 e^{-i\alpha} z$ is the complex potential for a uniform flow with velocity V_0 inclined at an angle α to the x axis. We take the angle α to be the principal value of the argument (or amplitude) of $\dfrac{d\Omega}{dz}$. That is, $-\pi \le \alpha \le \pi$.

If $\alpha = 0$ (special case), then

$$\Omega = V_0 z$$

is the complex potential for a uniform stream with velocity V_0 in the positive x-direction.

5.2 Source, Sink, and Vortex

Let $\Omega = m \ln z$ where m is a real constant. Then

$$\Omega = m \ln r e^{i\theta} = m \ln r + m \ln e^{i\theta}$$

$$= m \ln r + \ln e^{im\theta}$$

$$= m \ln r + i m\theta .$$

Thus $\phi = m \ln r$, $\psi = m\theta$.

The streamlines (ψ = constant) are straight lines passing through the origin. The equipotential lines (ϕ = constant) are a family of concentric circles with center at the origin.

The velocity is tangent to the streamline at any point. The stream function ψ has the property that the rate of flow across any curve C is given by

$$\psi_C = \int_C d\psi .$$

In particular, if we take C to be a closed curve containing the origin and if $\phi = m\theta$, then

$$\psi_C = \int_0^{2\pi} m \, d\theta = 2\pi m .$$

Thus if m is positive we see that fluid is emanating from the origin (it emanates from the origin because C can be <u>any</u> curve containing the origin) at a rate equal to $2\pi m$. We thus have a <u>point source</u> at the origin. If m is negative, the origin is a <u>point sink</u>. For either case m is called the <u>strength</u> (of the source or sink.)

Suppose that we next consider $\int_C d\phi$, which is defined as the circulation around the curve C. For our special example, the circulation is

$$\int_C m \, d(\ln r)$$

which equals zero for m real and constant and C any closed curve.

#PP But next consider $\Omega = -i\kappa \ln z$, where κ is real and constant.

Then $\Omega = -i\kappa \left[\ln(r \cdot e^{i\theta})\right] = -i\kappa\left[\ln r + i\theta\right] = \kappa\theta - i\kappa \ln r$,

and

$$\phi = \kappa\theta , \quad \psi = -\kappa \ln r .$$

The equipotential lines, $\phi =$ constant, are straight lines passing through the origin. The stream lines, $\psi =$ constant, are concentric circles with centers at the origin. The rate of flow across the closed curve C is now given by

$$\oint_C d\varphi = \oint_C -\kappa \, d(\ln r) = 0$$

because the value of $\ln r$ at the same point (beginning and end point of path of integration) is the same.

Emphasize 2-D character of source and sink

However, the circulation is

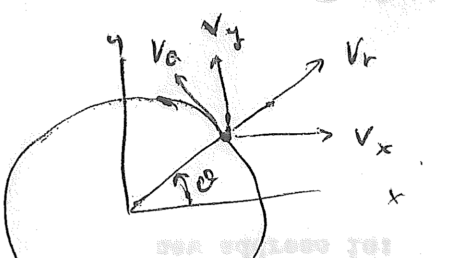
$$\oint_C d\phi = \oint_C \kappa \, d\theta = 2\pi\kappa.$$

This type of flow is produced by a ~~point~~ vortex of strength κ located at the origin. The velocity is tangent to the streamline so the velocity is always normal to a radial line drawn from the point ~~vortex~~ vortex to the point in question. This can also be shown in the following way:

$$\phi = \kappa \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{where } z = x + iy$$

and
$$V_x = \frac{\partial\phi}{\partial x} = \kappa\left[\frac{\frac{d(y/x)}{dx}}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{-\kappa y}{x^2+y^2} = -\frac{\kappa \sin\theta}{r}$$

$$V_y = \frac{\partial\phi}{\partial y} = \frac{\kappa x}{x^2+y^2} = \frac{\kappa\cos\theta}{r}.$$



Now from the figure
$$V_r = V_x \cos\theta + V_y \sin\theta$$
and
$$V_\theta = V_y \cos\theta - V_x \sin\theta.$$

Thus for our problem

$$V_r = \frac{-\kappa\sin\theta}{r}\cos\theta + \frac{\kappa\cos\theta}{r}\sin\theta = 0$$

$$V_\theta = \frac{\kappa\cos\theta}{r}\cos\theta - \frac{-\kappa\sin\theta}{r}\sin\theta = \frac{\kappa}{r}. \qquad \text{Q.E.D.}$$

We could also have written V_θ in polar coordinates as $\frac{1}{r}\frac{\partial\phi}{\partial\theta}$, which immediately gives $V_\theta = \frac{\kappa}{r}$. Also $V_r = \frac{\partial\phi}{\partial r}$ gives $V_r = 0$.

In summary, if a source (sink) of strength m is located at the point $z = a$, then the complex potential is $\Omega = m \ln(z-a)$. The flow is radial from the point $z = a$. If a vortex of strength κ is located at $z = a$, the complex potential is $\Omega = -i\kappa \ln(z-a)$. The streamlines are concentric circles with center at $z = a$.

Example 1. Source and uniform stream

Adding the expressions for the complex potential for a source and a uniform stream, we have

$$\Omega = m \ln(z-a) + V_0 z .$$

This corresponds physically to a point source of strength m located at $z = a$ placed in a uniform velocity field given by $V_0 x$.

Assume a is real. Then

$$\Omega = m \ln\sqrt{(x-a)^2+y^2} + V_0 x + i\left[m \tan^{-1}\frac{y}{x-a} + V_0 y\right]$$

and

$$\phi = m \ln\sqrt{(x-a)^2+y^2} + V_0 x$$

$$\psi = m \tan^{-1}\left(\frac{y}{x-a}\right) + V_0 y .$$

The velocity components are

$$V_x = \frac{\partial\phi}{\partial x} = V_0 + \frac{m(x-a)}{(x-a)^2+y^2}$$

$$V_y = \frac{\partial\phi}{\partial y} = \frac{my}{(x-a)^2+y^2} .$$

The streamlines are the curves

$$\psi = V_0 y + m \tan^{-1}\left(\frac{y}{x-a}\right) = \text{constant} .$$

The particular streamline on which $\psi = 0$ is

$$V_0 y = -m \tan^{-1}\left(\frac{y}{\ }\right)$$

$$V_0\, y = -m \tan^{-1}\left(\frac{y}{x-a}\right)$$

$$y = -\frac{1}{k}\tan^{-1}\left(\frac{y}{x-a}\right)$$

$$\frac{-y}{x-a} = \tan ky \qquad \text{or} \qquad x-a = \frac{-y}{\tan ky}$$

Let $x = a$. Then $0 = \dfrac{-y}{\tan ky}$ which requires that $y = \pm\dfrac{\pi}{2k}$.

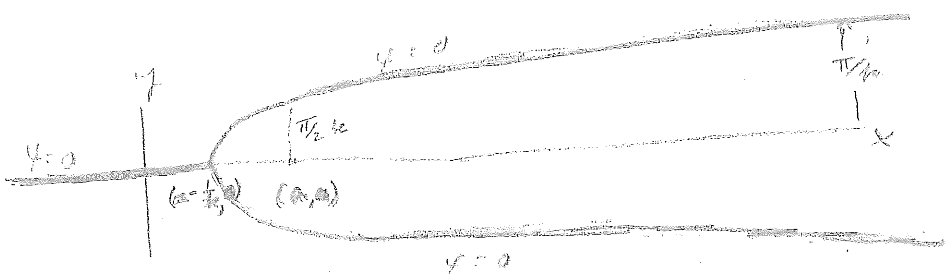
Note that $y = 0$ does not satisfy this equation, because $\lim\limits_{y\to 0}\dfrac{y}{\tan ky} \neq 0$.

$x - a = \dfrac{-ky}{k\tan ky} \to -\dfrac{1}{k}$ as $y \to 0$

$\therefore x = a - \frac{1}{k}$ for $y = 0$

Let $x = a - \frac{1}{k}$. Then $-\dfrac{1}{k} = \dfrac{-y}{\tan ky} = \dfrac{-ky}{k\tan ky} = -\dfrac{1}{k}$ if $y = 0$.

Let $x \to \infty$. Then $\dfrac{-y}{\tan ky} \to \infty$ as $y \to \dfrac{\pi}{k}$ or $-\dfrac{\pi}{k}$.



Consider $x < a - \frac{1}{k}$. Then from (5.24)

$$\frac{y}{(a-\frac{1}{k}-A)-a} + \tan ky = 0 \qquad \text{where } A > 0$$

or

$$\frac{-y}{\frac{1}{k}+A} + \tan ky = 0$$

$$y = (\tfrac{1}{k} + A)\tan ky = \frac{A'}{k}\tan ky \qquad \text{where } A' = 1 + Ak$$

which is satisfied only if $y = 0$.

If $k \equiv \frac{V_0}{m}$, the eqn. becomes

$$ky = -\cancel{A}\tan^{-1}\left(\frac{y}{x-a}\right) \quad \text{or} \quad \frac{y}{x-a} + \tan ky = 0 \quad . \qquad (5.2.4)$$

If $y \neq 0$, this last equation can be written as

$$\frac{ky}{\tan ky} = k(a-x) \ . \qquad (5.2.5)$$

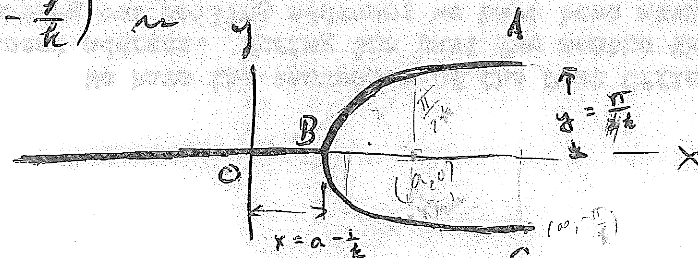
Because $\frac{\theta}{\tan\theta} < 1$ ~~implies that~~ ~~for all real values of θ,~~ for all real values of θ,

~~by~~ (5.2.5) implies that non-zero values of y exist for the streamline $\psi = 0$ if

$$k(a-x) < 1 \quad \text{or} \quad x > \left(a - \frac{1}{k}\right).$$

$$\begin{cases} a - x < \frac{1}{k} \\ a - x + x < \frac{1}{k} + x \\ a < \frac{1}{k} + x \\ a - \frac{1}{k} < \frac{1}{k} - \frac{1}{k} + x \\ x > \left(a - \frac{1}{k}\right) \end{cases}$$

A curve showing the streamline $\psi = 0$ for $x > \left(a - \frac{1}{k}\right)$ is



When $y = \pm\frac{\pi}{k}$, $\frac{y}{x-a} + \tan ky = 0$ gives $\frac{\frac{\pi}{k}}{x-a} + 0 = 0$ or $x = \infty$

When $x = a - \frac{1}{k}$, $\frac{y}{x-a} + \tan ky = 0$ gives $\frac{y}{-\frac{1}{k}} + \tan ky = 0$ or $-ky = \tan ky$, $y = 0$

When y is negative, both terms in $\frac{y}{x-a} + \tan ky = 0$ change sign.

Therefore the curve corresponding to $\psi = 0$ is symmetric with respect to x-axis

If $x < a - \frac{1}{k}$, we have from (5.2.5) that

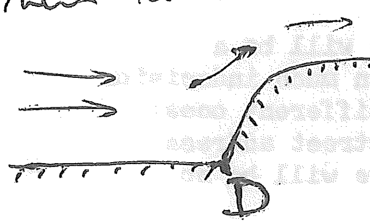
$$\frac{ky}{\tan ky} > 1 \quad \text{which is impossible for real } ky.$$

Therefore (5.2.4) represents the line $y = 0$ if $x < a - \frac{1}{k}$.

Problem (a) If the streamline whose equation is given by (5.2.4) for $x > a - \frac{1}{k}$ is replaced by a rigid boundary the flow could be interpreted as an approximation to the flow past an airplane wing. The choice of ~~two~~ values for the constants a and m will determine how good an approximation this is to the wing.

Problem (b). — Flow of air over a cliff —

Replace the streamline $y = 0$ to the left of B by a rigid boundary. Also replace the streamline BA by a rigid boundary. Then we have an approximation of flow of air over a cliff.



At the particular point

$$\begin{cases} x = a - \frac{1}{k} \\ y = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = a - m/V_0 \\ y = 0 \end{cases}$$

$$V_x = V_0 + \frac{m(x-a)}{(x-a)^2 + y^2} = V_0 + \frac{m(a - m/V_0 - a)}{(a - \frac{m}{V_0} - a)^2 + \cancel{y^2}} = V_0 - \frac{m^2/V_0}{m^2/V_0^2} = 0$$

$$V_y = \frac{my}{(x-a)^2 + y^2} = 0$$

A point **D** where the stream velocity is zero is called a <u>stagnation point</u>.

Example 2. Source and Source.

Let two sources of equal strength be separated by a distance $2a$. Arbitrarily set up an x-y coordinate system such that their coordinates are $(\pm a, 0)$. The complex potential for the system is

$$\Omega = m \ln (z-a) + m \ln (z+a) = m \ln (z^2 - a^2)$$

$$= m \ln [x^2 - y^2 - a^2 + 2ixy] = m \ln R + i m \theta$$

where $R = \sqrt{(x^2-y^2-a^2)^2 + 4x^2y^2}$, $\theta = \tan^{-1} \dfrac{2xy}{x^2-y^2-a^2}$.

Thus

$$\phi = Real(\Omega) = \frac{m}{2} \ln \{ (x^2-y^2-a^2)^2 + 4x^2y^2 \}$$

$$\psi = Im(\Omega) = m \tan^{-1} \frac{2xy}{x^2-y^2-a^2} .$$

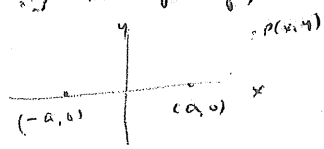
The components of the velocity ~~potential~~ are

$$V_x = \frac{\partial \phi}{\partial x} = \frac{m}{2} \frac{2(x^2-y^2-a^2)\cdot 2x + 8xy^2}{[(x^2-y^2-a^2)^2 + 4x^2y^2]} = 2mx \left[\frac{x^2-y^2-a^2+2y^2}{x^4+2x^2y^2+y^4-2x^2a^2+2a^2y^2+a^4} \right]$$

$(x^2-y^2-a^2)^2+4x^2y^2$

$$= 2mx \left\{ \frac{x^2+y^2-a^2}{[(x^2-y^2-a^2)^2 + 4x^2y^2]} \right\}$$

[This expression is correct, and eq. (5.2.7) in Irving & Mullineux is incorrect.] As a proof, use $V_x = \frac{m}{r} \cos(r,x)$ for a source, and add for the 2 sources. This gives



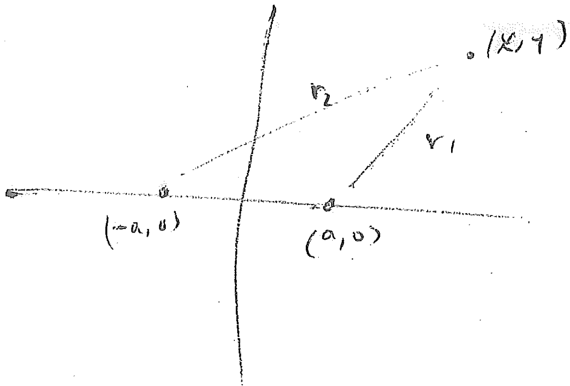
$$V_x = m \left\{ \frac{x-a}{(x-a)^2+y^2} + \frac{(x+a)}{(x+a)^2+y^2} \right\}.$$ This expression is equal to the one above.

$$V_y = \frac{\partial \phi}{\partial y} = \frac{m}{2} \frac{2(x^2-y^2-a^2)(-2y) + 8x^2y}{[(x^2-y^2-a^2)^2 + 4x^2y^2]} = +2my \left\{ \frac{x^2+y^2+a^2}{[(x^2-y^2-a^2)^2+4x^2y^2]} \right\},$$

Consider the streamline $\psi = 0$. From the expression for ψ above, this tells us that $x = 0$ can correspond to this streamline. If $x = 0$,

$(V_x)_{x=0} = 0$ and $(V_y)_{x=0} = \dfrac{2my}{y^2+a^2}$. If we replace the streamline by a rigid boundary, then the effect of the boundary on the flow due to a point source at (a, 0) only is the same as if there were sources of equal strength at both (a,0) and ... could speak of the imaginary source at $(-a,0)$ as an

70 a



$$V_x = \frac{-m(a-x)}{r_1^2} + \frac{m(a+x)}{r_2^2} = m\left\{ \frac{-(a-x)}{(a-x)^2+y^2} + \frac{(a+x)}{(a+x)^2+y^2} \right\}$$

$a^2+2ax+x^2+y^2$
$a+x$
$a^3 - a^2x - ax^2 + ay^2 + x^3 + xy^2$

$a^2+2ax+x^2+y^2$
$a-x$
$a^3 + a^2x - ax^2 + ay^2 - x^3 - xy^2$

$(a-x)^2+y^2$
$(a+x)^2+y^2$

$(a-x)^2(a+x)^2 + y^2[(a+x)^2 + (a-x)^2] + y^4$

$= (a^2-2ax+x^2)(a^2+2ax+x^2) + y^2[a^2+\cancel{2ax}+x^2+a^2-\cancel{2ax}+x^2] + y^4$

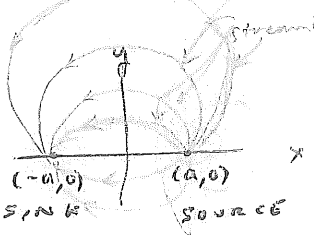
$a^2+2ax+x^2$

$$\frac{2a^2x - \cancel{2ax^2} - 2x^3 - 2xy^2}{a^4 - \cancel{2a^3x} - 2a^2x^2 + \cancel{2ax^3} + x^4 + 2a^2y^2 + 2x^2y^2 + y^4}$$

$= 2x(-x^2 + a^2 + y^2)$

Example 3 Source and Sink.

Let the source and sink ~~b~~ have numerically the same magnitude, and be separated by a distance $2a$. Let the x axis pass through the source and sink, and place the origin midway between them.



The complex potential is

$$\Omega = m \ln(z-a) - m \ln(z+a) = m \ln\left(\frac{z-a}{z+a}\right).$$

The potential is

$$\phi = m \ln\left|\frac{z-a}{z+a}\right|$$

and the stream function is

$$\psi = m \arg\left(\frac{z-a}{z+a}\right).$$

One can show that the streamlines are portions of non-concentric circles with all the centers lying on the y-axis. The lines of constant potential are non-concentric circles, with centers on the x-axis. The centers of the circles ~~fall on the same $y=0$~~ lie between $+a$ and $+\infty$ or $-a$ and $-\infty$.

In electrostatics the source and sink are line charges, of linear charge density Q. Let the medium surrounding the source and sink have a dielectric constant of Q. Then one can show that

$$m = \frac{2Q}{K},$$

so that

$$\phi = \frac{2Q}{K} \ln\left|\frac{z-a}{z+a}\right|, \quad \psi = \frac{2Q}{K} \arg\left(\frac{z-a}{z+a}\right).$$

Irving and Mullineux give expressions for the centers and radius of curvature of the ~~streamlines~~ lines of force and the equipotentials. They apply to the fluid field as well as to the electrostatic field.

5.3 Doublet or Dipole

For the case of the source and sink whose strengths have the same magnitude, let the distance $2a \to 0$ and the strength $m \to \infty$ so that $2ma \to \mu$, a finite limit.

The complex potential for the doublet is

$$\Omega = \lim_{\substack{a \to 0 \\ m \to \infty \\ 2ma \to \mu}} \left[m \ln \frac{z-a}{z+a} \right] = \lim_{\substack{a \to 0 \\ m \to \infty \\ 2ma \to \mu}} \left[m \ln \left(\frac{1-a/z}{1+a/z} \right) \right] \text{ if } z \neq 0 .$$

The series expansion of $\ln x$ for $x > 0$ is

$$\ln x = 2\left[\left(\frac{x-1}{x+1} \right) + \frac{1}{3} \left(\frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left(\frac{x-1}{x+1} \right)^5 + \cdots \right]$$

Thus

$$\ln \left(\frac{1-a/z}{1+a/z} \right) = 2\left[\left(\frac{\frac{1-\frac{a}{z}}{1+a/z} - 1}{\frac{1-a/z}{1+a/z} + 1} \right) + \frac{1}{3} \left(\quad \right)^3 + \cdots \right]$$

$$= 2 \left[\left(-\frac{a}{z} \right) + \frac{1}{3} \left(-\frac{a}{z} \right)^3 + \frac{1}{5} \left(-\frac{a}{z} \right)^5 + \cdots \right]$$

$$= -2 \left[\frac{a}{z} + \frac{1}{3} \left(\frac{a}{z} \right)^3 + \cdots \right]$$

Proceeding to the limit,

$$\Omega = -2m \, a/z = -\mu/z .$$

This is the complex potential for a two dimensional dipole or doublet.

5.4 Uniform flow + doublet + vortex. Flow past a cylinder.

Example 4. Find the complex potential for the combination of a uniform flow with velocity V_0 in the negative x-direction, a doublet at the origin, and a point vortex at the origin.

Since superposition is valid,

$$\Omega = -V_0 z - \mu/z - i\kappa \ln z .$$

Write $z = re^{i\theta}$. Then

$$\Omega = -V_0 r e^{i\theta} - \mu/r \, e^{-i\theta} - i\kappa(\ln r + i\theta)$$

and

$$\psi = Im(\Omega) = -V_0 r \sin\theta + \frac{\mu}{r}\sin\theta - \kappa \ln r .$$

Consider a particular streamline, $\psi = \psi_a \equiv -\kappa \ln a$. If, in addition, we define $\mu = V_0 a^2$, then

$$\psi_a = -\kappa \ln a = -V_0 r \sin\theta + \frac{V_0 a^2 \sin\theta}{r} - \kappa \ln r .$$

If we put $r = |z| = a$ in the equation above, then the only variable is θ. Thus the coefficients of $\sin\theta = 0$ in that equation, and

$$\psi_a = -\kappa \ln a .$$

Write $\mu = V_0 a^2$. Then

This is not an assumption, ~~because~~ μ and V_0 are real, and the equation serves to define the value of a^2 in terms of given quantities; i.e., $\mu/V_0 \equiv a^2$

$$\psi = -V_0 r \sin\theta + \frac{V_0 a^2}{r}\sin\theta - \kappa \ln r .$$

Let ψ_a be the value of ψ for $|z| = a$. Then

$$\psi_a = V_0 a \sin\theta(-1+1) - \kappa \ln a = -\kappa \ln a .$$

Thus the stream function ψ is constant on the circle $|z| = a$. We may therefore replace the streamline $\psi = \psi_a$ by a rigid circular cylinder whose equation is $|z| = a$. Therefore Ω is the complex potential for a flow past an infinite circular cylinder situated

Let us next find the force exerted on this rigid circular cylinder placed in the velocity field (given by Ω) of this ideal fluid. An ideal fluid can support no shearing stresses, so the stress at any point of the cylinder will be ~~normal~~ to its surface. The stress (pressure), p, and velocity, V, are related by Bernoulli's equation, namely

$$p + \rho \frac{V^2}{2} = h \qquad (V \text{ is velocity at any point in fluid})$$

where ρ is the fluid density (a constant) and h is also a constant.

Let $\vec{F}$ be the total force per unit length acting in the direction of the outward normal to the surface. Then the force on a strip of unit length subtending an angle $d\theta$ at the axis of the cylinder is

$$d\vec{F} = -p\, a\, d\theta\, \vec{e}_r = \left\{ \rho \frac{V^2}{2} - h \right\} \vec{e}_r\, a\, d\theta .$$

[This should really be a vector equation.]

In order to find F by integrating the equation above, we must express V as a function of θ. Now

$$V_x = \frac{\partial \phi}{\partial x}, \quad V_y = \frac{\partial \phi}{\partial y} \quad \text{where } \phi = -V_0 x + \kappa \tan^{-1}\left(\frac{y}{x}\right) - \mu x/(x^2+y^2)$$

and $\mu = V_0 a^2$.

Then

$$V_x = -V_0 - \frac{\kappa y}{x^2+y^2} - V_0 a^2 \left\{ \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} \right\}$$

$$V_y = \frac{\kappa x}{x^2+y^2} + \frac{2 V_0 a^2 x y}{(x^2+y^2)^2} .$$

On the surface of the cylinder $a^2 = x^2+y^2$, $\tan\theta = \frac{y}{x}$, and thus

$$(V_x)_{|z|=a} = -V_0 - \frac{\kappa \sin\theta}{a} - V_0 + 2V_0 \cos^2\theta = -\sin\theta \left(2V_0 \sin\theta + \frac{\kappa}{a}\right)$$

$$(V_y)_{|z|=a} = \frac{\kappa \cos\theta}{a} + 2V_0 \sin\theta \cos\theta = \cos\theta \left(2V_0 \sin\theta + \frac{\kappa}{a}\right) .$$

Therefore

$$(V^2)_{|z|=a} = (V_x^2 + V_y^2)_{|z|=a} = \left(2V_0 \sin\theta + \frac{\kappa}{a}\right)^2 .$$

We can substitute this expression for V^2 in terms of θ in the equation for dF above.

On the surface of the cylinder we thus have

$$(F_x)_{|z|=a} = \int_C \cos\theta \, dF = \frac{\rho}{2}\int_0^{2\pi} (2V_0 \sin\theta + \frac{K}{a})^2 \cos\theta \, a \, d\theta - h\int_0^{2\pi} \cos\theta \, a \, d\theta$$

$$= \left[\frac{1}{2}\rho a \left(\frac{4V_0^2}{3}\sin^3\theta - \frac{4V_0 K}{a}\frac{\sin^2\theta}{2} + \frac{K^2}{a^2}\sin\theta\right) - ah\sin\theta\right]_0^{2\pi} = 0$$

and

$$(F_y)_{|z|=a} = \int_C \sin\theta \, dF = \frac{\rho a}{2}\int_0^{2\pi} (2V_0\sin\theta + \frac{K}{a})^2 \sin\theta \, d\theta - ah\int_0^{2\pi}\sin\theta \, d\theta$$

$$= \frac{\rho a}{2}\int_0^{2\pi}\left\{4V_0^2\sin^3\theta + \frac{2V_0K}{a}(1-\cos 2\theta) + (\frac{K}{a^2} - 2\rho h)\sin\theta\right\} d\theta$$

$$= \frac{\rho a}{2}\left[4V_0^2\left\{-\frac{1}{3}\cos\theta(\sin^2\theta + 2)\right\} + \frac{2V_0K}{a}\left\{\theta - \frac{1}{2}\sin 2\theta\right\} + (\frac{K}{a^2} - 2\rho h)\cos\theta\right]_0^{2\pi}$$

$(F_y)_{|z|=a} = 2\pi\rho V_0 K$. This result is known as the Kutta-Joukowski law

It is ~~important~~ in aerodynamics, and gives ~~the~~ an expression for the lift on a structure such as an airplane wing.

Let us return to the expression for the fluid velocity, and locate the stagnation points, i.e., those points at which $V=0$.

$$V_r = V_x\cos\theta + V_y\sin\theta = \cos\theta\left\{-V_0 - \frac{K\sin\theta}{r} - V_0a^2\left[\frac{1}{r^2} - \frac{2\cos^2\theta}{r^2}\right]\right\}$$

$$+ \sin\theta\left\{\frac{K\cos\theta}{r} + \frac{2V_0a^2\sin\theta\cos\theta}{r^2}\right\} = -V_0\cos\theta\left(1 - \frac{a^2}{r^2}\right)$$

$$V_\theta = V_y\cos\theta - V_x\sin\theta = \cos\theta\left\{\frac{K\cos\theta}{r} + \frac{2V_0a^2\sin\theta\cos\theta}{r^2}\right\}$$

$$- \sin\theta\left\{-V_0 - \frac{K\sin\theta}{r} - V_0a^2\left[\frac{1}{r^2} - \frac{2\cos^2\theta}{r^2}\right]\right\}$$

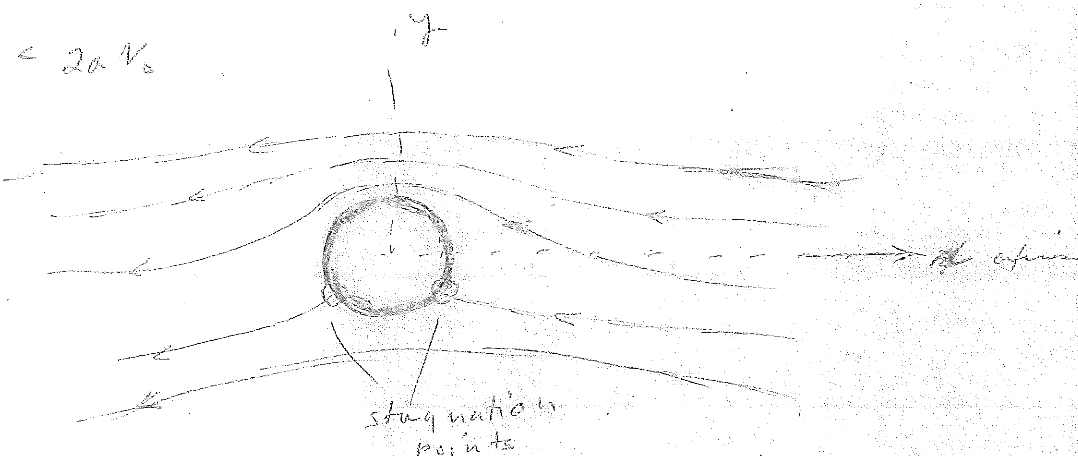
$$V_\theta = \frac{K}{r} + V_0\sin\theta\left(1 + \frac{a^2}{r^2}\right).$$

On the cylinder $r=a$, so $(V_r)_{r=a} = 0$ and $(V_\theta)_{r=a} = \frac{K}{a} + 2V_0\sin\theta$.

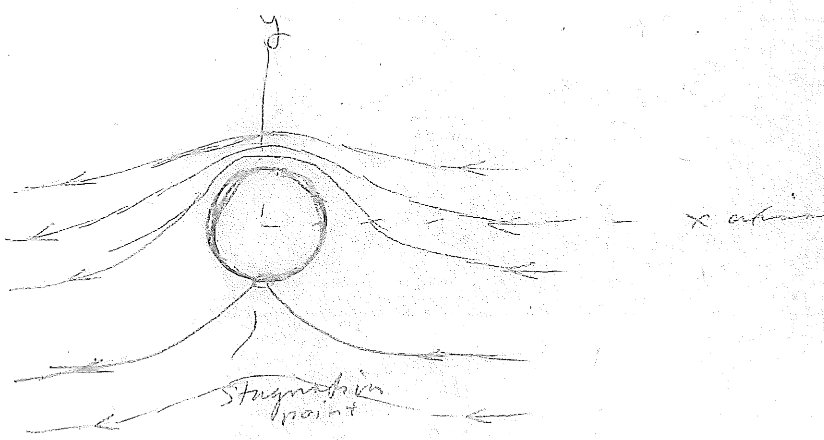
Therefore stagnation points occur on the cylinder if $\theta = \sin^{-1}\left(\frac{K}{2aV_0}\right)$.

If $K < 2aV_0$, there are two stagnation points on the surface of cylinder. If $K = 2aV_0$, there is one stagnation point on the cylindrical surface, and if $K > 2aV_0$ there are no stagnation points on this surface.

a) $K < 2aV_0$



b) $K = 2aV_0$



c) $K > 2aV_0$

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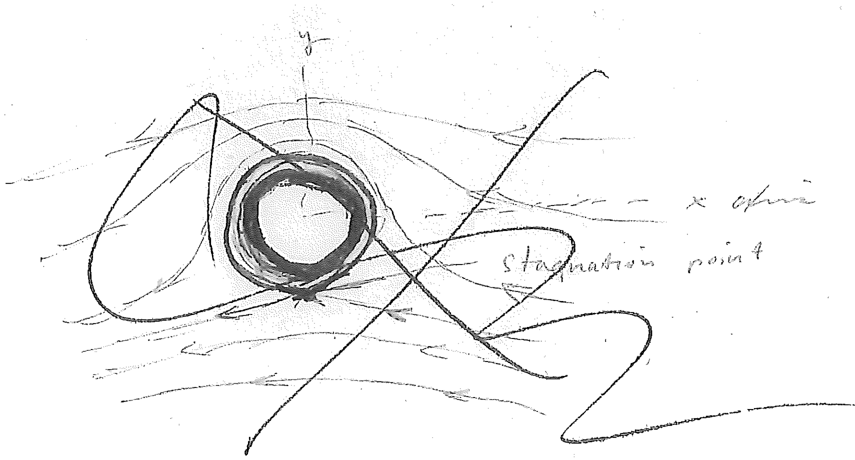
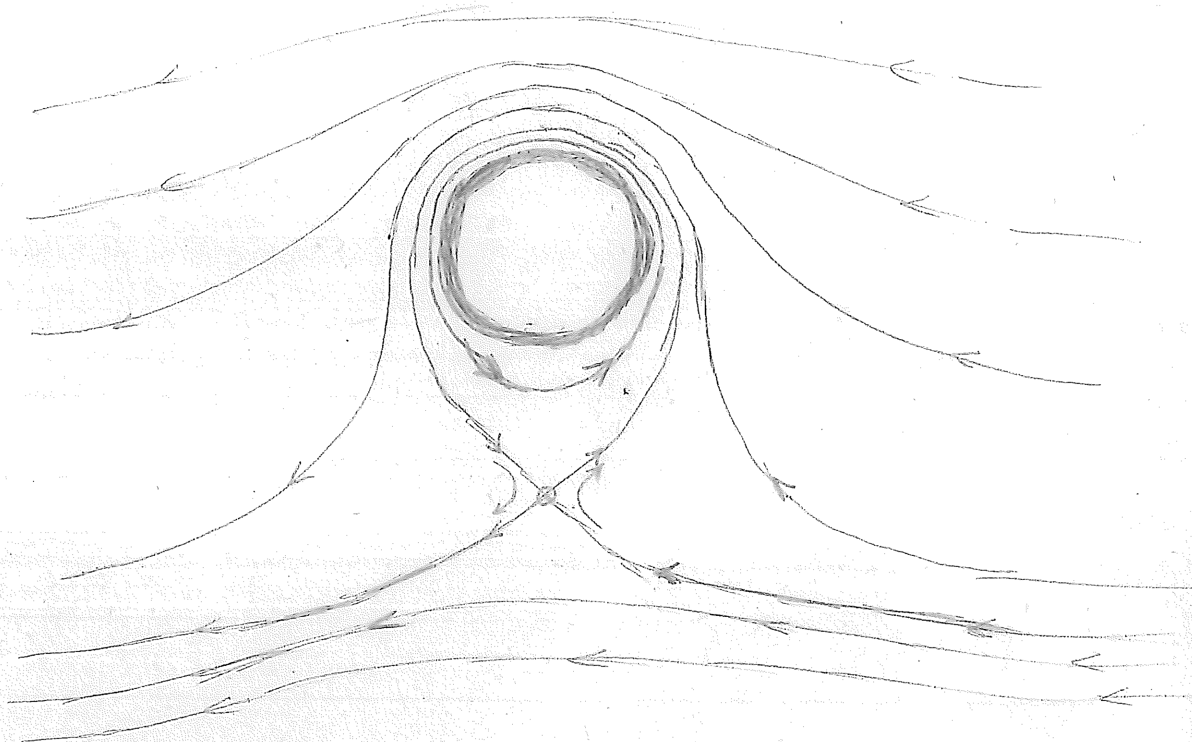


c) $K > 2aV_0$

o = stagnation point



6. Conformal Transformation

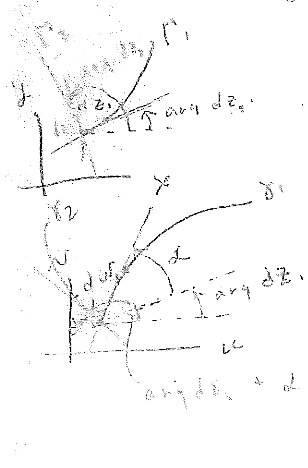
The equation $w = f(z)$ enables us to map a locus in the z-plane into a locus in the w-plane. We shall assume that the function $f(z)$ is analytic. It is often possible to find such a transformation in which a complicated curve maps into a simple one. This is useful in solving ~~boundary~~ value problems in 2-dimensions where the boundary curves can be transformed into simple ones. One solves the boundary value problem in the transformed variables, and then simply takes the inverse to obtain the solution of the original problem.

ensure that Laplace's eqn is satisfied in the transformed variables.

Another approach is to solve a simple boundary value problem in the z plane, and then ~~make~~ some transformation into the w-plane which gives a different boundary-value problem, whose solution is ~~already~~ then known. ~~Thus by~~ This process we can construct complicated boundary value problems with known solutions from very simple problems. In some cases the more difficult problems have a physical interpretation which is a real break. The properties of various transformations, together with experience and intuition, can be very important.

Suppose that we put ~~$w = f(z)$~~ $\frac{dw}{dz} = f'(z)$ equal to $a\,e^{i\alpha}$, where $a = |f'(z)|$ ($|dw/dz|$) is not ~~neither~~ zero ~~nor infinite~~. Equate the moduli and the arguments of the expression above, namely $dw = a e^{i\alpha} dz$. This gives

$$|dw| = a\,|dz| \quad , \arg dw = \alpha + \arg dz \tag{6-3}$$



~~arg that~~

where the second equation uses the fact that the argument of the product of two complex functions is the sum of their arguments (see Morse & Mott, 1993)

Let the locus of $P(z)$ be indicated by Γ, and the corresponding locus of $Q(w)$ by δ. Then the eq. (6.3) tell us:

1) an infinitesimal arc of γ is "a" times an infinitesimal arc of Γ, or, in other words, the magnification introduced by the transformation is a.

2) the arc dw is turned through the angle α in a positive sense, but there is no increase or decrease in the angle itself. This corresponds to a rigid body rotation of amount α. Thus in the transformation angles are preserved in magnitude and sense. This is a conformal transformation, in which infinitesimal figures in the z-plane transform to similar, but reoriented, figures in the w-plane.

~~The transformation is not conformal at points of Γ where (say z_1)~~

$$~~|f'(z_1)| = 0 \text{ or infinity.}~~$$

~~These are called critical points of the transformation. At these critical points no single-valued inverse of $w = f(z)$ exists.~~

The analytic function $w = f(z) = u + iv$ has a single-valued inverse (i.e., x and y can be found as single-valued functions of u and v) only if the Jacobian of the functions u and v does not vanish or become infinite. [See, e.g., Goursat-Hedrick, Mathematical Analysis, vol I, pg. 45.] The Jacobian of the transformation is defined as

$$\frac{\partial(u,v)}{\partial(x,y)} \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

From the Cauchy-Riemann conditions, ~~since~~ (w is assumed analytic)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2.$$

Therefore if $|f'(z)|^2 \neq$ zero (or infinity) at a point (x_0, y_0), a single-valued inverse $z = F(w)$ exists at that point. The points at which $|f'(z)|^2 = 0$ or ∞ are called ~~critical points of the transformation~~. At these ~~points~~ points the mapping is not conformal, because at them w is not a single-valued function of z. ~~them~~.

Suppose that $\phi(x,y)$ is a solution of Laplace's equn. in two-dimensions, i.e., $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$. If we make the conformal transformation $w=u+iv=f(z)\equiv f(x+iy)$, then it is easy to show that $\nabla^2_{uv}\phi=0$. Thus

$$\frac{\partial^2\phi}{\partial x^2}=\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\phi}{\partial u^2}+\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2\phi}{\partial v^2}+2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2\phi}{\partial u\partial v}+\frac{\partial^2 u}{\partial x^2}\frac{\partial\phi}{\partial u}+\frac{\partial^2 v}{\partial x^2}\frac{\partial\phi}{\partial v}$$

$$\frac{\partial^2\phi}{\partial y^2}=\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2\phi}{\partial u^2}+\left(\frac{\partial v}{\partial y}\right)^2\frac{\partial^2\phi}{\partial v^2}+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2\phi}{\partial u\partial v}+\frac{\partial^2 u}{\partial y^2}\frac{\partial\phi}{\partial u}+\frac{\partial^2 v}{\partial y^2}\frac{\partial\phi}{\partial v}$$

Using the Cauchy-Riemann conditions that $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{\partial^2\phi}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]+\frac{\partial^2\phi}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]+2\frac{\partial^2\phi}{\partial u\partial v}\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\right]$$

$$+\frac{\partial\phi}{\partial u}\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right].$$

But $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=|f'(z)|^2$. Thus

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=|f'(z)|^2\left(\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}\right)=0\text{, or}$$

$$\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}=0\text{, provided that } |f'(z)|^2\neq 0.$$

Transformation of source, sink, and Doublet

Let us look at the conformal transformation of the complex potential for (1) a source or sink (2) a ~~vortex~~ doublet.

Consider first the complex potential of a source of strength m located at $z = a$. It is

$$\Omega = m \ln (z - a).$$

Let the transformation between the z and w-planes be given by $z = F(w)$

If $w = b$ is the point corresponding to $z = a$, then

$$a = F(b)$$

and

$$F'(b) = \lim_{w \to b} \left[\frac{F(w) - F(b)}{w - b} \right] = \lim_{w \to b} \left[\frac{z - a}{w - b} \right].$$

This may be rewritten as

$$z - a = (w - b) \{ F'(b) + \epsilon \} \quad \text{where } \epsilon \to 0 \text{ as } w \to b.$$

Therefore the complex potential in the w plane in the neighborhood of $z = b$ is

$$\Omega = m \ln (z - a) = m \ln \left[(w - b) \{ F'(b) + \epsilon \} \right]$$

$$= m \ln (w - b) + m \ln \{ F'(b) + \epsilon \}.$$

The last term, $\{F'(b) + \epsilon\}$, is a constant. Therefore, the complex potential in the w plane is due to a source of strength m at $w = b$. ~~Therefore~~ We can conclude that the transform of a source of strength m at $z = a$ is a source of the same strength at $w = b$. [The term $\{F'(b) + \epsilon\}$ contributes nothing to the velocity field, because it is a constant and therefore its gradient equals zero.]

A similar result holds for the sink.

Consider next the complex potential of a doublet of strength μ at $z=a$. If $w=b$ corresponds to $z=a$, then in the neighborhood of $z=b$

$$\Omega = \frac{-\mu}{(w-b)\{F'(b)+\epsilon\}}$$

which can be rewritten as

$$\Omega = \frac{-\mu}{(w-b)\{F'(b)\}+(w-b)\epsilon} = \frac{-\mu}{(w-b)F'(b)\left[1+\frac{\epsilon}{F'(b)}\right]}$$

Expand $\frac{1}{1+\frac{\epsilon}{F'(b)}}$ in a series. This gives

$$\frac{1}{1+\frac{\epsilon}{F'(b)}} = 1-\frac{\epsilon}{F'(b)}+\left[\frac{\epsilon}{F'(b)}\right]^2-\left[\frac{\epsilon}{F'(b)}\right]^3+\cdots\cdots$$

Therefore

$$\Omega = \frac{-\mu}{|F'(b)|}\frac{e^{-i\beta}}{(w-b)} + \text{ terms which are constant and very small in the neighborhood of } w=b,$$

when $F'(b) = |F'(b)|e^{i\beta}$.

Therefore the transformation of a doublet of strength μ at $z=a$ is a doublet of strength $\frac{\mu}{|F'(b)|}$ at $w=b$. The axis of the transformed doublet is rotated in a negative (clockwise) sense through an angle β.

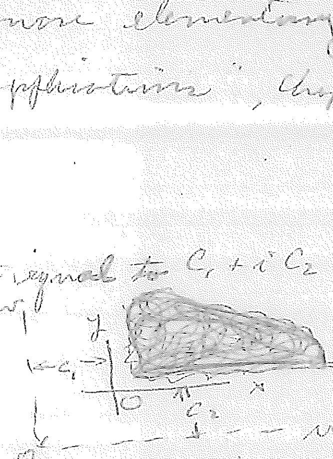
-53- Before discussing some of the more general transformations presented in Irving and Mullineux, let us look at a few more elementary ones given in Churchill "Intro. to Complex Variable and Applications", Chap 4.

1. Linear functions

Consider $w = z + C$, where C is a complex constant equal to $C_1 + iC_2$.

Thus $w = u + iv = x + iy + C_1 + iC_2$ gives

$u = x + C_1$, $v = y + C_2$.



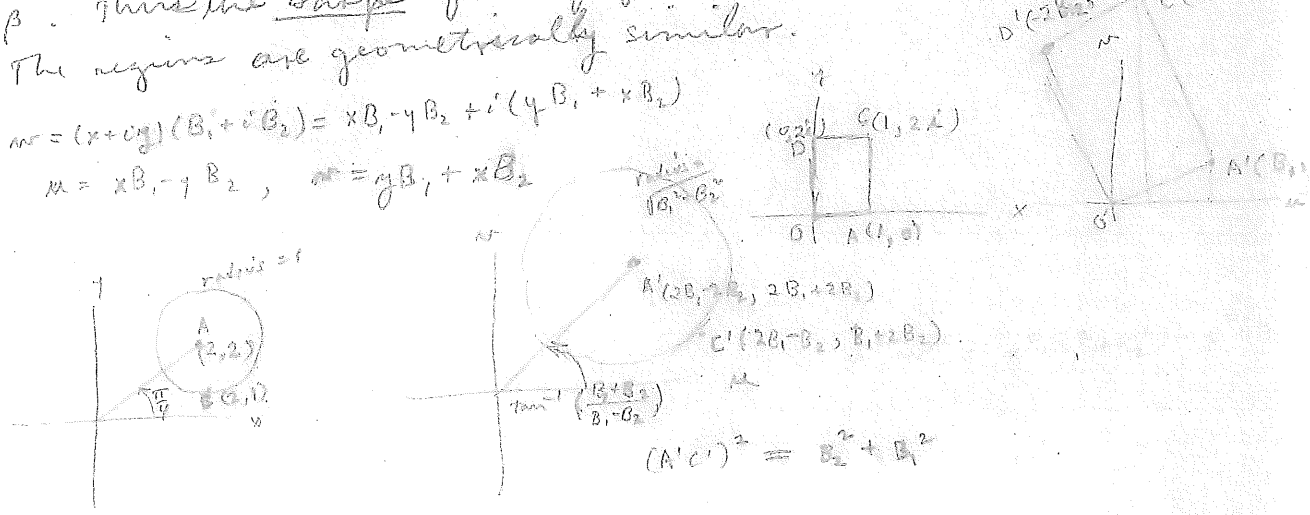
This amounts to a simple translation. That is, a region in the z-plane will have the same shape, size, and orientation in the w-plane.

Next let $w = Bz$ where B is a complex constant equal to $be^{i\beta}$.

Then if $z = re^{i\theta}$, $w = Bz = bre^{i(\theta+\beta)}$. This mapping consists of an expansion or contraction by a multiplier $b = |B|$, and a rotation through an angle β. Thus the shape of the figure is preserved, but its area is not. The regions are geometrically similar.

$w = (x+iy)(B_1+iB_2) = xB_1 - yB_2 + i(yB_1 + xB_2)$

$u = xB_1 - yB_2$, $v = yB_1 + xB_2$



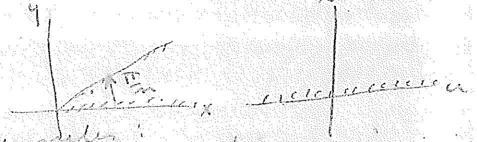
$(A'C')^2 = B_2^2 + B_1^2$

$f(z) = u + iv = ix - y$, $u = -y$, $v = x$

The transformation $w = Bz + C$ consists of a rotation through the argument of B, a magnification of $|B|$, and a translation of C. This transformation is called the general linear transformation. As an example, consider the mapping of the infinite strip between $x = 0$ and $x = 1$ under the transformation $w = iz = ze^{i\pi/2}$. The magnification is unity, and the region is rotated through an angle $\pi/2$. Thus in the w-plane the region becomes the strip bounded by $v = 0$ and $v = 1$.

2. <u>Powers of z. $w = z^n$</u>

Under this transformation, $z = re^{i\theta}$ becomes $w = r^n e^{in\theta}$. If we write $w = \rho e^{i\phi}$, then $\rho = r^n$ and $\phi = n\theta$. Thus there is both a magnification and a rotation. If $n > 1$ and $0 \le \theta \le 2\pi$, then w is multi-valued. The angular region $0 \le \theta \le \frac{\pi}{n}$ (n integral) is transformed into the half-plane $0 \le \phi \le \pi$.

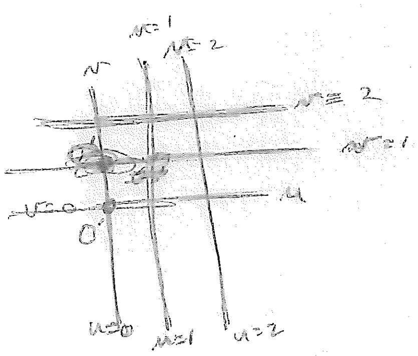


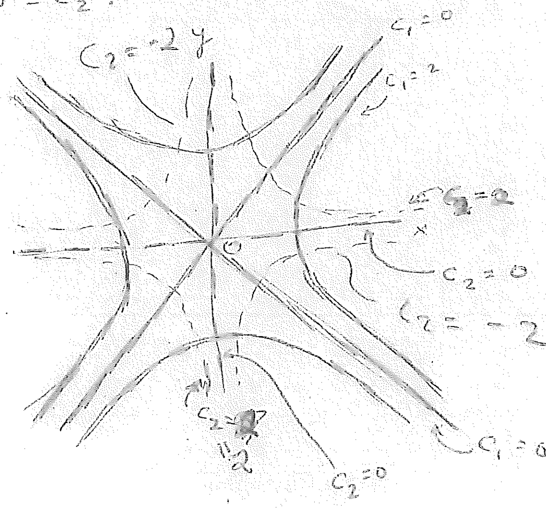
Examples: wedge problems in geophysics — dipping layers.

Consider as an example $w = z^2$. In rectangular coordinates

$$w = u + iv = x^2 - y^2 + 2ixy.$$

The hyperbolas $x^2 - y^2 = c_1$ and $2xy = c_2$ map into the vertical lines $u = c_1$ and the horizontal lines $v = c_2$.





Note that $|f'(z)| = 0$ at $z = 0$.

3. <u>Function $w = \frac{1}{z}$.</u>

In polar coordinates $w = \frac{1}{z}$ becomes $\rho e^{i\phi} = \frac{1}{r} e^{-i\theta}$.

In cartesian coordinates it has the form

$$w = u + iy = \frac{1}{x + iy}, \quad u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}, \quad x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}.$$

Now if a, b, c, d are real numbers, the equation

$$a(x^2 + y^2) + bx + cy + d = 0$$

represents any circle or line, depending on whether $a \ne 0$ or $a = 0$.

Under the transformation $w = 1/z$, the equation above becomes

$$a\left[\frac{u^2+v^2}{(u^2+v^2)^2}\right] + \frac{b\,u}{u^2+v^2} + \frac{c\,(-v)}{u^2+v^2} + d = 0$$

or

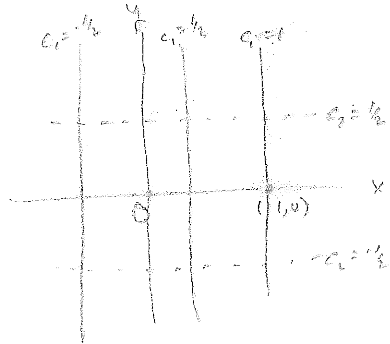
$$d\,(u^2+v^2) - c\,v + b\,u + a = 0\,.$$

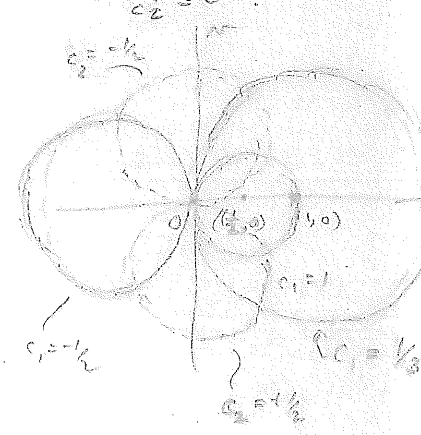
Therefore if a and d are different from zero, both the curve in the z plane and its map in the w plane are circles. We conclude that circles not passing through the point $z=0$ transform into circles not passing through $w=0$.

~~In this limiting case of a circle equalling a~~

A straight line, which is the limiting case of a circle, will transform into a circle ~~in~~ going from the z to the w plane or from the w plane into the z plane. As an example, consider the lines $x = c_1$ which transform into the circles $u^2+v^2-\frac{u}{c_1}=0$. $\left[\begin{array}{l} x=c_1,\ y=0 \\ x=c_1=\frac{u}{u^2+v^2}\end{array}\right]$

Likewise the ~~points~~ lines $y=c_2$ transform to the circles $u^2+v^2+\frac{v}{c_2}=0$.

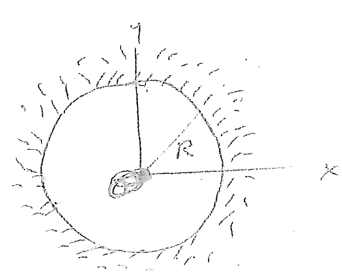


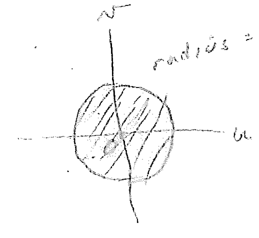


Note that $|f'(z)| = \infty$ at $z=0$.

Now let us return to the polar form $w = \rho e^{i\varphi} = \frac{1}{r} e^{-i\theta}$.

The points in the z plane exterior to the circle $r=R$ map into points interior to the circle $\rho = \frac{1}{R}$. The map of the point 0 in the w plane is the circle $R=\infty$ in the z plane.





6 ✓

6.1. Bilinear (Möbius or linear fractional) transformation.

The bilinear transformation is of the type

$$(6.1.1) \qquad w = \frac{az+b}{cz+d}$$ where a, b, c, d are complex constants and $c \neq 0$ ($c=0$ would reduce it to a simple linear transformation)

It is called bilinear because if we write it in the form

$$Awz + Bz + Cw + D = 0,$$

the equation is linear in z alone or in w alone, but it is not linear in zw.

Eq. (6.1.1) can be rewritten as $w = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}$ because

$$\frac{a}{c} + \frac{bc-ad}{c(cz+d)} = \frac{acz+ad+bc-ad}{c(cz+d)} = \frac{az+b}{cz+d}.$$ In order for the transformation to be non-trivial, $bc-ad \neq 0$.

The bilinear transformation has in general two fixed points, that is, points where $z=w$. They can be found by substituting z for w in (6.1.1), which gives

$$cz^2 + z(d-a) - b = 0$$

and $$z = \frac{-(d-a) \pm \sqrt{(d-a)^2+4bc}}{2c}.$$ In the case where $\sqrt{(d-a)^2+4bc} = 0$, there is one fixed point.

From eq. (6.1.1), we can write (if $z_1 \neq z_2$):

$$\frac{w-w_1}{w-w_2} = \frac{\frac{az+b}{cz+d} - \frac{az_1+b}{cz_1+d}}{\frac{az+b}{cz+d} - \frac{az_2+b}{cz_2+d}} = \frac{\frac{(cz_1+d)(az+b) - (cz+d)(az_1+b)}{(cz+d)(cz_1+d)}}{\frac{(cz_2+d)(az+b) - (cz+d)(az_2+b)}{(cz+d)(cz_2+d)}}$$

or

$$\frac{w-w_1}{w-w_2} = \frac{(cz_2+d)}{(cz_1+d)}\left(\frac{z-z_1}{z-z_2}\right) = K\left(\frac{z-z_1}{z-z_2}\right) \quad \text{where } K \equiv +\left(\frac{cz_2+d}{cz_1+d}\right).$$

$(zz_2)(ad-bc)$

~~We can write~~ $\left(\frac{z-z_1}{z-z_2}\right) = \frac{1}{K}\left(\frac{w-w_1}{w-w_2}\right)$

~~and~~ $\left|\frac{z-z_1}{z-z_2}\right| = \lambda$ ~~where λ is a constant, because~~

Consider a circle Γ in the z-plane. Its equation may be written as

$$\left|\frac{z-z_1}{z-z_2}\right| = \lambda \quad \text{where } \lambda \text{ is a constant.} \quad [\lambda = 1 \text{ corresponds to a straight line}]$$

The corresponding curve γ in the w-plane is

$$\left|\frac{w-w_1}{w-w_2}\right| = \lambda |K| \quad \text{represents a circle if } \lambda|K| \neq 1 \text{ and a straight line if } \lambda|K| = 1.$$

This follows because $\left|\frac{z-z_1}{z-z_2}\right| = \left|\frac{1}{K}\right| \left|\frac{w-w_1}{w-w_2}\right| = \lambda$ if one has a circle Γ in the z-plane.

Therefore, the bilinear transformation maps circles into circles or straight lines.

From eq. (6.1.1.) we ~~can~~ see that w is specified for a given z, if the ratio of $a:b:c:d$ is given. Three independent complex numbers are required to specify the ratio. Therefore, a bilinear transformation can be found which transforms any 3 points in the z plane into three given points in the w-plane.

Example of the preceding idea:

Let $z_1 = 1$, $z_2 = 0$, $z_3 = -1$, $w_1 = i$, $w_2 = 1$, $w_3 = \infty$.

If we would multiply ~~out~~ the terms in the equation

$$(z-z_3)(w-w_1)(z_2-z_1)(w_2-w_3) = (z-z_1)(w-w_3)(z_2-z_3)(w_2-w_1),$$

it would take the form of the bilinear transformation.

Then

$$(z+1)(w-i)(0-1)(1-\infty) = (z-1)(w-\infty)(0+1)(1-i).$$

The term $w_3 = \infty$ gives us trouble. Let us rewrite the transform equation, replacing w_3 by $\frac{1}{w_3'}$. Then

$$(z-z_3)(w-w_1)(z_2-z_1)\left(w_2 - \frac{1}{w_3'}\right) = (z-z_1)\left(w - \frac{1}{w_3'}\right)(z_2-z_3)(w_2-w_1)$$

or

$$(z-z_3)(w-w_1)(z_2-z_1)(w_2 w_3' - 1) = (z-z_1)(w w_3' - 1)(z_2-z_3)(w_2-w_1)$$

... and more thorough method

- 88 -

Substituting in the numerical value again,

$$(z+1)(w-i)(0-1)(0-1) = (z-1)(0-1)(0+1)(1-i)$$

or $$w - i = \frac{(z-1)(-1)(1-i)}{(z+1)(-1)(-1)} = \frac{-(z-1)(1-i)}{z+1}.$$

This gives as our desired transformation

$$w = \frac{(z+1)i - (z-1)(1-i)}{z+1} = \frac{zi + i - z + 1 + zi - i}{z+1} = \frac{1 + z(2i-1)}{z+1}$$

Irving & Mullineux

Example 6. Find the most general bilinear transformation which maps the upper-half z-plane into the interior of the unit circle in the w-plane.

~~Write~~ the transformation will be of the ~~in~~ ~~general~~ form $w = \frac{az+b}{cz+d}$.

The region ~~Im(z)~~ $\text{Im}(z) = y > 0$ is to be transformed into the interior of the unit circle $|w| = 1$.

~~The line $y = 0$ must map into a circle because the region in the w-plane is of finite extent.~~

Let us first make use of the fact that the boundary $y = 0$ corresponds to the boundary $|w| = 1$. Therefore

$$|w| = 1 = \left|\frac{ax+b}{cx+d}\right| = \left|\frac{a}{c}\right| \left|\frac{x + b/a}{x + d/c}\right|$$ because $y = 0$.

Everywhere on the boundary ~~of the~~ curve $|w| = 1$. Therefore, in particular,

$\lim_{x \to \infty} |w| = 1 = \left|\frac{a}{c}\right|$. Therefore $\frac{a}{c} = e^{i\sigma}$ where σ is a real constant.

Because $\left|\frac{a}{c}\right| = 1$ for all values of x on $y = 0$, we have that on this line $\left|x + \frac{b}{a}\right| = \left|x + \frac{d}{c}\right|$. (6.1.5) If we now consider the particular value $x = 0$, we have $\left|\frac{b}{a}\right| = \left|\frac{d}{c}\right|$, which can be expressed as $\frac{d}{c} = \rho e^{i(\beta+\gamma)}$ (6.1.6) if we write $\frac{b}{a} = \rho e^{i\beta}$.

Square (6.1.5) and use (6.1.6). This gives

The square of the ~~modulus~~ of $x + \frac{b}{a}$.

$$(x + \rho\cos\beta)^2 + \rho^2\sin^2\beta = \{x + \rho\cos(\beta+\gamma)\}^2 + \rho^2\sin^2(\beta+\gamma)$$

which simplifies to $\cos\beta = \cos(\beta+\gamma)$ if $x \neq 0$, $\rho \neq 0$. This comes from ~~eq~~ doing the algebra indicated in the equation above.

Then $\rho = 0$ if $x = 0$

This tells us that

1) $\beta \equiv \beta + \delta$ or simply $\delta = 0$

or 2) $\beta + \delta = 2\pi - \beta$.

If (2) is the case, then ~~both~~ $\frac{b}{a}$ and $\frac{d}{c}$ are conjugate complex ~~complex conjugate~~ numbers since ~~their arguments are not zero~~.

$$\frac{b}{a} = \rho e^{i\beta} = \rho(\cos\beta + i\sin\beta)$$

and
$$\frac{d}{c} = \rho e^{i(2\pi-\beta)} = \rho\left[\cos(2\pi-\beta) + i\sin(2\pi-\beta)\right]$$
$$= \rho\left[\cos\beta + i(-\sin\beta)\right].$$

refers to bottom of preceding page

Because x can take on all values, we can say in general that $x \neq 0$. If $\rho = 0$ then $|\frac{b}{a}| = |\frac{d}{c}| = 0$ and our transformation would be the trivial one $w = a/c$. Finally if $\delta = 0$ the transformation would again be the trivial one $w = \frac{a}{c}$.

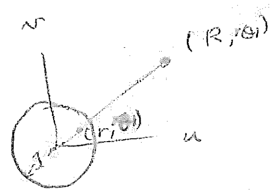
Therefore we conclude that the <u>most general</u> transformation satisfying the given conditions is

$$w = e^{i\sigma}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right) \quad \text{where } \alpha \equiv -\frac{b}{a}. \tag{6.1.7}$$

We still have not determined whether the half plane $y>0$ maps into the <u>interior</u> or the <u>exterior</u> of the w plane. All we have shown so far is that $y=0$ maps into $|W|=1$, i.e., a line maps into a circular curve. We want ~~it~~ the half space to be in the interior. To show that it does so it is only necessary to show that one point of the upper half of the z-plane transforms into the interior of the circle $|w|=1$. It is sufficient to take one point in the half-space $y>0$ and show ~~that~~ that its transform is inside $|w|=1$. Let us take in particular $z=\alpha$ where $\alpha_2 > 0$ (α_2 is defined to be the imaginary part of α — thus $\alpha \equiv \alpha_1 + i\alpha_2$). Then this point definitely lies in the space $\mathrm{Im}(z) > 0$. Its transforms

$$w = e^{i\sigma}\left(\frac{\alpha-\alpha}{\alpha-\bar{\alpha}}\right) = \frac{e^{i\sigma}}{2i\alpha_2}(\alpha-\alpha) = 0 .$$ Therefore $z=\alpha$ ~~...~~ when $\mathrm{Im}(\alpha)>0$ transforms into $w=0$, which lies *within* the circle $|w|=1$. Thus our transformation (6.1.7) maps the half-space into the interior of the unit circle.

If we wished to find a transformation that maps $y>0$ into the *exterior* of the circle $|w|=1$, we could do so by noting that $w' = \frac{1}{\bar{w}}$ (6.1.8) maps the interior of the circle $|w|=1$ into the exterior of the circle. In fact, w' and $\bar{w}$ are *inverse points*. That is , $\left|\frac{1}{\bar{w}}\right| = |w'|$ and $\arg \bar{w} = \arg w'$



[Note that $\bar{w} = re^{-i\theta}$ — small θ — where $w = re^{i\theta}$ and $w' = Re^{i\Theta}$. Thus, $\frac{1}{\bar{w}} = w'$ implies that $\frac{1}{r} = R$ and $\theta = -\Theta$.]

Thus, successive transformations of the type (6.1.7) and (6.1.8) will map $y>0$ into the exterior of $|w|=1$. We could combine them and write

$$w = e^{-i\sigma}\left(\frac{z-\bar{\alpha}}{z-\alpha}\right).$$

As a check we could note that $z=\alpha$ with $\mathrm{Im}(\alpha)>0$ would map into $w=\infty$.

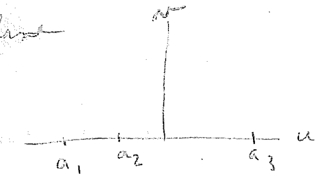
Triangles and Rectangles (Churchill pg 175)

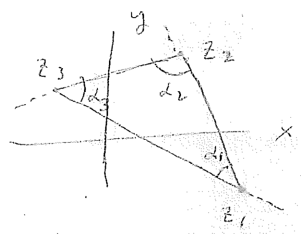
Let us consider the polygon to be a triangle with vertices at the points z_1, z_2, z_3. The transformation can be written as

$$(a) \qquad z = k \int (w-a_1)^{\left(\frac{\alpha_1}{\pi}-1\right)} (w-a_2)^{\left(\frac{\alpha_2}{\pi}-1\right)} (w-a_3)^{\left(\frac{\alpha_3}{\pi}-1\right)} dw + C$$

$\alpha_1 + \alpha_2 + \alpha_3 = \pi$

w-plane





Suppose that we are given the coordinates of the triangular region in the z-plane, and that we arbitrarily select the coordinates a_1, a_2, a_3 on the u axis which correspond to the vertices z_1, z_2, z_3 of the triangle in the z-plane. Then we shall have to find the values of k and C which will map the triangular region into the upper half of the w-plane.

If we would take a_3 ~~at~~ to be at infinity, then

$$(b) \qquad z = k \int (w-a_1)^{\left(\frac{\alpha_1}{\pi}-1\right)} (w-a_2)^{\left(\frac{\alpha_2}{\pi}-1\right)} dw + C.$$

The integration of eqs. (a) or (b) is difficult, and in general will not involve elementary functions. In fact, this is the principal disadvantage of the Schwarz-Christoffel transformation. Comparatively simple cases of (b) arise when the triangle is equilateral $(\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{3})$ or when the triangle is a right triangle with one of the angles equal to $\frac{\pi}{3}$ or $\frac{\pi}{4}$. In these cases it can be shown that (b) reduces to an elliptic integral. For all other nondegenerate triangles the ~~[illegible]~~ solution is more difficult.

Let us next consider the case where the polygon is a rectangle. Then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{\pi}{2}$. The transformation becomes

$$z = k \int [(w-a_1)(w-a_2)(w-a_3)(w-a_4)]^{-\frac{1}{2}} dw + C .$$

~~Let $a_1 = -1$, $a_2 = \frac{\pi}{\alpha-\pi}$~~

Let $a_1 = \frac{\pi}{\alpha-\pi} = \frac{\pi}{\frac{\pi}{2}-\pi} = -2$, $a_2 = -1$, $a_3 = +1$. Then by symmetry, because we are dealing with a rectangle, $a_4 = +2$. Then

$$z = k \int_0^w [(w+2)(w+1)(w-1)(w-2)]^{-\frac{1}{2}} dw + C ,$$

or

$$z = k \int_0^w \frac{dw}{\sqrt{(w^2-1)(w^2-4)}} + C = \frac{k}{2} \int_0^w \frac{dw}{\sqrt{(1-w^2)(1-(\frac{w}{2})^2)}} + C$$

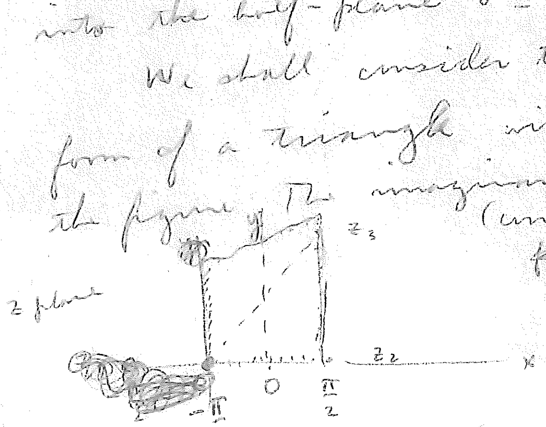
which is an elliptic integral (See Byerly, # 528).

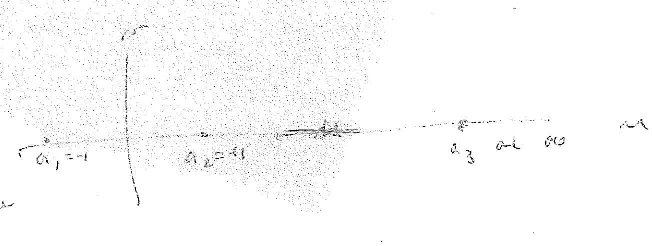
Degenerate Polygons

The Schwarz-Christoffel transformation is more readily suitable to transforming degenerate polygons into a half-plane, because the integrals which must be evaluated usually involve only elementary functions.

Example 1. Map the semi-infinite strip $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $y \geq 0$ into the half-plane $v \geq 0$.

We shall consider the strip in the z-plane to be the limiting form of a triangle with vertices z_1, z_2, and z_3 as shown in the figure. The imaginary part of z_3 is at infinity. Choose a_1 and a_2 (corresponding to z_1 and z_2) to be at $u_1 = -1$, $u_2 = 1$ and for w_3 (corresponding to z_3) at infinity.





Let $U = X\,T$

$$\frac{X''}{X} = \frac{T'}{T} = c^2$$

$$\left.\begin{aligned} X'' - c^2 X &= 0 \\ T' - c^2 T &= 0 \end{aligned}\right\}$$

$(D^2 - c^2)\,X = 0 \,, \quad D = \pm c$

$$X = Ae^{cx} + Be^{-cx}$$

$(D - c^2)\,T = 0$

$$T = De^{c^2 t}$$

$$U = \left(Ae^{cx} + Be^{-cx}\right)e^{c^2 t}$$

To make it bounded, make c^2 negative $= -k^2$

$$U = \left(A \sin kx + B \cos kx\right)e^{-k^2 t}$$

$U(0,t) = 0 \;\rightarrow\; B = 0$

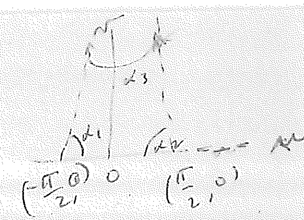
$$U(x,0) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x \geq 1 \end{cases}$$

The latter condition, for $x = 1$, gives $k = n\pi$. Thus

$$U(x,t) = \sum_{n=0}^{\infty} A_n \sin n\pi x \; e^{-(n\pi)^2 t}$$

where $A_n = 2\int_0^1 \sin n\pi x \, dx$

The limiting values of the angles α are:



$$\alpha_1 = \frac{\pi}{2}, \quad \alpha_2 = \frac{\pi}{2}, \quad \alpha_3 = 0.$$

Thus the equation for the transformation is

$$z = k\int^{w} [(w+1)(w-1)]^{-1/2}\, dw + C$$

$$z = ik\int^{w} \frac{dw}{\sqrt{1-w^2}} + C$$

and $z = ik \sin^{-1} w + C$.

To complete the solution, we must evaluate k and C. Let us first, we, $z = -\frac{\pi}{2}$ when $w = -1$ and $z = \frac{\pi}{2}$ when $w = 1$. This gives

$$-\frac{\pi}{2} = ik\sin^{-1}(-1) + C \quad \text{and} \quad \frac{\pi}{2} = ik\sin^{-1}(1) + C.$$

or

$$-\frac{\pi}{2} = ik\left(-\frac{\pi}{2}\right) + C \quad \text{and} \quad \frac{\pi}{2} = ik\frac{\pi}{2} + C$$

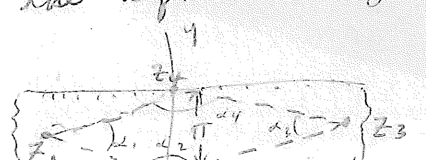
Adding these two equations tells us that $C = 0$. Then subtract the ~~second~~ first from the second, which gives $\pi = ik\pi$ or $ik = 1$.

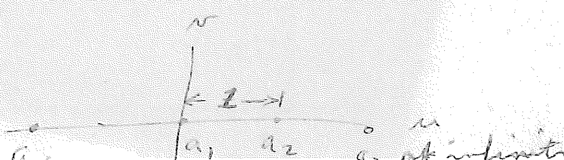
Therefore, the transformation is

$z = \sin^{-1} w$ or $w = \sin z$. We can verify by inspection that the semi infinite strip in the z plane transforms into the half plane.

Example 2. Map the infinite strip $0 \le y \le \pi$ into the half-plane $v \ge 0$.

We shall consider the infinite strip $0 \le y \le \pi$ as the limiting form of a rhombus with vertices at $z_1, z_2 = 0$ and $z_3, z_4 = \pi i$ as the points z_1 and z_3 are moved infinitely far to the left and right, respectively.





In the limit, $\alpha_1 = 0$, $\alpha_2 = \pi$, $\alpha_3 = 0$, $\alpha_4 = +\pi$. Let us assign the values $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$ and let a_4 to be determined. The transformation becomes

$$z = k\int^{z} (w-0)^{-1}(w-1)^{0}(w-a_4)^{0}\,dw + B$$

$$= k\int^{z}\frac{dw}{w} + C = k \ln w + C.$$

(z_2 transforms to a_2)

To evaluate ~~const~~ C, use the fact that $w = 1$ when $z = 0$.

This gives $0 = k \ln 1 + C = 0 + C$. Therefore $C = 0$.

Where should k? The constant k must be real because the point z lies on the real axis when $w = u$ and $u \geq 0$, i.e. ~~for~~ the points a_1, a_2, a_3 lie on the u-axis between 0 and $+\infty$ and the corresponding points ~~z~~ z_1, z_2, z_3 lie on the x-axis between $-\infty$ and $+\infty$. The point corresponding to $w = a_4$ is $z = \pi i$, where a_4 is a negative number. ($z_4 = \pi i$ given in statement of problem.)

~~This follows from substituting~~ $w = a_4$ and $z = \pi i$ into $z = k \ln w$.

[Note: for $w = a_4$, $z_4 = k \ln a_4 = \pi i$

The substitution gives

$$\pi i = k \ln a_4.$$

Now a_4 lies on the u axis. In order for it to ~~have~~ be a pure imaginary ~~part~~ (this is required because of the expression on the left of the equation) $k \ln a_4$ must equal $k\ln|a_4| + i\pi k$.

Therefore $k = 1$ and $|a_4| = 1$, and $w_4 = a_4 = -1 = 1e^{i\pi}$.

We conclude that the transformation which maps the semi-infinite strip of width π in the z-plane into the half space $v \geq 0$ in the w plane is

$$\cancel{w = \ln z} \quad z = \ln w \quad \text{or} \quad w = e^{z}$$

First, let ~~us~~ show that k is real.

The points z_1, z_2, z_3 lie on the x-axis, and w_1, w_2, w_3 on the positive w axis. ~~Then so~~ Then $z = k \ln w$ gives

1) for z_1 and w_1 $(-\infty, 0)$ and $(0,0)$

$$-\frac{\pi}{2} \cancel{= k \ln (0)} = \quad -\infty = k \ln w_1 = k(-\infty)$$

Only real terms are contained in this eqn., so k must be real and positive

2) for z_2 and w_2, $(0,0)$ and $(1,0)$

$$0 = k \ln 1$$

again k must be real and positive

3) for z_3 and w_3, $(\infty, 0)$ and $(\infty, 0)$

$$\infty = k \ln \infty$$

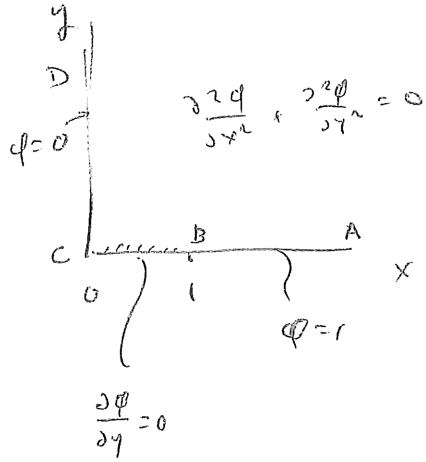
again k must be real and positive.

Now consider z_4 and w_4, $(0, \pi)$ and w_4.

$$0 + i\pi = k \left[\ln |w_4| + i \theta \right]$$ where θ is either 0 or π because w_4 is on the w axis

This gives, equating reals that $k \ln |w_4| = 0$ or $|w_4| = 1$

and, equating imaginaries that $\pi = k\theta$ or $\theta = \pi$ and $k = 1$.



$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0 \quad \text{for } x>0,\ y>0$$

$\left(\frac{\partial\varphi}{\partial y}\right)_{y=0}=0$ $\qquad 0<x<1$

$\varphi(x,0)=1$ $\qquad x>1$

$\varphi(0,y)=0$ $\qquad y>0$

$\sin iv = \frac{-\sinh v}{i} = i\sinh v$

Use the transformation $z=\sin w$

$x+iy = \sin(u+iv) = \sin u\cos iv + \cos u\sin iv = \sin u(\cosh v) + i\cos u(\sinh v)$

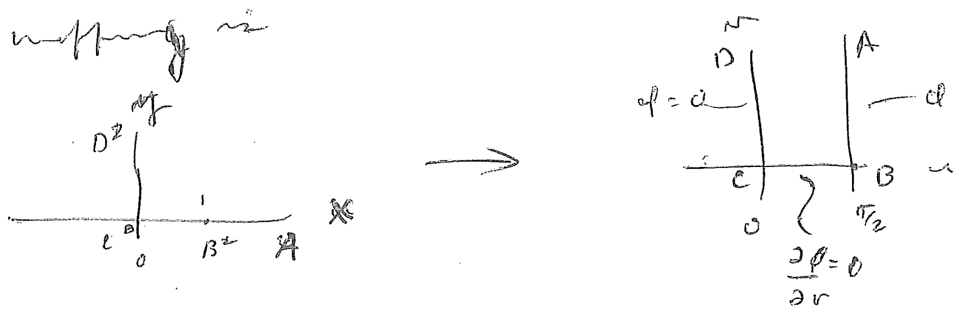
So $x=\sin u\cosh v$

$y=\cos u\sinh v$

and $\frac{x^2}{\sin^2 u}-\frac{y^2}{\cos^2 u}=\cosh^2 v-\sinh^2 v=1$ (hyperbolas)

(See Fig 10 of Churchill, [illegible], pg 207)

The mapping is



A solution to this transformed problem is $\varphi=\frac{2}{\pi}u$

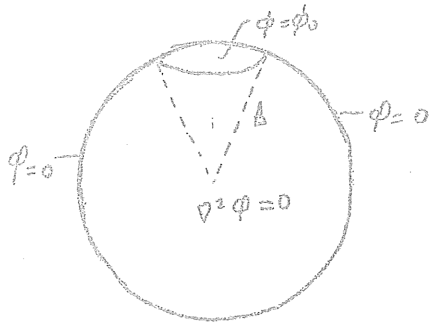
(satisfies all boundary conditions and $\frac{\partial^2\varphi}{\partial u^2}+\frac{\partial^2\varphi}{\partial v^2}=0$)

Going back to x and y, $= \frac{2}{\pi}\text{Real}(\sin^{-1}z)$

$$\varphi=\frac{2}{\pi}u=\frac{2}{\pi}\sin^{-1}\left\{\frac{1}{2}\left[\sqrt{(x+1)^2+y^2}-\sqrt{(x-1)^2+y^2}\right]\right\}.$$

POTENTIAL THEORY (GPH 204)

1. GIVEN $\nabla^2 \phi = 0$ INSIDE A SPHERE OF RADIUS b. ON THE SURFACE $r = b$, $\phi = \phi_0$ FOR $0 \le \theta < \alpha$ AND $\phi = 0$ FOR $\alpha < \theta \le \pi$.



SHOW THAT

$$\phi(r,\theta) = \frac{\phi_0}{2}\left\{1 - \cos\alpha - \sum_{n=1}^{\infty}\left[P_{n+1}(\cos\alpha) - P_{n-1}(\cos\alpha)\right]\left(\frac{r}{b}\right)^n P_n(\cos\theta)\right\}$$

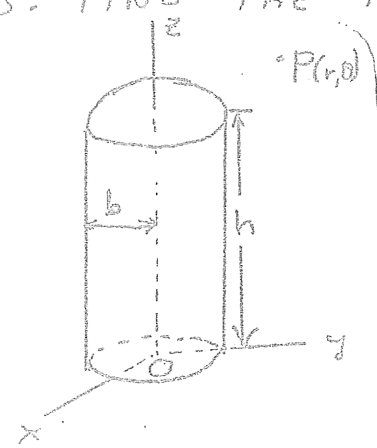
2. IRVING AND MULLINEUX, #15, PG. 202.

3. IRVING AND MULLINEUX, #16, PG. 202.

4. GIVEN $\nabla^2 \phi = 0$ OUTSIDE THE SPHERICAL SURFACE $r = b$, AND $\phi = f(\theta, \psi)$ ON $r = b$.

   FIND $\phi(r, \theta, \psi)$ FOR $r > b$.

5. FIND THE POTENTIAL AT POINTS $P(r, \theta)$ OUTSIDE A CIRCULAR CYLINDRICAL SURFACE OF RADIUS b, HEIGHT h AND CONSTANT SURFACE DENSITY σ.



(HINT: FIRST FIND THE POTENTIAL AT A POINT ON THE AXIS OF THE CYLINDRICAL SURFACE.)

1. Laplace Transform

Let $f(x)$ be some function of x. Then the function $g(p)$, which is called the Laplace transform of $f(x)$, is given by

(1.1) $$g(p) = \int_0^\infty e^{-px} f(x)\,dx \quad \text{for } p>0 \quad \text{if the integral exists.}$$

symbollically one writes $\mathcal{L} f(x) = g(p)$ (1.2)

or, for the inverse transform,

$$f(x) = \mathcal{L}^{-1} g(p) \qquad (1.3)$$
$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} g(p)\,e^{px}\,dp$$

[All singularities to left of $p=\gamma$]

Tables have been constructed which relate $f(x)$ and $g(p)$. See, e.g. ~~table~~ table I on pg. 207.

2. Laplace Transforms and Some General Properties

One can differentiate $g(p)$ with respect to p. The order of differentiation and integration may be interchanged if the integral, after the integration has formally been carried out, is uniformly convergent. If these conditions are satisfied, then

$$\frac{dg}{dp} = \frac{d}{dp}\int_0^\infty e^{-px} f(x)\,dx = \int_0^\infty (-x)\,e^{-px} f(x)\,dx$$

or, more generally

$$\frac{d^r g}{dp^r} = \int_0^\infty (-x)^r e^{-px} f(x)\,dx \qquad r = 1, 2, 3, \ldots \qquad (2.1)$$

Next suppose that $f(x)$ involves a constant a. One may then differentiate $g(p)$ with respect to either p or a, and in the process obtain transform pair tables involving the parameter "a". Consider an example, namely $f(x) = \sinh ax$. Then

$$g(p) = \int_0^\infty e^{-px} \sinh ax\,dx = \int_0^\infty e^{-px}\left[\tfrac{1}{2}(e^{ax} - e^{-ax})\right]dx$$

$$\int_0^\infty \tfrac{1}{2}\left[e^{-(p-a)x} - e^{-(p+a)x}\right]dx = \frac{1}{2(p-a)} - \frac{1}{2(p+a)} = \frac{(p+a)-(p-a)}{2(p^2-a^2)}$$

$$= \frac{a}{\ldots}$$

Differentiate $g(p)$ with respect to p. This gives

$$\frac{\partial g}{\partial p} = \int_0^\infty (-x)\, e^{-px} \sinh ax \, dx = \frac{\partial}{\partial p}\left[\frac{a}{p^2-a^2} \right] = \frac{-2ap}{(p^2-a^2)^2}$$

which gives
$$\int_0^\infty e^{-px}\, x \sinh ax \, dx = \frac{2ap}{p^2-a^2}. \qquad (2.3)$$

Differentiate $g(p)$ with respect to a. This gives

$$\frac{\partial g}{\partial a} = \int_0^\infty e^{-px}\, x \cosh ax \, dx = \frac{\partial}{\partial a}\left[\frac{a}{p^2-a^2} \right] = \frac{1}{p^2-a^2} + \frac{2a^2}{(p^2-a^2)^2}$$

or

$$\int_0^\infty e^{-px}\, x \cosh ax \, dx = \frac{p^2+a^2}{(p^2-a^2)^2}. \qquad (2.4)$$

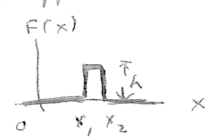
We can consider (2.3) and (2.4) as two additional pairs for Table I.

That is

$g(p)$	$f(x)$
$\frac{2ap}{p^2-a^2}$	$x \sinh ax$
$\frac{p^2+a^2}{(p^2-a^2)^2}$	$x \cosh ax$.

Next consider as one final example of this discussion the impulse, or Dirac delta, function which is defined as

$$F(x) = \begin{cases} 0 & \text{for } 0 \le x < x_1 \\ h & \text{for } x_1 < x < x_2 \\ 0 & \text{for } x_2 < x \end{cases}$$ where x_2 approaches x_1 in the limit.



The Laplace transform of $F(x)$ is

Over for other properties of Dirac δ-function

$$\mathcal{L}\, F(x) = \int_{x_1}^{x_2} h\, e^{-px}\, dx = \frac{h}{p}\left(e^{-px_1} - e^{-px_2}\right).$$

no paragraph indention For the special case of $x_1 = 0$, $\mathcal{L}\, F(x) = \frac{h}{p}(1 - e^{-px_2})$. Next let x_2 also approach zero, and in such a way that

$$\lim_{\substack{x_2 \to 0 \\ h \to \infty}} h x_2 = 1. \qquad \left(\text{e.g., let } h = \frac{1}{x_2}\right).$$

In the equation 4 lines above, we have $\mathcal{L}\, F(x) = \frac{h}{p}(1 - e^{-px_2})$. Let us expand e^{-px_2} in a series. This gives

Do this after the top

Simple properties of Dirac δ-function:

$$\int_c^d f(x)\,\delta(a-x)\,dx = f(a) \text{ when } c < a < d.$$

$$\delta(x-a) = \delta(a-x)$$

$$x\,\delta(x) = 0$$

$$x\,\delta'(x) = -\delta(x)$$

$$\delta(ax) = \frac{1}{a}\,\delta(x) \text{ for } a > 0$$

$$\delta(x^2-a^2) = \frac{2}{a}\left[\delta(x-a) + \delta(x+a)\right] \text{ for } a > 0.$$

$$\mathcal{L}\, F(x) = \frac{h}{p}\left[1 - \left\{1 - p x_2 + \frac{p^2 x_2^2}{2!} - \cdots\right\}\right]$$

so that

$$\lim_{\substack{x_2 \to 0 \\ h \to \infty}} \mathcal{L}\, F(x) = \lim_{\substack{x_2 \to 0 \\ h \to \infty}} \left[h x_2 - \tfrac{1}{2}(h x_2)(p x_2) + \cdots\right] = 1 - 0 + 0 - \cdots = .$$

Thus, the δ function $\delta(x) = 0$ for $x > 0$, $\delta(x) = \infty$ for $x = 0$

have the Laplace transform $\mathcal{L}\,\delta(x) = 1$. (2.5)

Some theorems:

1. The Laplace transform of the sum or difference of two functions equals the sum or difference of the transforms of the individual function

$$\mathcal{L}\left[f_1(x) \pm f_2(x)\right] = \mathcal{L}\, f_1(x) \pm \mathcal{L}\, f_2(x) \qquad (2.6)$$

2. The Laplace transform of the derivative of f with respect to x is

$f(0)$ means the value of f at $x = 0$,

$$\mathcal{L}\,\frac{df}{dx} = p\, g(p) - f(0) \quad \text{if } \lim_{x \to \infty} e^{-px} f(x) = 0$$

where g is the Laplace transform of $f(x)$.

give proof over. Proof follows by integration by parts.

The Laplace transform of $\frac{d^2 f}{dx^2}$ is

$$\mathcal{L}\, f''(x) = -\{p f(0) + f'(0)\} + p^2 g(p) \quad \text{if } \lim_{x \to \infty} f'(x) e^{-px} = 0 \qquad (2.8)$$

and for the r'th derivative

$$\mathcal{L}\, f^r(x) = -\{p^{r-1} f(0) + p^{r-2} f'(0) + \cdots + p f^{r-2}(0) + f^{r-1}(0)\} + p^r g(p) \qquad (2.9)$$

for $r = 1, 2, \cdots n$ if $\lim_{x \to \infty} f^r(x) e^{-px} = 0$.

3. If $g(p)$ is the transform of $f(x)$, then $g(p+b)$ is the transform of $e^{-bx} f(x)$, provided that $p + b > 0$ and $g(p)$ exists: This can be written as

$$\mathcal{L}\left[e^{-bx} f(x)\right] = \int_0^\infty e^{-(p+b)x} f(x)\, dx = g(p+b) \qquad (2.10)$$

Proof of (2):

Integrate by parts.

$$\mathcal{L}\left(\frac{df}{dx}\right) = \int_0^\infty \frac{df}{dx} e^{-px}\,dx = \left[f(x)e^{-px}\right]_0^\infty + p\int_0^\infty f(x)e^{-px}\,dx$$

$$= -f(0) + p\,g(p)$$

For

$$\mathcal{L}\left(\frac{d^2f}{dx^2}\right) = \int_0^\infty f''(x)e^{-px}\,dx = \left[f'(x)e^{-px}\right]_0^\infty + p\int_0^\infty f'(x)e^{-px}\,dx$$

$$= -f'(0) + p\left[-f(0) + pg(p)\right]$$ provided that $\lim_{x\to\infty} f'(x)e^{-px} = 0$

$\cancel{-pf'(0)}$

$$= -\left\{p\,f(0) + f'(0)\right\} + p^2 g(p)$$

1

Theorem 4. If $g_1(p)$ and $g_2(p)$ are polynomials of degree m and n in p respectively, where $m \leq (n-1)$ and if, in addition, the n zeroes of $g_2(p)$, namely $a_1, a_2, a_3, \dots, a_n$, are all different or distinct,

$$\mathcal{L}^{-1}\left[\frac{g_1(p)}{g_2(p)}\right] = \sum_{r=1}^{n} \frac{g_1(a_r)}{g_2'(a_r)} e^{a_r x} \qquad (2.11)$$

Eq. (2.11) is Heaviside's expansion theorem. It follows from writing $\frac{g_1(p)}{g_2(p)}$ in a series using partial fractions and no. 4 of table I ($g(p) = \frac{1}{p-a}$ with $a > 0$, $f(x) = e^{ax}$).

3. Solution of linear differential equations with constant coefficients.

A linear diff. eqn. with constant coefficients a_r can be written as

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) x = F(t) \quad \text{for } t > 0, \qquad (3.1)$$

where $D^r = \frac{d^r}{dt^r}$.

We want a solution of (3.1) which satisfies given values of $x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^{n-1}x}{dt^{n-1}}$ when $t = 0$. ~~When~~ We shall call $(x)_{t=0}$ the quantity x_0, $\left(\frac{dx}{dt}\right)_{t=0}$ the quantity x_1, ...,

$\left(\frac{d^{n-1}x}{dt^{n-1}}\right)_{t=0}$ the quantity x_{n-1}.

To obtain a solution, multiply eq. (3.1) by e^{-pt} and integrate with respect to t from $t=0$ to $t=\infty$. This gives

$$\int_0^\infty e^{-pt}\left[D^n x + a_1 D^{n-1} x + \dots + a_{n-1} Dx + a_n x\right] dt = \int_0^\infty e^{-pt} F(t)\, dt. \qquad (3.2)$$

Let us next write out a few terms which will result from integrating terms on the left hand

$D \equiv \frac{d}{dt}$

side of the equation. For example, integrating by parts,

$$\int_0^\infty e^{-pt} Dx \, dt = \left[x e^{-pt}\right]_0^\infty + p\int_0^\infty e^{-pt} x \, dt = -x_0 + p\int_0^\infty e^{-pt} x \, dt$$

where we assume $\lim_{t\to\infty} (e^{-pt} x) = 0$ and that $\int_0^\infty e^{-pt} x \, dt$ exists

when p is greater than some fixed positive number, say α.

Next consider $\int_0^\infty e^{-pt} D^2 x \, dt$. This integral is given by

$$\int_0^\infty e^{-pt} D^2 x \, dt = \left[e^{-pt} Dx\right]_0^\infty + p\int_0^\infty e^{-pt} Dx \, dt$$
$$= -x_1 + p\int_0^\infty e^{-pt} Dx \, dt = -x_1 - p x_0 + p^2 \int_0^\infty e^{-pt} x \, dt$$

assuming that $\lim_{t\to\infty} (e^{-pt} Dx) = 0$.

Considering now a general term, we have

$$\int_0^\infty e^{-pt} D^r x \, dt = -(p^{r-1} x_0 + p^{r-2} x_1 + \cdots + p^{r-(r-1)} x_{r-2} + p^{r-r} x_{r-1}) + p^r \int_0^\infty e^{-pt} x \, dt$$

for $r \le n$ where we assume that $\lim_{t\to\infty} e^{-pt} D^r x = 0$.

or $$\int_0^\infty e^{-pt} D^r x \, dt = -(p^{r-1} x_0 + p^{r-2} x_1 + \cdots + p x_{r-2} + x_{r-1}) + p^r \bar{x}(p)$$

where $\bar{x}(p) \equiv \int_0^\infty e^{-pt} x(t) \, dt$ [$\bar{x}(p)$ is the transform of x].

Therefore $\int_0^\infty e^{-pt} (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) x \, dt$

$$= -(p^{n-1} x_0 + p^{n-2} x_1 + \cdots + p x_{n-2} + x_{n-1}) + p^n \bar{x}(p)$$
$$- a_1 (p^{n-2} x_0 + p^{n-3} x_1 + \cdots + p x_{n-3} + x_{n-2}) + a_1 p^{n-1} \bar{x}(p)$$
$$- \cdots\cdots\cdots$$
$$- a_{n-2} (p x_0 + x_1) + a_{n-2} p^2 \bar{x}(p)$$
$$- a_{n-1} x_0 + a_{n-1} p \, \bar{x}(p) \qquad = \int_0^\infty e^{-pt} F(t) \, dt$$

Reanranging terms, and making use of the definition of $\phi(p)$, namely $\phi(p) = p^n + a_1 p^{n-1} + a_2 p^{n-2} + \dots + a_{n-1} p + a_n$, we have

$$\bar{x}(p)\left[p^n + a_1 p^{n-1} + a_2 p^{n-2} + \dots + a_{n-1} p + a_n\right] = \bar{x}(p)\,\phi(p)$$

$$\begin{aligned}
&= p^{n-1} x_0 + p^{n-2} x_1 + \dots + p\, x_{n-2} + x_{n-1} \\
&+ a_1 \left(p^{n-2} x_0 + p^{n-3} x_1 + \dots + p\, x_{n-3} + x_{n-2}\right) \\
&+ \dots \\
&+ a_{n-2}\,(p\, x_0 + x_1) \\
&+ a_{n-1}\, x_0 \\
&+ \int_0^{\infty} e^{-pt} F(t)\, dt \qquad (3.4)
\end{aligned}$$

Equation (3.4), which determines $\bar{x}(p)$, is called the subsidiary equation. The solution of eq. (3.1) which satisfies the given initial conditions is the inverse transform of $\bar{x}(p)$, namely

$$\mathcal{L}^{-1}\,\bar{x}(p) = x(t) .$$

The inverse transform of $\bar{x}(p)$ can be found from tables of transform pairs. For a physical system the function $x(t)$ is called the <u>response</u> to the <u>excitation function</u> $F(t)$.

Examples of use of Laplace transform to solve ordinary linear differential equations with constant coefficients.

Ex. (a): $(D+1)x = 1$ for $t > 0$ and $x = 0$ when $t = 0$, i.e. $x_0 = 0$.

The subsidiary equation is

$$(p+1)\bar{x}(p) = p^{n-1}x_0 + \dots + \int_0^\infty e^{-pt}F(t)\,dt$$ where $n=1$, $x_0 = 0$, $F(t) = 1$

$$= 0 + \int_0^\infty e^{-pt}\,dt = \frac{1}{p}$$ from entry (1), table (I).

Thus

$$\bar{x}(p) = \frac{1}{p(p+1)} = \frac{1}{p} - \frac{1}{p+1}.$$

(Use partial fractions)

$$\frac{1}{p(p+1)} = \frac{A}{p} + \frac{B}{p+1}$$

$$A(p+1) + Bp = 1$$

Equate coefficients of p: $A = -B$

" " of p_0: $A = 1$, $B = -1$

The solution is

$$x(t) = \mathcal{L}^{-1}\bar{x}(p) = 1 - e^{-t}$$

(from entries 1 and 4, table I.)

Ex. (b): $(D^2 - 5D + 4)x = 12 + 9e^t + 5\sin 2t$ for $t > 0$

and $x_0 = 1$, $x_1 = -2$.

For this problem n is 2 and $F(t) = 12 + 9e^t + 5\sin 2t$.

Thus the subsidiary equation is: $= p x_{n-2} + x_{n-1} + a_{n-1}x_0 + \int_0^\infty e^{-pt}F(t)\,dt$

$$(p^2 - 5p + 4)\bar{x}(p) = p(1) + (-2) + (-5)(1) + \int_0^\infty e^{-pt}[12 + 9e^t + 5\sin 2t]\,dt$$

$$= p - 7 + \frac{12}{p} + \frac{9}{p-1} + \frac{5\times 2}{p^2+4}$$ from entries (1), (4), & (7) of table I.

Therefore

$$\bar{x}(p) = \frac{p - 7 + \frac{12}{p} + \frac{9}{p-1} + \frac{10}{p^2+4}}{p^2 - 5p + 4}.$$

By partial fractions, this can be written as

$$\bar{x}(p) = \frac{3}{p} - \frac{11}{3(p-1)} + \frac{7}{6(p-4)} - \frac{3}{(p-1)^2} + \frac{p}{2(p^2+4)}.$$

Therefore

$$x(t) = \mathcal{L}^{-1}\bar{x}(p) = 3 - \frac{11}{3}e^t + \frac{7}{6}e^{4t} - 3te^t + \frac{1}{2}\cos 2t.$$

Ex. C Find the solution of the simultaneous differential equas.

$$(D^2 - 3D + 2)x + (D-1)y = 0$$
$$(D-1)x - (D^2 - 5D + 4)y = 0$$

which satisfy the conditions $x = 0,\ y = 1,\ Dx = 0,\ Dy = 0$ at $t = 0$.

The subsidiary equation gives:

$$(p^2 - 3p + 2)\bar{x} + (p-1)\bar{y} = p x_0 + x_1 - 3x_0 + y_0$$
$$(p-1)\bar{x} - (p^2 - 5p + 4)\bar{y} = x_0 - p y_0 - y_1 + 5y_0$$

where $\bar{x} \equiv \int_0^\infty e^{-pt} x(t)\,dt$

$\bar{y} \equiv \int_0^\infty e^{-pt} y(t)\,dt$

Using the initial conditions,

$$(p^2 - 3p + 2)\bar{x} + (p-1)\bar{y} = 1$$
$$(p-1)\bar{x} - (p^2 - 5p + 4)\bar{y} = -p + 5.$$

These are simultaneous equations in $\bar{x}$ and $\bar{y}$. There solutions are:

$$\bar{x} = \frac{1}{(p-1)(p-3)^2} = \frac{1}{4}\left[\frac{1}{p-1} - \frac{1}{p-3} + \frac{2}{(p-3)^2}\right]$$

$$\bar{y} = \frac{p^2 - 7p + 11}{(p-1)(p-3)^2} = \frac{1}{4}\left\{\frac{5}{p-1} - \frac{1}{p-3} - \frac{2}{(p-3)^2}\right\}.$$

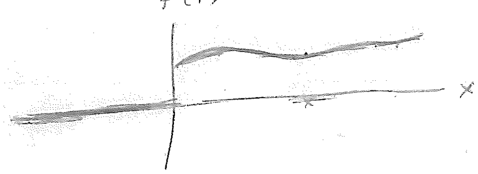
By inversion,

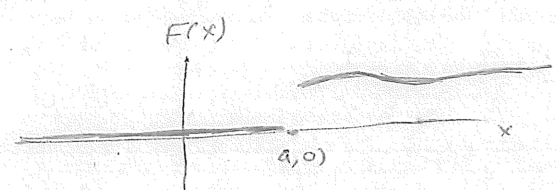
$$x = \mathcal{L}^{-1}\bar{x} = \frac{1}{4}\left[e^t - e^{3t}(1-2t)\right]$$
$$y = \mathcal{L}^{-1}\bar{y} = \frac{1}{4}\left[5e^t - e^{3t}(1+2t)\right].$$

Theorem 5. If $g(p)$ is the Laplace transform of $f(x)$, then $e^{-ap}\, g(p)$, where $a > 0$, is the Laplace transform of $F(x)$, when $F(x)$ is defined as

$$F(x) = 0 \quad \text{for} \quad 0 < x < a \qquad (4.1)$$
$$F(x) = f(x-a) \quad \text{for} \quad x > a .$$

The factor e^{-ap} acts as a shift operator, where the origin of coordinates is moved along the x-axis a distance a.





a) Example. Unit step-function.

Consider a function whose value is zero for $x < a$ and unity for $x > a$. Call this function $u(x-a)$. It can be defined by the relations

$$u(x-a) = \begin{cases} 0 & \text{for } 0 < x < a \\ 1 & \text{for } x > a \end{cases} \qquad (4.2)$$

Its Laplace transform is e^{-ap}/p, which follows from

$$\mathcal{L}[u(x-a)] \equiv \int_0^a e^{-px}(0)\,dx + \int_a^\infty e^{-px}(1)\,dx = \left[-\frac{1}{p}e^{-px}\right]_a^\infty = \frac{e^{-ap}}{p} .$$

b) Example. A type of pulse function.

Evaluate $\mathcal{L}^{-1}\left\{ \frac{p-2}{p^3} e^{-2p} + \frac{p+2}{p^3} e^{-3p} \right\} = F(x)$

$$\frac{p-2}{p^3} = \frac{1}{p^2} - \frac{2}{p^3} \quad \text{and} \quad \frac{p+2}{p^3} = \frac{1}{p^2} + \frac{2}{p^3} .$$

Now $\mathcal{L}^{-1}\left\{\frac{1}{p^2} - \frac{2}{p^3}\right\} = x - x^2$ from entry 2, table I.

and $\mathcal{L}^{-1}\left\{\frac{1}{p^2} + \frac{2}{p^3}\right\} = x + x^2$ " " " "

Using theorem 5,

$$F(x) = \mathcal{L}^{-1}\left[\frac{(p-2)}{p^3}e^{-2p}\right] = \begin{cases} 0 & \text{for } 0<x<2 \\ (x-2)-(x-2)^2 = (x-2)(3-x) & \text{for } x>2 \end{cases}$$

and

$$F(x) = \mathcal{L}^{-1}\left[\frac{p+2}{p^3}e^{-3p}\right] = \begin{cases} 0 & \text{for } 0<x<3 \\ (x-3)+(x-3)^2 = (x-3)(x-2) & \text{for } x>3 . \end{cases}$$

Combine these results, ~~using~~ that $\mathcal{L}^{-1}(a) + \mathcal{L}^{-1}(b) = \mathcal{L}^{-1}(a+b)$ ~~and note that F(x) is not zero~~

This gives

$$F(x) = \begin{cases} 0 & \text{for } 0<x<2 \\ (x-2)(3-x) & \text{for } 2<x<3 \\ (x-2)(3-x)+(x-3)(x-2) = 0 & \text{for } x>3 . \end{cases}$$

Thus F(x) is called a <u>type of pulse function</u> because it has non-zero values only between x = 2 and x = 3.

Theorem 6. <u>Convolution Theorem</u>. [See next page]

Let $g_1(p)$ and $g_2(p)$ be the Laplace transforms of $f_1(x)$ and $f_2(x)$, respectively. Then it can be proved that $g_1(p)g_2(p)$ is ~~also~~ equal to

$$g_1(p)\,g_2(p) = \mathcal{L}\int_0^x f_1(x')\,f_2(x-x')\,dx'$$

$$\text{or } = \mathcal{L}\int_0^x f_1(x-x')\,f_2(x')\,dx' .$$

These equations ~~are called~~ describe the convolution theorem.

<u>Exercise 4.</u> Evaluate $\mathcal{L}^{-1}\left[\int_0^\infty \frac{e^{-px}G(x)}{p^2+a^2}\,dx\right]$

In this problem $g_1(p) = \int_0^\infty e^{-px}G(x)\,dx$ and $g_2(p) = \frac{1}{p^2+a^2}$.

Therefore $f_1(x) = \mathcal{L}^{-1}\{g_1(p)\} = G(x)$ and $f_2(x) = \mathcal{L}^{-1}\{g_2(p)\} = \mathcal{L}^{-1}\left(\frac{1}{p^2+a^2}\right) = \frac{1}{a}\sin ax$

Using the convolution theorem,

$$\mathcal{L}^{-1}\left[\int_0^\infty \frac{e^{-px}G(x)\,dx}{p^2+a^2}\right] = \frac{1}{a}\int_0^x G(x-x')\sin ax'\,dx' \text{ or } = \frac{1}{a}\int_0^x G(x')\sin a(x-x')\,dx'$$

III 6a -

Proof of Convolution Theorem

Using the definition of the Laplace transform,

$$g_1(p)\,g_2(p) = \left\{ \int_0^\infty e^{-pu} f_1(u)\,du \right\} \left\{ \int_0^\infty e^{-pv} f_2(v)\,dv \right\} \qquad (4.2)$$

$$= \iint_A e^{-p(u+v)} f_1(u)\,f_2(v)\,du\,dv \qquad (4.3)$$

where A is the area of the positive quadrant of the (u, v) plane.

Make a change in variables. Let

$$u = x', \qquad v = x - x'.$$

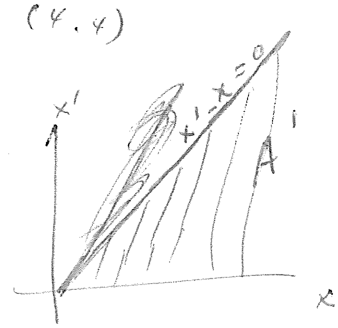
Then ~~$du = dx'$,~~

$$du\,dv = \left| \frac{\partial(u,v)}{\partial(x', x)} \right| dx\,dx' = \begin{vmatrix} \frac{\partial x'}{\partial x'} & \frac{\partial x'}{\partial x} \\ \frac{\partial(x-x')}{\partial x'} & \frac{\partial(x-x')}{\partial x} \end{vmatrix} dx\,dx' = \begin{vmatrix} +1 & 0 \\ -1 & +1 \end{vmatrix} dx\,dx'$$

$$= dx\,dx'$$

Thus

$$g_1(p)\,g_2(p) = \iint_{A'} e^{-px} f_1(x')\,f_2(x-x')\,dx\,dx' \qquad (4.4)$$



The new area of integration is the triangular region indicated in the figure, because

$u > 0$ gives $x' > 0$

and $v > 0$ gives $x - x' > 0$.

Thus eq (4.4) becomes

$$g_1(p)\,g_2(p) = \int_0^\infty e^{-px} \left\{ \int_0^x f_1(x')\,f_2(x-x')\,dx' \right\} dx$$

$$= \mathcal{L} \int_0^x f_1(x')\,f_2(x-x')\,dx'.$$

which is the desired proof. Note that $\int_0^x f_1(x')\,f_2(x-x')\,dx' = \int_0^x f_1(x-x')\,f_2(x')\,dx'$

which can be seen by letting $x - x' = y$.

5. Solution of the equation $\Phi(D)\,x(t) = F(t)$ by means of convolution theorem.

The equation to be solved is the nth order linear differential equation with constant coefficients and excitation function $F(t)$. That is

$$(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)\, x(t) \equiv \phi(D)\, x(t) = F(t). \qquad (5.1)$$

The subsidiary equation corresponding to (5.1) is given by (3.4) which may be written as

$$\phi(p)\,\bar{x}(p) = \psi(p) + \int_0^\infty e^{-pt} F(t)\, dt$$

where $\psi(p)$ corresponds to the terms in (3.4) containing $x_0, x_1, \ldots x_{n-1}$. Write out, if necessary

Dividing the equation above by $\phi(p)$,

$$\bar{x}(p) = \frac{\psi(p)}{\phi(p)} + \frac{\int_0^\infty e^{-pt} F(t)\, dt}{\phi(p)} \qquad (5.2)$$

with $x(t) = \mathcal{L}^{-1}\, \bar{x}(p)$.

The inverse of $\bar{x}(p)$ is the solution of (5.1).

The inverse of $\frac{\psi(p)}{\phi(p)}$ can be found readily because $\phi(p)$ and $\psi(p)$ are both polynomials in p by theorem 4, eq. (2.11), pg 212

The inverse of $\frac{\int_0^\infty e^{-pt} F(t)\, dt}{\phi(p)}$ can be found by using the convolution theorem

This gives

$$\mathcal{L}^{-1}\left[\frac{\int_0^\infty e^{-pt} F(t)\, dt}{\phi(p)}\right] = \int_0^t F(t')\, f(t-t')\, dt' \text{ or } = \int_0^t F(t-t')\, f(t')\, dt'$$

where $f(t) \equiv \mathcal{L}^{-1}\left[\frac{1}{\phi(p)}\right]$.

Then the solution to the diff. eqn. is

$$x(t) = \mathcal{L}^{-1}\left[\frac{\psi(p)}{\phi(p)}\right] + \int_0^t F(t')\, f(t-t')\, dt' \qquad (5.3)$$

$g_1(p)\, g_2(p) = \mathcal{L}\int_0^x f_1(x')\, f_2(x-x')\, dx'$

$\mathcal{L}^{-1}\{g_1(p)\, g_2(p)\} = \int_0^x f_1(x')\, f_2(x-x')\, dx'$

let $g_1(p) = \int_0^\infty e^{-pt} F(t)\, dt$

$f(x) = \mathcal{L}\left\{\frac{1}{\phi(p)}\right\}$

 Exercise:

Solve the integro-differential equation

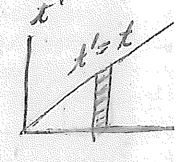
$$Di + \omega^2 \int_0^t i(t')\,dt' = F(t) \qquad \text{with } i=0 \text{ at } t=0 \text{ and } D \equiv d/dt .$$

This is called an integro-differential eqn. because the dependent variable i appears as both a derivative (Di) and in the integrand.

Consider first the Laplace transform of the integral, i.e.,

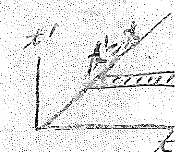
$$\int_0^\infty e^{-pt}\left\{\int_0^t i(t')\,dt'\right\}dt .$$

t' from 0 to t
t from 0 to ∞



Interchange the ~~limits~~ order of integration (see sketch to right for limits of integration).

t from t' to ∞
t' from 0 to ∞



$$\int_0^\infty e^{-pt}\left\{\int_0^t i(t')\,dt'\right\}dt = \int_0^\infty i(t')\left\{\int_{t'}^\infty e^{-pt}\,dt\right\}dt'$$

$$= \int_0^\infty i(t')\left[-\frac{1}{p}e^{-pt}\right]_{t'}^\infty dt' = \frac{1}{p}\int_0^\infty i(t')\,e^{-pt'}\,dt' \quad \left(\text{provided } \lim_{t\to\infty} e^{-pt} = 0\right)$$

$$= \frac{\bar{i}(p)}{p} .$$

~~Our integro-differential eqn. becomes~~

$$\cancel{Di + \frac{\omega^2}{p}\int_0^\infty i(t')\,e^{-pt'}\,dt' = F(t)}$$

~~Our integro-differential equation becomes~~

$$Di \cancel{+ \frac{\omega^2 \bar{i}(p)}{p} = F(t)} .$$

The Laplace transform of the integro-differential equation is

$$= D\bar{i}(p) + \frac{\omega^2}{p}\bar{i}(p)$$

$$\int_0^\infty e^{-pt}\,Di\,dt + \frac{\omega^2}{p}\bar{i}(p) = \int_0^\infty e^{-pt}F(t)\,dt .$$

The subsidiary equation is, from (3.4)

$$\left(p + \frac{\omega^2}{p}\right)\bar{i}(p) = \int_0^\infty e^{-pt}F(t)\,dt$$

which may be written as

$$\bar{i}(p) = \frac{p}{p^2+\omega^2}\int^\infty e^{-pt}F(t)\,dt$$

In the notation of eq. (5.2), the terms in the last equation are

$$\cancel{\psi(p)} \quad \frac{1}{\Phi(p)} = \frac{p}{p^2+\omega^2} \quad \text{and thus } \psi(p) = 0 .$$

We have previously shown (eqn. just before ~~after~~ 5.3) that

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{\Phi(p)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{p}{p^2+\omega^2}\right] \text{ in our case.}$$

But from entry 8, table I (pg 207),

$$\mathcal{L}^{-1}\left[\frac{p}{p^2+\omega^2}\right] = \cancel{\ldots} \cos \omega t .$$

Thus, from (5.3),

$$A(t) = \int_0^t F(t') \cos \omega (t-t')\, dt' . \qquad (5.4)$$

This is the desired solution.

As a particular case, let $F(t)$ be a periodic function

such that $F(t + 2r\tau) = F(t) \qquad r = 1, 2, 3, \ldots$

and suppose that $x(t)$ is required at time $t = 2n\tau + T$

where $0 < T < 2\tau$ and n is a positive integer.

Divide the interval of integration from 0 to $2n\tau$ and from $2n\tau$ to T. Then (5.4) ~~may~~ is ~~be~~ written as

$$x(t) = \sum_{r=0}^{n-1} \int_{2r\tau}^{2(r+1)\tau} F(t') \cos \omega (t-t')\, dt' + \int_{2n\tau}^{T} F(t') \cos \omega (t-t')\, dt'$$

Substitute $t' = u + 2rT$ in the first integral. This gives

$$\int_{2rT}^{2(r+1)T} F(t') \cos\omega(t-t')\,dt' = \int_0^{2T} F(u+2rT)\cos\omega(t - 2rT - u)\,du.$$

Substitute $t' = v + 2nT$ in the second integral. This gives

$$\int_{2nT}^{T} F(t')\cos\omega(t-t')\,dt' = \int_0^{T} F(v + 2nT)\cos\omega(t - 2nT - v)\,dv.$$

when the upper limit has T instead of $T-2nT$ because the integrand is a periodic function of period $2nT$. Note also that

$$F(u+2rT) = F(u) \quad , \quad F(v+2nT) = F(v) \quad . \text{ Thus}$$

$$x(t) = \sum_{r=0}^{n-1} \int_0^{2T} F(u)\cos\omega(t-2rT-u)\,du + \int_0^{T} F(v)\cos\omega(t-2nT-v)\,dv.$$

For illustration consider the "square-wave" function

$$F(t) = \begin{cases} 1 & \text{for } 0 \le t < T \\ 0 & \text{for } T < t \le 2T \end{cases}$$

of period $2T$; i.e. $F(t+2rT) = F(t)$, $r = 1, 2, 3, \ldots$

Then

$$\int_0^{2T} F(u)\cos\omega(t-2rT-u)\,du = \int_0^{T} \cos\omega(t-2rT-u)\,du$$

$$= \left[-\frac{1}{\omega}\sin\omega(t-2rT-u)\right]_0^T$$

$$= -\frac{1}{\omega}\left[\sin\omega(t-2rT-T) - \sin\omega(t-2rT)\right]$$

$$= \frac{1}{\omega}\left[\sin\omega(t-2rT) - \sin\omega\{t-(2r+1)T\}\right]$$

Use $\sin a - \sin b = 2\sin\frac{1}{2}(a-b)\cos\frac{1}{2}(a+b)$. Then

$$\int_0^T \cos\omega(t-2rT-u)\,du = \frac{2}{\omega}\sin\frac{\omega}{2}\{t-2rT-t+(2r+1)T\}\cos\frac{\omega}{2}\{t-2rT+t-(2r+1)T\}$$

$$= \frac{2}{\omega}\sin\frac{\omega T}{2}\cos\omega\{t-(2r+\tfrac{1}{2})T\}.$$

Next consider

$$\int_0^T F(v)\cos\omega(t-2nT-v)\,dv = \int_0^{T}\cos\omega(T-v)\,dv = \left[-\frac{1}{\omega}\sin\omega(T-v)\right]_0^T$$

$$= \frac{1}{\omega}\sin\omega T \qquad \text{for } 0 \le T \le T \quad (\text{otherwise } F(v) = 0)$$

Thus

$$x(t) = \frac{1}{\omega}\left[2\sum_{r=0}^{n-1} \cos\omega\left(t - \frac{T}{2}\right)\right.$$

$$x(t) = \frac{1}{\omega}\left[2\sum_{r=0}^{n-1}\cos\omega\left(t-\frac{T}{2}-2rT\right)\sin(\omega T/2) + \sin\omega T\right]$$

$$= \frac{2}{\omega}\left[\left\{\cos\omega\left(t-\frac{T}{2}\right) + \cos\omega\left(t-\frac{T}{2}-2T\right) + \cos\omega\left(t-\frac{T}{2}-4T\right) + \cos\omega\left(t-\frac{T}{2}-2(n-1)T\right)\right\}\sin\frac{\omega T}{2} + \sin\omega T\right]$$

$t = u + 2rT$
$2(r+1)T = u + 2rT$
$2T = u$

On summing the series, it can be shown that

$$x(t) = \frac{\sin\omega\left(t+\frac{T}{2}\right) + \sin\omega\left(T-\frac{T}{2}\right)}{2\omega\cos\frac{\omega T}{2}} \qquad \text{for } 0 \le T \le T \qquad (5.5)$$

and

$$x(t) = \frac{\sin\omega\left(t+\frac{T}{2}\right) + \sin\frac{\omega T}{2}}{2\omega\cos\frac{\omega T}{2}} \qquad \text{for } T \le T \le 2T.$$

<u>Exercise 2</u> Solve the previous problem without the use of the convolution theorem for the above periodic excitation function.

In this case we have to find the transform of $F(t)$. Our initial condition problem is

$$Di + \omega^2\int_0^t i(t')\,dt' = F(t) \qquad \text{where } i = 0 \text{ at } t = 0$$

$$F(t) = \begin{matrix} 0 & \text{for } 0 \le t < T \\ 1 & \text{for } T < t \le 2T. \end{matrix}$$

Because $F(t)$ is periodic, we have

$$\int_0^\infty e^{-pt}F(t)\,dt = \sum_{r=0}^{\infty}\int_{2rT}^{2(r+1)T} e^{-pt}F(t)\,dt = \sum_{r=0}^{\infty} e^{-2prT}\int_0^{2T} e^{-pu}F(u)\,du$$

where $t = u + 2rT$ has been used.

Summing the series

$$\int_0^\infty e^{-pt}F(t)\,dt = \frac{\int_0^{2T} e^{-pu}F(u)\,du}{1-e^{-2pT}} \qquad (5.6)$$

The above result holds for any periodic function $F(t)$. For the particular function in our problem

$$\int_0^{2T} e^{-pu}F(u)\,du = \int_0^{T} e^{-pu}\,du = \frac{1-e^{-pT}}{p}$$

Thus

$$\int_0^\infty e^{-pt}F(t)\,dt = \frac{(1-e^{-pT})}{p(1-e^{-2pT})} = \frac{1-e^{-pT}}{p(1-e^{-pT})(1+e^{-pT})} = \frac{1}{p(1+e^{-pT})}.$$

-over-

We showed earlier that the subsidiary equation is

$$\bar{i}(p) = \frac{p}{p^2+\omega^2}\int_0^\infty e^{-pt}F(t)\,dt.$$

Thus

$$\bar{i}(p) = \frac{1}{(p^2+\omega^2)(1+e^{-pT})} \qquad (5.7)$$

To solve our problem, we must find $i(t)$, the inverse of $\bar{i}(p)$.

We can do this

a) by expanding the right hand side of (5.7) in a series

$$\bar{i}(p) = \frac{1-e^{-pT}+e^{-2pT}-e^{-3pT}}{p^2+\omega^2}$$

and inverting each term separately, and then summing the resulting series

or b) by using the inversion theorem (to be established later), namely

$$i(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{pt}\,dp}{(p^2+\omega^2)(1+e^{-pT})}$$

$$= \frac{\sin\omega(t+T/2)}{2\omega\cos\frac{\omega T}{2}} + \frac{2}{T}\sum_{r=0}^{\infty}\frac{\cos(2r+1)\frac{\pi t}{T}}{\omega^2-\left(\frac{(2r+1)^2\pi^2}{T^2}\right)}$$

SAINT LOUIS UNIVERSITY

INSTITUTE OF TECHNOLOGY
DEPARTMENT OF GEOPHYSICS
AND GEOPHYSICAL ENGINEERING

"FIRST WEST OF THE MISSISSIPPI"

3621 OLIVE STREET
SAINT LOUIS, MISSOURI 63108

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Chap 4. section 6 (Irving and Mullineux)

Application of Laplace Transform to Solution of Diffusion Equation

Example:

Diff. eqn.: $\dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{1}{K}\dfrac{\partial \theta}{\partial t}$ for $x > 0$, $t > 0$ (6.1)

Bdy. Cond.: (a) $\theta = \theta_0$ at $x = 0$ for $t > 0$

(b) $\theta = 0$ at $t = 0$ for $x > 0$.

To solve this problem, we shall ~~find~~ write the Laplace transform of (6.1) and then find a solution to the resulting diff. eqn. To evaluate the constants which appear in this solution, we shall have to find the transform of the boundary conditions. This will give us an expression for the transform of the dependent variable θ. By taking the inverse of this we shall obtain an expression for θ in terms of x and t, which is our desired result.

First take the transform of the diff. eq. (6.1), i.e. multiply by e^{-pt} ~~and~~ ~~[illegible]~~ integrate ~~[illegible]~~ with respect to t for t varying from zero to infinity. This can be written as

$$\int_0^\infty e^{-pt}\frac{\partial^2\theta}{\partial x^2}\,dt = \frac{d^2}{dx^2}\int_0^\infty \theta e^{-pt}\,dt = \frac{d^2\bar\theta}{dx^2}, \text{ where } \bar\theta \equiv \int_0^\infty \theta e^{-pt}\,dt.$$

Note that $\bar\theta$ is a function of both p and x, because θ is a function of x and t.

Also $$\int_0^\infty e^{-pt}\frac{\partial\theta}{\partial t}\,dt = \left[\theta e^{-pt}\right]_0^\infty - \int_0^\infty \theta(-p)e^{-pt}\,dt$$ from integration by parts.

$$= 0 - (\theta)_{t=0} + p\bar\theta = p\bar\theta$$ because by (b) we have $\theta_{t=0} = 0$.

Thus the transform of eq. (6.1) is

$$\frac{d^2\bar\theta}{dx^2} - \frac{p}{K}\bar\theta = 0 \quad \text{for } x > 0 \qquad (6.2)$$

Notice that this is an ordinary diff. eq.

Next get the transform of the boundary condition (a). ~~It~~ is

$$\bar\theta_{x=0} = \int_0^\infty (\theta)_{x=0}e^{-pt}\,dt = \theta_0\int_0^\infty e^{-pt}\,dt = \theta_0\left[-\frac{1}{p}e^{-pt}\right]_0^\infty = \theta_0/p \qquad (6.3)$$