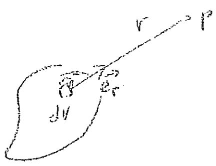
The Newtonian Field – Some Elementary Concepts

Point mass $\vec{F} = -G\frac{m}{r^2}\vec{e}_r$, where origin is taken at point mass

Distributed Mass



$$\vec{F} = -G\int_V \frac{\rho(\vec{\xi})\,\vec{e}_r\,dV}{r^2}$$

[difficult to integrate because $\vec{e}_r$ is a variable]

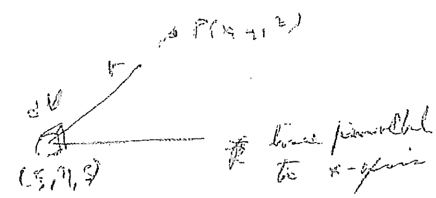
Thus

$$F_x = \vec{F}\cdot\vec{e}_x = -G\int_V \frac{\rho(\vec{\xi})\cos(\vec{e}_r,\vec{e}_x)\,dV}{r^2}$$

etc.

where (ξ,η,ζ) are coordinates of dV and (x,y,z) of point where F_x is being calculated

Now $\cos(r,x) = \frac{x-\xi}{r}$;



therefore $F_x = G\int_V \frac{\rho(\vec{\xi})(\xi - x)}{r^3}\,dV$.

For the gravitational field the potential at $P(x,y,z)$ is defined as the work done by the field in bringing a unit mass from infinity to P (work independent of path taken). Thus

$$U(x,y,z) = \int_\infty^{P(x,y,z)} \left\{-G\int_V \frac{\rho(\vec{\xi})\,\vec{e}_r\,dV}{r^2}\right\}\cdot d\vec{r}$$

If P is outside the body of volume V, r is never zero and we can interchange order of integration. Thus

$$U(x,y,z) = -G\int_V \rho(\vec{\xi})\int_\infty^{P(x,y,z)} \frac{\vec{e}_r\cdot d\vec{r}}{r^2}\,dV = -G\int_V \rho(\vec{\xi})\left\{\int_\infty^{P(x,y,z)} \frac{dr}{r^2}\right\}dV .$$

But $-\frac{dr}{r^2} = d\left(\frac{1}{r}\right)$. Thus

$$U(x,y,z) = G\int_V \rho(\vec{\xi})\left[\frac{1}{r}\right]_\infty^{P(x,y,z)} dV = G\int_V \frac{\rho(\vec{\xi})}{r}\,dV .$$

For points within the body,

$$U = G \lim_{V_0 \to 0} \int_{V-V_0} \frac{\rho}{r}\, dV$$

where V_0 is a small volume containing P

and

$$\vec{F} = -G \lim_{V_0 \to 0} \int_{V-V_0} \frac{\rho\, \vec{e}_r\, dV}{r^2} .$$

It is simple to show that $\nabla U = \frac{\partial U}{\partial x}\vec{e}_x + \frac{\partial U}{\partial y}\vec{e}_y + \frac{\partial U}{\partial z}\vec{e}_z = \vec{F}$, by differentiating expression for U and showing $\frac{\partial U}{\partial x} = F_x$, etc.

Next consider $\nabla^2 U = \nabla \cdot \nabla U$. Now $\nabla^2 U = \nabla \cdot \vec{F}$. Then consider first points outside the body (points of no mass). Then

$$\nabla^2 U = \frac{\partial}{\partial x}\left\{G \int_V \frac{\rho(\xi - x)\, dV}{r^3}\right\} + \frac{\partial}{\partial y}\left\{G \int_V \frac{\rho(\eta - y)\, dV}{r^3}\right\} + \frac{\partial}{\partial z}\left\{G \int_V \frac{\rho(\zeta - z)\, dV}{r^3}\right\}$$

If r is never zero (point of no mass), we can interchange order of differentiation and integration. Then

$$\nabla^2 U = G \int_V \rho \left\{\frac{\partial}{\partial x}\left(\frac{\xi - x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{\eta - y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{\zeta - z}{r^3}\right)\right\} dV$$

$$= G \int_V \rho \left\{\frac{r^3(-1) - 3r^2(\xi - x)\frac{\partial r}{\partial x}}{r^6} + \frac{-r^3 - 3r^2(\eta - y)\frac{\partial r}{\partial y}}{r^6} + \cdots\right\} dV$$

$$= G \int_V \rho \left\{\frac{-r^3 - 3r^2(\xi - x)\left(\frac{\xi - x}{r}\right)}{r^6} + \cdots + \cdots\right\} dV$$

$$= 0 . \quad \text{(Laplace's Equation)}$$

Divergence theorem

$$\int_V (\nabla \cdot \vec{f}\,)\, dV = \int_S \vec{f} \cdot \vec{e}_n \, dS$$

Stokes Theorem

$$\int_C \vec{f} \cdot \vec{dr} = \int_S (\nabla \times \vec{f}\,) \cdot \vec{e}_n \, dS \;.$$

[The condition that a vector function $\vec{f}$ has a scalar potential ϕ is that $\oint_C \vec{f} \cdot \vec{dr} = 0$ for all possible closed paths. From Stoke's theorem we see that if $\oint_C \vec{f} \cdot \vec{dr} = 0$ for all paths, then either $\nabla \times \vec{f} = 0$ or $(\nabla \times \vec{f}\,) \perp \vec{e}_n$ normal to at all points on S. But the latter is not possible for all surfaces enclosed by the path C. Therefore the conclusion is $\nabla \times \vec{f} = 0$, which is another equation for testing if $\vec{f}$ has a potential ϕ.]

Poisson's Equation

At a point of mass, write the potential as

$$U(P) = G\int_{V-V_0} \frac{\rho\, dV}{r} + G\int_{V_0} \frac{\rho\, dV}{r}$$

, where V_0 is a small spherical volume with center at the point P.

Now P is not contained within the body of volume $V-V_0$. Therefore, from Laplace's equation,

$$\nabla^2\left\{G\int_{V-V_0} \frac{\rho\, dV}{r}\right\} = 0 .$$

Take the volume of V_0 so small that within it ρ may be treated as a constant. Then

$$\nabla^2 U(P) = G\rho\, \nabla^2 \int_{V_0} \frac{dV}{r}$$

The integral is the expression for the potential at an arbitrary point inside a sphere of ~~radius~~ volume V_0 and unit density. This potential can be shown to equal $2\pi G\rho(b^2 - r^2) + \frac{2}{3}\pi\rho G r^2$

Thus

$$\nabla^2 U(P) = G\rho\, \nabla^2\left[\pi G\rho\left(-2r^2 + \tfrac{2}{3}r^2\right)\right] = -\frac{2}{3}\pi G\rho\,(\underbrace{\nabla^2 r^2}_{=6}) = -4\pi G\rho$$

Two Dimensional Fields

Consider the Newtonian field intensity of a homogeneous straight wire of infinite length.

$$\vec{F} = \sigma_L G \lim_{L\to\infty} \int_{-L}^{L} \frac{(0-x)\vec{e}_x + (0-y)\vec{e}_y + (\zeta - z)\vec{e}_z}{r^3} \, d\zeta$$

if the wire is placed along the z-axis and σ_L is linear density.

where $r = \sqrt{x^2+y^2+(\zeta-z)^2}$.

Integration gives

$$\vec{F} = G\sigma_L \lim_{L\to\infty} \left[-\frac{(x\vec{e}_x + y\vec{e}_y)\,\zeta}{(x^2+y^2)\sqrt{x^2+y^2+(\zeta-z)^2}} - \frac{\vec{e}_z}{\sqrt{x^2+y^2+(\zeta-z)^2}} \right]_{-L}^{L}$$

$$= G\sigma_L \lim_{L\to\infty} \left[-\frac{(x\vec{e}_x + y\vec{e}_y)}{x^2+y^2} \left\{ \frac{L}{\sqrt{x^2+y^2+(L-z)^2}} + \frac{L}{\sqrt{x^2+y^2+(L+z)^2}} \right\} - \vec{e}_z \left\{ \frac{1}{\sqrt{x^2+y^2+(L-z)^2}} - \frac{1}{\sqrt{x^2+y^2+(L+z)^2}} \right\} \right]$$

$$= -G\sigma_L \left[\frac{x\vec{e}_x + y\vec{e}_y}{x^2+y^2}(1+1) \right] = -\frac{2G\sigma_L}{x^2+y^2}(x\vec{e}_x + y\vec{e}_y)$$

$$= \frac{2G\sigma_L \vec{e}_r}{r}$$

Notice that the field intensity is independent of the z-~~component~~ coordinate of the point P. $\vec{F}$ depends on only 2 space coordinates, x and y, & this is called a 2-dimensional field.

Let us see if this particular $\vec{F}$ has a potential. Take

$$\nabla \times \vec{F} = \vec{e}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = \vec{e}_z(-2G\sigma_L)\left[\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) \right]$$

$$= \vec{e}_z(-2G\sigma_L)\left[\frac{-2xy}{(x^2+y^2)^2} - \frac{-2xy}{(x^2+y^2)^2} \right] = 0$$

Therefore a pot. exists.

To find potential, take

$$U = \int_{\infty}^{P(x,y)} \frac{-2G\sigma_L}{x^2+y^2}\left(x\vec{e}_x + y\vec{e}_y\right)\cdot\left(\vec{e}_x\,dx + \vec{e}_y\,dy\right)$$

$$= -2G\sigma_L \int_{\infty}^{P(x,y)} \frac{x\,dx + y\,dy}{x^2+y^2} = -2G\sigma_L \int_{\infty}^{P(x,y)} \frac{r\,dr}{r^2} = -2G\sigma_L \int_{\infty}^{P(x,y)} \frac{dr}{r}$$

$$= -2G\sigma_L \left[\log r\right]_{\infty}^{P(x,y)} .$$

Here we run into trouble, because $\log\infty = \infty$, which makes the potential infinite at all points $P(x,y)$. This expression is of no physical value.

Take reference point to be somewhere other than infinity, say $P_0(x_0,y_0)$. Then

$$U = -2G\sigma_L\left(\log r - \log r_0\right).$$

It is customary to take $r_0 = 1$, so that $\log r_0 = 0$. Then

$$= -2G\sigma_L \log r$$

$$U_L = 2G\sigma_L \log\frac{1}{r}. \quad \text{(called logarithmic potential).}$$

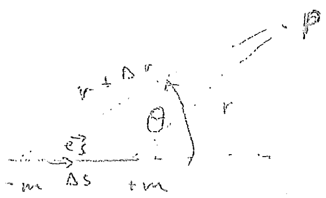
Write expressions for distributed masses.

Problem: Δg for finite slab.

$$U_L = 2G\int_S \rho \log\frac{1}{r}\,dS$$

$$F = 2G \int \rho \int \frac{\vec{e}_x(\xi-x)}{r^2} \ldots \quad r^2 = (\xi-x)^2 + (\eta-y)^2$$

Magnetic or Electric Dipole.



The potential at P owing to the magnetic poles +m and −m is

$$W = \frac{m}{\mu_0 r} - \frac{m}{\mu_0 (r+\Delta r)}$$ where μ_0 is the mag. permeability of free air.

$$= \frac{m}{\mu_0}\left(\frac{1}{r} - \frac{1}{r+\Delta r}\right) = \frac{m\,\Delta r}{\mu_0 r (r+\Delta r)}.$$

As $\Delta s \to 0$, Δr also $\to 0$. Thus $\lim_{\Delta r \to 0} W = 0$. Again, this is a physically undesirable result which does not conform to physical observations and experiments. To avoid the problem, define $\vec{M} = m \Delta s\, \vec{e}_s$ to be a constant, such that m increases as Δs decreases, their product staying constant.

Then, as $\Delta s \to 0$,

$$W = \lim_{\substack{\Delta s \to 0 \\ \Delta r \to 0}} \left\{ \frac{M\, \Delta r/\Delta s}{\mu_0 r (r+\Delta r)} \right\} = \frac{M}{\mu_0 r^2}\frac{dr}{ds}.$$

The derivative $\frac{dr}{ds}$ is a directional derivative, and is equal to the cosine of the angle between r and $\vec{e}_s$. Call it $\cos\theta$. Then

$$W = \frac{M \cos\theta}{\mu_0 r^2}.$$

For volume distributions of magnetized matter, $W = \frac{1}{\mu_0}\int_V \frac{M_v}{r^2}\cos\theta\, dV$ where M_v is magnetic moment per unit volume.

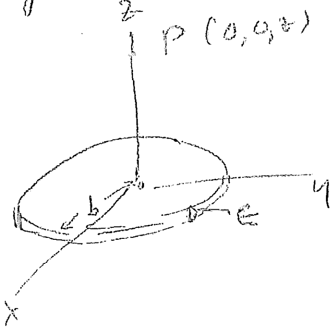
We can build up magnetized bodies from dipoles. The dipoles are aligned in the direction of the inducing field. Then $W = \frac{k\vec{F}_0}{\mu_0} \cdot \int_V \nabla\left(\frac{1}{r}\right) dV$ where k is mag. susceptibility, $\vec{F}_0$ is intensity of inducing field.

Poisson's Relation: $W = -\frac{k\vec{F}_0}{\rho G} \cdot \nabla_{e_0} U$

Problems.

1. Consider a thin circular disc of thickness ϵ and radius b. (The density $\rho = \epsilon\sigma$, where σ is called the surface density and has ~~units of~~ dimensions ML^{-2}.)

a. Derive expressions for the Newtonian potential and field intensity at a point P on the axis of the disc, assuming σ is constant.



b. Set up expressions, in the form of definite integrals, for the Newtonian potential and field intensity of the disc at some general point $P(x, y, z)$. What difficulties will you encounter in attempting to carry out the integrations?

2. Consider a sphere in which the density $\rho = \rho_0 (1 - kr)$ where ρ_0 and k are constants and r is distance from the centre of the sphere.

a. Derive expressions for the Newtonian potential and field intensity at points outside the sphere.

b. Derive expressions for the Newtonian potential and field intensity at points within the sphere.

- over -

c. Discuss the ~~continuity~~ continuity of U, $\frac{\partial U}{\partial x}$ and $\frac{\partial^2 U}{\partial x^2}$, at the ... body.

3) a. Show that $\lim_{R \to \infty} U(P) = 0$, where R is the distance from a point in the body to the point P.

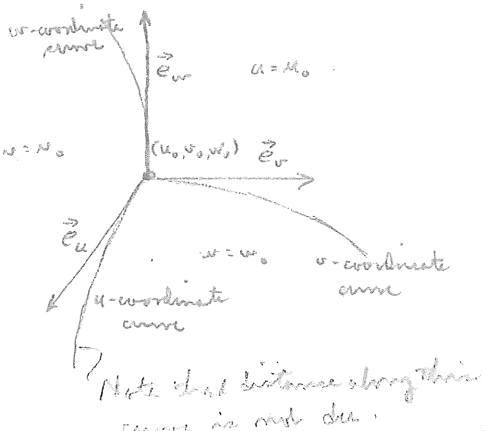
b. Show that $\lim_{R \to \infty} R\,U(P) = GM$, where G is the constant of gravitation and M is the mass of the body.

c. Show that $\lim_{R \to \infty} \vec{F}(P) = 0$.

In proving these results, use $U = G\int_V \frac{\rho\, dv}{r}$ and $\vec{F} = -G\int_V \frac{\vec{e}_r\, \rho\, dv}{r^2}$

Curvilinear Coordinates   Dettmann pg 137 ff.

Consider the coordinate surfaces $u = u_0$, $v = v_0$, $w = w_0$ which intersect orthogonally at the point (u_0, v_0, w_0). That is, the tangents to the curves of intersection of these surfaces at (u_0, v_0, w_0) are mutually perpendicular.



The orthogonality of the coordinate system can be expressed in three different ways.

The first, very simply, is

$\vec{e}_u \cdot \vec{e}_v = \vec{e}_u \cdot \vec{e}_w = \vec{e}_v \cdot \vec{e}_w = 0$.

The second makes use of the fact that $\vec{e}_u, \vec{e}_v, \vec{e}_w$ are vectors normal to the surfaces $u = u_0$, $v = v_0$, $w = w_0$, respectively. Therefore

$$\vec{e}_u = \frac{\nabla u}{|\nabla u|}$$, i.e., the gradient of u is normal to the surface u = constant $= c$

$$\vec{e}_v = \frac{\nabla v}{|\nabla v|}$$

$$\vec{e}_w = \frac{\nabla w}{|\nabla w|} ,$$

where the gradients are evaluated at (u_0, v_0, w_0).

Thus, at (u_0, v_0, w_0), ∇u has the direction of $\vec{e}_u$, ∇v that of $\vec{e}_v$, and ∇w that of $\vec{e}_w$. Therefore, $\nabla u \cdot \nabla v = 0$, etc. But $\nabla u = \frac{\partial u}{\partial x}\vec{e}_x + \frac{\partial u}{\partial y}\vec{e}_y + \frac{\partial u}{\partial z}\vec{e}_z$. Thus

$$\nabla u \cdot \nabla v = 0 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}$$

and

$$\nabla u \cdot \nabla w = 0 = \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z}$$

$$\nabla v \cdot \nabla w = 0 = \frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} .$$

The third manner of expressing the orthogonality of the coordinate system is by expressing the unit tangent vectors $\vec{e}_u, \vec{e}_v, \vec{e}_w$ in terms of the rectangular coordinates x, y, z or their derivatives. ~~Let ds be any element of arc length, and~~

~~See pg. 8 to get the direction of $\vec{e}_u$~~

Thus,

Gives unit vectors in curvilinear coordinates in terms of $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$

$$\vec{e}_u = \left(\frac{\partial x}{\partial u} \vec{e}_x + \frac{\partial y}{\partial u} \vec{e}_y + \frac{\partial z}{\partial u} \vec{e}_z \right) \Big/ \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}$$

$$\vec{e}_v = \left(\frac{\partial x}{\partial v} \vec{e}_x + \frac{\partial y}{\partial v} \vec{e}_y + \frac{\partial z}{\partial v} \vec{e}_z \right) \Big/ \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2}$$

$$\vec{e}_w = \left(\frac{\partial x}{\partial w} \vec{e}_x + \frac{\partial y}{\partial w} \vec{e}_y + \frac{\partial z}{\partial w} \vec{e}_z \right) \Big/ \sqrt{\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2} .$$

Therefore, $\vec{e}_u \cdot \vec{e}_v = 0$ gives

$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 0$$

and

$$\vec{e}_u \cdot \vec{e}_w = 0 = \frac{\partial x}{\partial u}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial w}$$

$$\vec{e}_v \cdot \vec{e}_w = 0 = \frac{\partial x}{\partial v}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial v}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial v}\frac{\partial z}{\partial w} .$$

We are now in a position to express the line element ds in terms of the coordinates (u, v, w). Into

$$ds^2 = dx^2 + dy^2 + dz^2$$

substitute $dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw$, etc.

This gives

$$\begin{aligned} ds^2 = & \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \right] (du)^2 \\ & + \left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \right] (dv)^2 \\ & + \left[\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2 \right] (dw)^2 \\ & + 2 \left[\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right) + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \right] du\, dv \\ & + 2 \left[\frac{\partial x}{\partial u}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial w} \right] du\, dw \\ & + 2 \left[\frac{\partial x}{\partial v}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial v}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial v}\frac{\partial z}{\partial w} \right] dv\, dw . \end{aligned}$$

But the last 3 terms in brackets are zero. Thus

h_i called scale factors

$$ds^2 = h_1^2 (du)^2 + h_2^2 (dv)^2 + h_3^2 (dw)^2$$

where $h_1^2 \equiv \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$

$h_2^2 =$ etc.

These are important equations in practical applications

We may think of $h_1\,du$ as an element of length along the u-axis. It is the projection of ds on the u-axis.

Note that $\dfrac{\partial u}{\partial s_u} = \dfrac{1}{\sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}} = \dfrac{1}{h_1}$, $\dfrac{\partial v}{\partial s_v} = \dfrac{1}{h_2}$, $\dfrac{\partial w}{\partial s_w} = \dfrac{1}{h_3}$

The elements of area in generalized coordinates are

$$dS_1 = h_2 h_3 \, dv \, dw$$

$$dS_2 = h_1 h_3 \, du \, dw$$

$$dS_3 = h_1 h_2 \, dv \, dw \, .$$

The element of volume is

$$dV = h_1 h_2 h_3 \, du \, dv \, dw \, .$$

Next let me write an expression for the gradient of a scalar φ in curvilinear coordinates. In rectangular coordinates,

$$\nabla \varphi = \frac{\partial \varphi}{\partial x} \vec{e}_x + \frac{\partial \varphi}{\partial y} \vec{e}_y + \frac{\partial \varphi}{\partial z} \vec{e}_z$$

$$= \left(\frac{\partial \varphi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial \varphi}{\partial w} \frac{\partial w}{\partial x} \right) \vec{e}_x$$

$$+ \left(\frac{\partial \varphi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial \varphi}{\partial w} \frac{\partial w}{\partial y} \right) \vec{e}_y$$

$$+ \left(\frac{\partial \varphi}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial \varphi}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial \varphi}{\partial w} \frac{\partial w}{\partial z} \right) \vec{e}_z$$

$$= \frac{\partial \varphi}{\partial u} \left(\frac{\partial u}{\partial x} \vec{e}_x + \frac{\partial u}{\partial y} \vec{e}_y + \frac{\partial u}{\partial z} \vec{e}_z \right) + \frac{\partial \varphi}{\partial v} \left(\frac{\partial v}{\partial x} \vec{e}_x + \frac{\partial v}{\partial y} \vec{e}_y + \frac{\partial v}{\partial z} \vec{e}_z \right)$$

$$+ \frac{\partial \varphi}{\partial w} \left(\frac{\partial w}{\partial x} \vec{e}_x + \frac{\partial w}{\partial y} \vec{e}_y + \frac{\partial w}{\partial z} \vec{e}_z \right)$$

$$= \frac{\partial \varphi}{\partial u} \nabla u + \frac{\partial \varphi}{\partial v} \nabla v + \frac{\partial \varphi}{\partial w} \nabla w$$

$$= \frac{\partial \varphi}{\partial u} |\nabla u| \vec{e}_u + \frac{\partial \varphi}{\partial v} |\nabla v| \vec{e}_v + \frac{\partial \varphi}{\partial w} |\nabla w| \vec{e}_w$$

$$= \frac{\partial \varphi}{\partial u} \frac{du}{ds_u} \vec{e}_u + \frac{\partial \varphi}{\partial v} \frac{dv}{\partial s_v} \vec{e}_v + \frac{\partial \varphi}{\partial w} \frac{\partial w}{\partial s_w} \vec{e}_w \, .$$

from back of pg. 5

~~Since~~ ~~where we move along the curve holding v and w constant.~~

Now

$$\vec{e}_u = \left(\frac{\partial x}{\partial u}\vec{e}_x + \frac{\partial y}{\partial u}\vec{e}_y + \frac{\partial z}{\partial u}\vec{e}_z\right)\frac{du}{ds_u}$$

$$= \left(\frac{\partial x}{\partial u}\vec{e}_x + \frac{\partial y}{\partial u}\vec{e}_y + \frac{\partial z}{\partial u}\vec{e}_z\right)\Big/\sqrt{\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2} .$$

Thus

$$\frac{du}{ds_u} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2}} = \frac{1}{h_1} .$$

Similarly,

$$\frac{dv}{ds_v} = \frac{1}{h_2} , \quad \frac{dw}{ds_w} = \frac{1}{h_3} .$$

unnecessary

Substituting for $\frac{du}{ds_u}$ from the equation on the top of pg. 7, ~~Therefore~~, the gradient in curvilinear coordinates is

$$\nabla\varphi = \frac{1}{h_1}\frac{\partial\varphi}{\partial u}\vec{e}_u + \frac{1}{h_2}\frac{\partial\varphi}{\partial v}\vec{e}_v + \frac{1}{h_3}\frac{\partial\varphi}{\partial w}\vec{e}_w .$$

Next we shall obtain an expression for the divergence of a vector in curvilinear coordinates. First note from above that

$$\vec{e}_u = h_1\nabla u , \quad \vec{e}_v = h_2\nabla v , \quad \vec{e}_w = h_3\nabla w .$$

~~Now~~ Write

$$\cancel{\vec{F} = \vec{F}_u}$$

$$\vec{F} = F_u\vec{e}_u + F_v\vec{e}_v + F_w\vec{e}_w .$$

Using the preceding equations, ~~we can write this as~~ and making use of the fact that $\vec{e}_u, \vec{e}_v, \vec{e}_w$ are orthogonal, we can write

$$\cancel{\vec{F} = F_u}$$

$$\vec{e}_u = \vec{e}_v\times\vec{e}_w = h_2h_3\,\nabla v\times\nabla w \quad , \text{etc}$$

$\nabla\cdot a\vec{b} = \nabla a\cdot\vec{b} + a\nabla\cdot\vec{b}$

and

$$\vec{F} = h_2h_3F_u(\nabla v\times\nabla w) + h_1h_3F_v(\nabla w\times\nabla u) + h_1h_2F_w(\nabla u\times\nabla v)$$

Thus the divergence of $\vec{F}$ is

$$\nabla\cdot\vec{F} = \nabla\cdot[h_2h_3F_u(\nabla v\times\nabla w)] + \nabla\cdot[h_1h_3F_v(\nabla w\times\nabla u)] + \nabla\cdot[h_1h_2F_w(\nabla u\times\nabla v)]$$

$$= \nabla(h_2h_3F_u)\cdot(\nabla v\times\nabla w) + \nabla(h_1h_3F_v)\cdot(\nabla w\times\nabla u) + \nabla(h_1h_2F_w)\cdot(\nabla u\times\nabla v)$$

$$+ h_2h_3F_u\nabla\cdot(\nabla v\times\nabla w) + h_1h_3F_v\nabla\cdot(\nabla w\times\nabla u) + h_1h_2F_w\nabla\cdot(\nabla u\times\nabla v)$$

[If potential exists, then the curl of its gradient is equal to zero.]

Now $\nabla\cdot(\nabla v\times\nabla w) = \nabla w\cdot(\nabla\times\nabla v) - \nabla v\cdot(\nabla\times\nabla w) = 0$ because the curl of a gradient is zero.

Therefore the last three terms in the expression for $\nabla\cdot\vec{F}$ are zero.

Consider the term $\nabla(h_2h_3F_u)\cdot(\nabla v\times\nabla w)$ which appears in the expression for $\nabla\cdot\vec{F}$.

$$\cancel{\nabla F_u\frac{h_2h_3}{} = \frac{h_2h_3}{h_1}\frac{\partial F_u}{\partial u}\vec{e}_u + \frac{h_3}{h_2}\frac{\partial(F_u h_2)}{\partial v}\vec{e}_v + \frac{h_2}{h_3}\frac{\partial(F_u h_3)}{\partial w}\vec{e}_w}$$

~~Thus~~ Now, from the expression for the gradient in curvilinear coordinates,

$$\nabla(h_2h_3F_u) = \frac{1}{h_1}\frac{\partial(h_2h_3F_u)}{\partial u}\vec{e}_u + \frac{1}{h_2}\frac{\partial(h_2h_3F_u)}{\partial v}\vec{e}_v + \frac{1}{h_3}\frac{\partial(h_2h_3F_u)}{\partial w}\vec{e}_w\,,$$

Also ∇v has the direction of $\vec{e}_v$ and ∇w that of $\vec{e}_w$. Thus

$$\nabla(h_2h_3F_u)\cdot(\nabla v\times\nabla w) = \frac{1}{h_1}\frac{\partial(h_2h_3F_u)}{\partial u}\vec{e}_u\cdot(\nabla v\times\nabla w)\,.$$

But $\frac{1}{h_1}\vec{e}_u = \nabla u$. Thus

$$\nabla(h_2h_3F_u)\cdot(\nabla v\times\nabla w) = \frac{\partial(h_2h_3F_u)}{\partial u}\nabla u\cdot(\nabla v\times\nabla w)$$

and therefore

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial u}(h_2h_3F_u)\nabla u\cdot(\nabla v\times\nabla w) + \frac{\partial}{\partial v}(h_1h_3F_v)\nabla v\cdot(\nabla w\times\nabla u)$$
$$+\frac{\partial}{\partial w}(h_1h_2F_w)\nabla w\cdot(\nabla u\times\nabla v)\,.$$

We may note that

$$1 = \vec{e}_u\cdot(\vec{e}_v\times\vec{e}_w) = \vec{e}_v\cdot(\vec{e}_w\times\vec{e}_u) = \vec{e}_w\cdot(\vec{e}_u\times\vec{e}_v)$$
$$= h_1h_2h_3\nabla u\cdot(\nabla v\times\nabla w) = h_1h_2h_3\nabla v\cdot(\nabla w\times\nabla u) = h_1h_2h_3\nabla w\cdot(\nabla u\times\nabla v),$$

$\nabla u\cdot(\nabla v\times\nabla w) = \frac{1}{h_1h_2h_3}$ etc.

remembering that $\vec{e}_u = h_1\nabla u$, etc.

Thus

$$\nabla\cdot\vec{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3F_u) + \frac{\partial}{\partial v}(h_1h_3F_v) + \frac{\partial}{\partial w}(h_1h_2F_w)\right].$$

The expression for the Laplacian $\nabla^2\phi$ follows immediately, since $\nabla^2\phi = \nabla\cdot\nabla\phi$

$(F_u = \frac{1}{h_1}\frac{\partial\phi}{\partial u})$ from the expression for the gradient)

Thus

$$\nabla\cdot\nabla\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial w}\right)\right].$$

Finally, let us derive an expression for the curl of $\vec{F}$ in curvilinear coordinates. Now

$$\vec{F} = F_u \vec{e}_u + F_v \vec{e}_v + F_w \vec{e}_w = h_1 F_u \nabla u + h_2 F_v \nabla v + h_3 F_w \nabla w .$$

$$\nabla \times \psi \vec{E} = \psi \nabla \times \vec{E} + \nabla \psi \times \vec{E}$$

Thus

$$\nabla \times \vec{F} = \nabla(h_1 F_u) \times \nabla u + \nabla(h_2 F_v) \times \nabla v + \nabla(h_3 F_w) \times \nabla w$$
$$+ h_1 F_u \nabla \times \nabla u + h_2 F_v \nabla \times \nabla v + h_3 F_w \nabla \times \nabla w .$$

But the last 3 terms are zero, because the ~~gradient~~ curl of a ~~curl~~ gradient is zero.

Consider the term $\nabla(h_1 F_u) \times \nabla u$. Using the expression for the gradient in curvilinear coordinates, we get

$$\nabla(h_1 F_u) \times \nabla u = \left[\frac{\vec{e}_u}{h_1} \frac{\partial (h_1 F_u)}{\partial u} + \frac{\vec{e}_v}{h_2} \frac{\partial (h_1 F_u)}{\partial v} + \frac{\vec{e}_w}{h_3} \frac{\partial (h_1 F_u)}{\partial w} \right] \times \nabla u \qquad \text{and } \frac{\vec{e}_u}{h_1} = \nabla u, \text{ etc}$$

$$= 0 + \frac{\partial (h_1 F_u)}{\partial v} \nabla v \times \nabla u + \frac{\partial (h_1 F_u)}{\partial w} \nabla w \times \nabla u .$$

Thus

$$\nabla \times \vec{F} = \frac{\partial (h_1 F_u)}{\partial v} (\nabla v \times \nabla u) + \frac{\partial (h_1 F_u)}{\partial w} (\nabla w \times \nabla u)$$
$$+ \frac{\partial (h_2 F_v)}{\partial u} (\nabla u \times \nabla v) + \frac{\partial (h_2 F_v)}{\partial w} (\nabla w \times \nabla v)$$
$$+ \frac{\partial (h_3 F_w)}{\partial u} (\nabla u \times \nabla w) + \frac{\partial (h_3 F_w)}{\partial v} (\nabla v \times \nabla w) .$$

But $\nabla v \times \nabla u = \dfrac{\vec{e}_v}{h_2} \times \dfrac{\vec{e}_u}{h_1} = \dfrac{-\vec{e}_w}{h_1 h_2}$.

Thus

$$\nabla \times \vec{F} = \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial v} (h_3 F_w) - \frac{\partial}{\partial w} (h_2 F_v) \right] \vec{e}_u$$
$$+ \frac{1}{h_1 h_3} \left[\frac{\partial}{\partial w} (h_1 F_u) - \frac{\partial}{\partial u} (h_3 F_w) \right] \vec{e}_v$$
$$+ \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial u} (h_2 F_v) - \frac{\partial}{\partial v} (h_1 F_u) \right] \vec{e}_w$$

$$= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{e}_u & h_2 \vec{e}_v & h_3 \vec{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 F_u & h_2 F_v & h_3 F_w \end{vmatrix} .$$

As an example, consider spherical coordinates. In this system

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta .$$

Thus, $u = r$, $v = \theta$, $w = \varphi$.

$$\cancel{(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = r^2\sin^2\theta\cos}$$

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2} = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} = 1$$

$$h_2 = \sqrt{r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta} = r$$

$$h_3 = \sqrt{r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi} = r\sin\theta .$$

The element of arc length is

$$ds = \sqrt{h_1^2\,du^2 + h_2^2\,dv^2 + h_3^2\,dw^2} = \sqrt{(dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\varphi)^2} .$$

The element of area is

$$dS'_u = h_2 h_3\, dv\, dw = r^2\sin\theta\, d\theta\, d\varphi .$$

The element of volume is

$$dV = h_1 h_2 h_3\, du\, dv\, dw = r^2\sin\theta\, dr\, d\theta\, d\varphi .$$

The gradient is

$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u}\vec{e}_u + \frac{1}{h_2}\frac{\partial\psi}{\partial v}\vec{e}_v + \frac{1}{h_3}\frac{\partial\psi}{\partial w}\vec{e}_w$$

$$= \frac{\partial\psi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\vec{e}_\varphi .$$

The divergence is

$$\nabla\cdot\vec{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3F_u) + \frac{\partial}{\partial v}(h_1h_3F_v) + \frac{\partial}{\partial w}(h_1h_2F_w)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[r^2\sin\theta\frac{\partial F_r}{\partial r} + 2r\sin\theta\, F_r + r\sin\theta\frac{\partial F_\theta}{\partial\theta} + r\cos\theta\, F_\theta + r\frac{\partial F_\varphi}{\partial\varphi}\right]$$

$$\cancel{= \frac{1}{r\sin\theta}\left[2\sin\theta\frac{\partial F_r}{\partial r} + \sin\theta\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_\varphi}{\partial\varphi}\right]}$$

$$\nabla\cdot\vec{F} = \frac{\partial F_r}{\partial r} + \frac{2}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\cot\theta}{r}F_\theta + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi} .$$

The Laplacian is

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial w}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[2r\sin\theta\frac{\partial\psi}{\partial r} + r^2\sin\theta\frac{\partial^2\psi}{\partial r^2} + \cos\theta\frac{\partial\psi}{\partial\theta} + \sin\theta\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]$$

$$= \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}$$

The curl is

$$\nabla \times \vec{F} = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial v}(h_3 F_w) - \frac{\partial}{\partial w}(h_2 F_v)\right]\vec{e}_u$$

$$+ \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial w}(h_1 F_u) - \frac{\partial}{\partial u}(h_3 F_w)\right]\vec{e}_v$$

$$+ \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u}(h_2 F_v) - \frac{\partial}{\partial v}(h_1 F_u)\right]\vec{e}_w .$$

$$= \frac{1}{r^2 \sin\theta}\left[r\cos\theta\, F_\varphi + r\sin\theta\,\frac{\partial F_\varphi}{\partial \theta} - r\,\frac{\partial F_\theta}{\partial \varphi}\right]\vec{e}_r$$

$$+ \frac{1}{r\sin\theta}\left[\frac{\partial F_r}{\partial \varphi} - r\sin\theta\,\frac{\partial F_\varphi}{\partial r} - \sin\theta\, F_\varphi\right]\vec{e}_\theta$$

$$+ \frac{1}{r}\left[r\,\frac{\partial F_\theta}{\partial r} + F_\theta - \frac{\partial F_r}{\partial \theta}\right]\vec{e}_\varphi .$$

One Method of Solving Laplace's Equation - Separation of Variables

1. Cartesian Coordinates

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

<u>Assume</u> $\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$ is a solution of $\nabla^2\phi=0$.

Substitute into diff. eqn.

$$YZ\frac{d^2X}{dx^2}+XZ\frac{d^2Y}{dy^2}+XY\frac{d^2Z}{dz^2}=0$$

or $$\frac{1}{X}\frac{d^2X}{dx^2}=-\left(\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}\right)\equiv -p^2 \quad \text{(a constant)}$$

Then $$\frac{1}{Y}\frac{d^2Y}{dy^2}=+\left(p^2-\frac{1}{Z}\frac{d^2Z}{dz^2}\right)=-q^2 \quad \text{(a constant)}$$

and $$\frac{1}{Z}\frac{d^2Z}{dz^2}=p^2+q^2$$

The three diff. eqns are ~~separa~~

$$\left.\begin{aligned}&\frac{d^2X}{dx^2}+p^2X=0\\&\frac{d^2Y}{dy^2}+q^2Y=0\\&\frac{d^2Z}{dz^2}-(p^2+q^2)Z=0\end{aligned}\right\}$$

In them the variables are separated. We'll obtain solutions to boundary value problems later.

2. Spherical coordinates

$$\nabla^2\phi = \frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\psi^2}$$

Assume $\varphi = R(r)\,\Theta(\theta)\,\Psi(\psi)$. The diff equation becomes, after substitution,

$$0 = \Theta\Psi\frac{d^2R}{dr^2} + \frac{2}{r}\Theta\Psi\frac{dR}{dr} + \frac{1}{r^2}R\Psi\frac{d^2\Theta}{d\theta^2} + \frac{\cot\theta}{r^2}R\Psi\frac{d\Theta}{d\theta} + \frac{1}{r^2\sin^2\theta}R\Theta\frac{d^2\Psi}{d\psi^2}$$

Divide by $R\Theta\Psi$ and multiply by r^2.

$$0 = \frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{\cot\theta}{\Theta}\frac{d\Theta}{d\theta} + \frac{1}{\sin^2\theta\,\Psi}\frac{d^2\Psi}{d\psi^2}$$

Then we have

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} = -\left(\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{\cot\theta}{\Theta}\frac{d\Theta}{d\theta} + \frac{1}{\sin^2\theta\,\Psi}\frac{d^2\Psi}{d\psi^2}\right)$$

$= +n(n+1)$ where n is a constant

Next multiply by $\sin^2\theta$.

$$\frac{\sin^2\theta}{\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{\sin\theta\cos\theta}{\Theta}\frac{d\Theta}{d\theta} + n(n+1)\sin^2\theta = -\frac{1}{\Psi}\frac{d^2\Psi}{d\psi^2} = m^2$$

Therefore $\nabla^2\varphi = 0$ separates into

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0$$

$$\frac{d^2\Psi}{d\psi^2} + m^2\Psi = 0$$

$$\sin^2\theta\frac{d^2\Theta}{d\theta^2} + \sin\theta\cos\theta\frac{d\Theta}{d\theta} + \left[n(n+1)\sin^2\theta - m^2\right]\Theta = 0$$

(dividing by $\sin^2\theta$)

or $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

$$\frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + [n(n+1) \quad - \frac{m^2}{\sin^2\theta}]\Theta = 0$$

$$\sin\theta\frac{d^2\Theta}{d\theta^2} + \cos\theta\frac{d\Theta}{d\theta}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

3. Circular cylindrical coordinates

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Assume $\phi = R(r)\,\Theta(\theta)\,Z(z)$.

$$\Theta Z\frac{d^2R}{dr^2} + \frac{\Theta Z}{r}\frac{dR}{dr} + \frac{RZ}{r^2}\frac{d^2\Theta}{d\theta^2} + \cancel{R\Theta}\, R\Theta\frac{d^2Z}{dz^2} = 0$$

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

(a) $$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -p^2$$

$$\frac{d^2Z}{dz^2} - p^2Z = 0$$

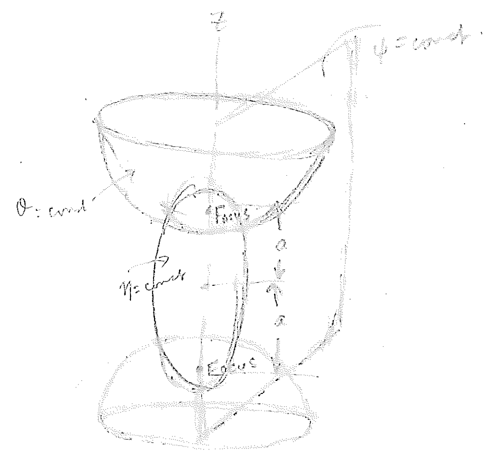
Multiply (a) by r^2. This gives

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + p^2r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = q^2.$$

Then the separated equations are:

$$\left.\begin{aligned}
&\frac{d^2Z}{dz^2} - p^2Z = 0\\
&\frac{d^2\Theta}{d\theta^2} + q^2\Theta = 0\\
&r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (p^2r^2 - q^2)R = 0
\end{aligned}\right\}$$

Laplace's Eqn. in Prolate Spheroidal Coordinates — Moon & Spencer, pg 236



Prolate spheroidal coordinates (η, θ, ψ) are related to rectangular coordinates by

$$x = a \sinh\eta \sin\theta \cos\psi$$
$$y = a \sinh\eta \sin\theta \sin\psi$$
$$z = a \cosh\eta \cos\theta .$$

The surfaces of constant η are prolate spheroids $\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $b = a \sinh\eta$, $c = a\cosh\eta$. Thus when $\eta = 0$, the prolate spheroid reduces to a line on the z axis from $-a$ to $+a$. As η increases, the spheroids become more spherical, and approach an infinite sphere as $\eta \to \infty$.

The surfaces of constant θ are hyperboloids of 2 sheets, namely

$$\frac{-x^2}{b^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{, where}$$

$b = a\sin\theta$, $c = a\cos\theta$. The angle θ is the angle between the asymptotic cone of the hyperboloid and the z-axis. For $\theta = 0$, the cone degenerates into the part of the z-axis from ∞ to $+a$; for $\theta = \pi$, the part of the z-axis from $-a$ to $-\infty$. At $\theta = \pi/2$ the hyperboloid becomes the xy plane.

The scale factors for this coordinate system are:

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2} = a\sqrt{\cosh^2\eta \sin^2\theta + \sinh^2\eta \cos^2\theta}$$
$$= a\sqrt{(1+\sinh^2\eta)\sin^2\theta + \sinh^2\eta(1-\sin^2\theta)}$$
$$= a\sqrt{\sin^2\theta + \sinh^2\eta}$$

$$h_1^2 = h_2^2 = a^2(\sinh^2\eta + \sin^2\theta)$$
$$h_3^2 = a^2 \sinh^2\eta \sin^2\theta .$$

Laplace's eqn becomes

$$\nabla^2\varphi = \frac{1}{a^2(\sinh^2\eta + \sin^2\theta)}\left\{\frac{\partial^2\varphi}{\partial\eta^2} + \coth\eta\frac{\partial\varphi}{\partial\eta} + \frac{\partial^2\varphi}{\partial\theta^2} + \cot\theta\frac{\partial\varphi}{\partial\theta}\right\} + \frac{1}{a^2\sinh^2\eta\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} = 0 .$$

Assume a separation of variables. Let

$$\varphi = H(\eta)\,\Theta(\theta)\,\Psi(\psi).$$

Substitution gives

$$0 = \frac{1}{a^2(\sinh^2\eta + \sin^2\theta)}\left\{\frac{1}{H}\left(\frac{d^2H}{d\eta^2} + \coth\eta\,\frac{dH}{d\eta}\right) + \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta}\right)\right\} + \frac{1}{a^2\sinh^2\eta\,\sin^2\theta}\cdot\frac{1}{\Psi}\frac{d^2\Psi}{d\psi^2}$$

or

$$\frac{\sinh^2\eta\,\sin^2\theta}{\sinh^2\eta + \sin^2\theta}\left\{\frac{1}{H}\left(\frac{d^2H}{d\eta^2} + \coth\eta\,\frac{dH}{d\eta}\right) + \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta}\right)\right\} = -\frac{1}{\Psi}\frac{d^2\Psi}{d\psi^2} = q^2$$

This separates the ψ term. There remains

$$\frac{\sinh^2\eta + \sin^2\theta}{\sinh^2\eta\,\sin^2\theta}\,q^2 = \frac{1}{H}\left(\frac{d^2H}{d\eta^2} + \coth\eta\,\frac{dH}{d\eta}\right) + \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta}\right)$$

or

$$\frac{q^2}{\sin^2\theta} - \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta}\right) = \frac{-q^2}{\sinh^2\eta} + \frac{1}{H}\left(\frac{d^2H}{d\eta^2} + \coth\eta\,\frac{dH}{d\eta}\right) = p(p+1).$$

Thus the separated equations are

$$\frac{d^2H}{d\eta^2} + \coth\eta\,\frac{dH}{d\eta} - \left[p(p+1) + \frac{q^2}{\sinh^2\eta}\right]H = 0$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta} + \left[p(p+1) - \frac{q^2}{\sin^2\theta}\right]\Theta = 0$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0.$$

Make a change in variables. Let $\xi = \cosh\eta$, $\mu = \cos\theta$. This gives

$$(\xi^2-1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - \left[p(p+1) + \frac{q^2}{\xi^2-1}\right]H = 0$$

$$(\mu^2-1)\frac{d^2\Theta}{d\mu^2} + 2\mu\frac{d\Theta}{d\mu} - \left[p(p+1) + \frac{q^2}{\mu^2-1}\right]\Theta = 0,$$

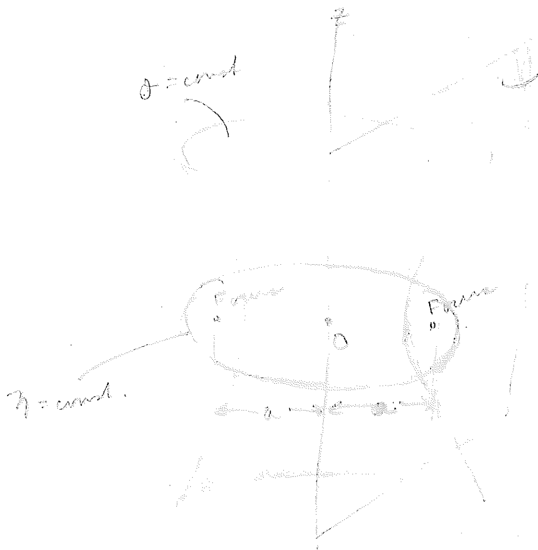
both of which are associated Legendre's equations. Thus

$$\varphi = \left.\begin{matrix} P_p^q(\cosh\eta) \\ Q_p^q(\cosh\eta)\end{matrix}\right\}\left.\begin{matrix} P_p^q(\cos\theta) \\ Q_p^q(\cos\theta)\end{matrix}\right\}\left\{\begin{matrix}\cos \\ \sin\end{matrix}\, q\psi\right\}.$$

In the case of axial symmetry,

$$\varphi = \left.\begin{matrix} P_p(\cosh\eta) \\ Q_p(\cosh\eta)\end{matrix}\right\}\left.\begin{matrix} P_p(\cos\theta) \\ Q_p(\cos\theta)\end{matrix}\right\}$$

Laplace's Equation in Oblate Spheroidal Coordinates (Moon & Spencer, pg 267 f.f.)



Oblate spheroidal coordinates are related to rectangular coordinates by

$x = a \cosh\eta \sin\theta \cos\varphi$

$y = a \cosh\eta \sin\theta \sin\varphi$

$z = a \sinh\eta \cos\theta$.

The surfaces of constant η are oblate spheroids $\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $b = a\cosh\eta$, $c = a\sinh\eta$. Thus when $\eta = 0$, the spheroid degenerates into a flat disc of radius a. The spheroids become more spherical and approach an infinite sphere as $\eta \to \infty$.

The surfaces of constant θ are hyperboloids of one sheet, namely

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

where $b = a\sin\theta$, $c = a\cos\theta$. The positive z axis corresponds to $\theta = 0$, and the negative z axis to $\theta = \pi$. For $\theta = \pi/2$ the hyperboloid flattens out into the entire x y axis except for a circular opening of radius a. The angle θ in general is the angle between the asymptotic cone and the z-axis.

The scale factors for this coordinate system are

$h_1^2 = h_2^2 = a^2(\cosh^2\eta - \sin^2\theta)$

$h_3^2 = a^2 \cosh^2\eta \sin^2\theta$.

Laplace's equation becomes

$$\nabla^2\varphi = \frac{1}{a^2(\cosh^2\eta - \sin^2\theta)}\left\{ \frac{\partial^2\varphi}{\partial\eta^2} + \tanh\eta\,\frac{\partial\varphi}{\partial\eta} + \frac{\partial^2\varphi}{\partial\theta^2} + \cot\theta\,\frac{\partial\varphi}{\partial\theta} \right\}$$

$$+ \frac{1}{a^2\cosh^2\eta\sin^2\theta}\,\frac{\partial^2\varphi}{\partial\varphi^2} = 0 .$$

Assume separation of variables. Let

$$\phi = H(\eta)\,\Theta(\theta)\,\Phi(\varphi).$$

Substitution gives

$$0 = \frac{1}{a^2(\cosh^2\eta - \sin^2\theta)}\left\{\frac{1}{H}\left(\frac{d^2H}{d\eta^2} + \tanh\eta\,\frac{dH}{d\eta}\right) + \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta}\right)\right\}$$

$$+ \frac{1}{a^2\cosh^2\eta\,\sin^2\theta}\,\frac{1}{\Phi}\,\frac{d^2\Phi}{d\varphi^2},$$

or

$$\frac{\cosh^2\eta\,\sin^2\theta}{\cosh^2\eta - \sin^2\theta}\left\{\frac{1}{H}\left(\frac{d^2H}{d\eta^2} + \tanh\eta\,\frac{dH}{d\eta}\right) + \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta}\right)\right\} = -\frac{1}{\Phi}\cdot\frac{d^2\Phi}{d\varphi^2} = q^2.$$

This separates the φ term. There remains

$$\frac{\cosh^2\eta - \sin^2\theta}{\cosh^2\eta\,\sin^2\theta}\,q^2 = \frac{1}{H}\left(\frac{d^2H}{d\eta^2} + \tanh\eta\,\frac{dH}{d\eta}\right) + \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta}\right)$$

or

$$\frac{q^2}{\sin^2\theta} - \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta}\right) = \frac{q^2}{\cosh^2\eta} + \frac{1}{H}\left(\frac{d^2H}{d\eta^2} + \tanh\eta\,\frac{dH}{d\eta}\right) = p(p+1).$$

Thus the separated equations are

$$\frac{d^2H}{d\eta^2} + \tanh\eta\,\frac{dH}{d\eta} + \left[-p(p+1) + \frac{q^2}{\cosh^2\eta}\right]H = 0$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta} + \left[p(p+1) - \frac{q^2}{\sin^2\theta}\right]\Theta = 0$$

$$\frac{d^2\Phi}{d\varphi^2} + q^2\Phi = 0.$$

In the first of these diff. eqns., let $\xi = i\sinh\eta$. It reduces to

$$(\xi^2-1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - \left[p(p+1) + \frac{q^2}{\xi^2-1}\right]H = 0$$

, which is the associated Legendre's eqn.

Therefore the solutions of Laplace's eqn. are:

$$\phi = \left.\begin{matrix} P_p^q(i\sinh\eta) \\ Q_p^q(i\sinh\eta) \end{matrix}\right\} \left.\begin{matrix} P_p^q(\cos\theta) \\ Q_p^q(\cos\theta) \end{matrix}\right\} \begin{matrix} \cos \\ \sin \end{matrix}\, q\varphi \Big\}.$$

For axial symmetry,

$$\phi = \left.\begin{matrix} P_p(i\sinh\eta) \\ Q_p(i\sinh\eta) \end{matrix}\right\} \left.\begin{matrix} P_p(\cos\theta) \\ Q_p(\cos\theta) \end{matrix}\right\}.$$

We shall discuss examples involving spheroidal coordinates after discussing [illegible] Legendre functions.

(23)

Fourier Integral

If the function $f(x)$ is non-periodic and defined over the interval $-\infty < x < \infty$; we cannot represent it by a Fourier series. Rather we must use the Fourier integral representation,

$$f(x) = \cancel{\frac{1}{\pi}\int_{-\infty}^{\infty} f(x')\left\{\int_0^{\infty} \cos \alpha(x'-x)\, d\alpha\right\} dx'}$$

$$= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(x') \cos \alpha\,(x'-x)\, dx'\right\} d\alpha \quad .$$

This can also be written, in exponential form, as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(x') e^{-i\alpha x'}\, dx'\right\} e^{i\alpha x}\, d\alpha \quad .$$

If $f(x)$ is an odd function, then

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\left\{\int_0^{\infty} f(x') \sin \alpha x'\, dx'\right\} \sin \alpha x\, d\alpha$$

and if $f(x)$ is an even function, then

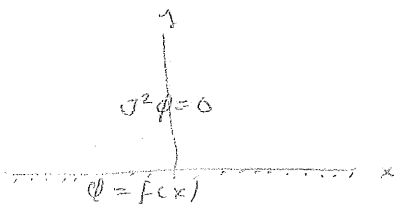
$$f(x) = \frac{2}{\pi}\int_0^{\infty}\left\{\int_0^{\infty} f(x') \cos \alpha x'\, dx'\right\} \cos \alpha x\, d\alpha \quad .$$

Recognizing the similarity of these equations to those of the Fourier transform, the Fourier sine transform and the Fourier cosine transform.

(24)

Example using Fourier Integral

Consider the half-plane $y \ge 0$. Given $\nabla^2 \phi = 0$ for $y > 0$, $\phi(x,0) = f(x)$ and $\phi(x,y) \to 0$ as $y \to \infty$. We wish to find $\phi(x,y)$ for $y > 0$.



A particular solution to Laplace's equation is

$$\phi(x,y) = (a+bx)(c+dy) + (A \sin px + B \cos px)(C e^{-py} + D e^{py}).$$

The homogeneous boundary condition gives

$\lim_{y\to\infty} \phi(x,y) = 0$ which requires that $ac = bc = d = 0$ and $D = 0$.

Therefore a particular solution satisfying this boundary condition is

$$\phi(x,y) = (A \sin px + B \cos px)\, e^{-py}.$$

To put this in the form where we can use the Fourier integral, let $p = \alpha$ and write

$$\phi(x,y) = e^{-\alpha y} \cos \alpha (x' - x)$$ where x' is independent of x and y

~~By substitution into Laplace's equation it is easy to verify that this is a particular solution~~

The integral of this particular solution is also a solution, provided it can be made to satisfy $\phi = f(x)$ on $y = 0$. The integral with respect to the parameters x' and α is

$$\phi(x,y) = \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^{\infty} f(x') e^{-\alpha y} \cos \alpha (x'-x)\, dx' \right\} d\alpha$$

where $\phi(x,0) = f(x) = \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^{\infty} f(x') \cos \alpha (x'-x)\, dx' \right\} d\alpha$

is the Fourier integral representation of $f(x)$.

(See 24a next)

(25) If the function $f(x)$ is given the particular value

$$f(x) = 0 \quad \text{for} \quad x < -L$$
$$f(x) = \phi_0 \quad \text{for} \quad -L < x < L$$
$$f(x) = 0 \quad \text{for} \quad x > L$$

we can further reduce the solution. Then

$$\phi(x,y) = \frac{\phi_0}{\pi}\int_0^\infty \int_{-L}^{L} e^{-\alpha y} \cos \alpha (x'-x)\, dx'\, d\alpha$$
$$= \frac{\phi_0}{\pi}\int_{-L}^{L} \int_0^\infty e^{-\alpha y} \cos \alpha (x'-x)\, d\alpha\, dx' .$$

But $\int_0^\infty e^{-mu} \cos pu \, du = \frac{m}{m^2+p^2}$ for $m > 0$ Peirce #506

Thus
$$\phi(x,y) = \frac{\phi_0}{\pi}\int_{-L}^{L} \frac{y\, dx'}{y^2+(x'-x)^2}$$

and $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$ Peirce #47

Thus
$$\phi(x,y) = \frac{\phi_0 \cancel{y}}{\pi}\left[\frac{1}{y}\tan^{-1}\left(\frac{x'-x}{y}\right)\right]_{-L}^{L}$$
$$= \frac{\phi_0}{\pi}\left[\tan^{-1}\left(\frac{L-x}{y}\right) + \tan^{-1}\left(\frac{L+x}{y}\right)\right].$$

2-1 [illegible] Theory for Engineers, [illegible]
Morse & Feshbach, Methods of Math. Physics, McGraw Hill

Chap II – Ordinary Diff. Equations

When we separated the variables in Laplace's eqn., we got ordinary diff. eqns. of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x) \; .$$

If $R(x)$ is zero, the equation is said to be homogeneous. If $P(x)$ and $Q(x)$ are finite at $x = x_0$, x_0 is called an ordinary point. But if either $P(x)$ or $Q(x)$ (or both) become infinite at $x = x_0$, x_0 is called a singular point.

Suppose x_0 is a singular point. Rewrite the equation as

$$(x-x_0)^2\frac{d^2y}{dx^2} + (x-x_0)\,p(x)\frac{dy}{dx} + q(x)\,y = r(x)$$

where $P(x) = \frac{p(x)}{x-x_0}$, $Q(x) = \frac{q(x)}{(x-x_0)^2}$, $R(x) = \frac{r(x)}{(x-x_0)^2}$.

Assume that the equation is homogeneous, so that $r(x) = 0$. If $p(x)$ becomes infinite at $x = x_0$, and/or $q(x)$ becomes infinite at $x = x_0$, then x_0 is called an irregular singular point. But if $p(x_0)$ and $q(x_0)$ are finite at $x = x_0$, then x_0 is a regular singular point.

There is a theorem (Fuchs) that if x_0 is an ordinary or a regular singular point, then there exists at least one solution to the equation which has the form of a convergent power series in $(x - x_0)$. If the point $x = x_0$ is an irregular singular point, the power series method of solution may fail. [The point x_0 may be the origin (it often is) or infinity.]

The method of solving diff. eqns. by integration in series is a powerful one. Certain diff eqns, when solved by this method, have series solutions which define the special functions. Some of these diff eqns are:

Bessel's $\quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\,y = 0$

Legendre's $\quad (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0$

Laguerre's $\quad x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0$

Hermite's $\quad \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$

Hypergeometric $\quad x(1-x)\frac{d^2y}{dx^2} + \{c - (a+b+1)x\}\frac{dy}{dx} - ab\,y = 0 \; .$

2-2

Bessel's Equn.

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \quad \text{where } n \geq 0 .$$

If we put it in standard form by dividing through by x^2, we see that $P(x) = \frac{1}{x}$, $Q(x) = \frac{x^2 - n^2}{x^2}$ and that $x = 0$ is a singular point. But $p(x) = x\,P(x)$ and $q(x) = x^2\,Q(x)$ remain finite at $x = 0$, so the origin is a regular singular point. Thus we can find at least one power series solution to Bessel's equation of the form $y = \sum_{r=0}^{\infty} a_r (x - x_0)^{s+r}$ where x_0 is 0, or $y = \sum_{r=0}^{\infty} a_r x^{s+r}$. The number s and the coefficients a_r have to be determined such that the assumed solution is a true or actual solution.

$$\text{Now} \quad y' = \frac{dy}{dx} = \sum_{r=0}^{\infty} (s+r) a_r x^{s+r-1}$$

$$\text{and} \quad y'' = \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (s+r)(s+r-1) a_r x^{s+r-2} .$$

Substitute into Bessel's equn. This gives

$$\cancel{x^2} \sum_{r=0}^{\infty} (s+r)(s+r-1) a_r x^{s+r} + \sum_{r=0}^{\infty} (s+r) a_r x^{s+r} + \sum_{r=0}^{\infty} (x^2 - n^2) a_r x^{s+r} = 0$$

or
$$\sum_{r=0}^{\infty} a_r \left[(s+r)\{(s+r-1)+1\} x^{s+r} + (x^2 - n^2) x^{s+r} \right] = 0$$

or
$$\sum_{r=0}^{\infty} a_r \left[\{(s+r)^2 - n^2\} x^{s+r} + x^{s+r+2} \right] = 0 .$$

~~Writing out the first two terms,~~

$$a_0 \left[(s^2 - n^2) x^s + x^{s+2} \right] + a_1 \left[\{(s+1)^2 - n^2\} x^{s+1} + x^{s+3} \right]$$

$$+ \sum_{r=2}^{\infty} a_r \left[\{(s+r)^2 - n^2\} x^{s+r} + x^{s+r+2} \right] = 0$$

$r' = r+2$

Rewrite the last equation as

$$\sum_{r=0}^{\infty} a_r \left[(s+r)^2 - n^2\right] x^{s+r} + \sum_{r=0}^{\infty} a_r x^{s+r+2} = 0$$

or $$\sum_{r=0}^{\infty} a_r \left[(s+r)^2 - n^2\right] x^{s+r} + \sum_{r=2}^{\infty} a_{r-2} x^{s+r} = 0 .$$

~~or~~ $$\cancel{\sum_{r=0}^{\infty} \left[a_r \{(s+r)^2 - n^2\} + a_{r-2} \right] x^{s+r} = 0} .$$

Write out the first two terms of the series. ~~setting that $a_{-2} = a_{-1} = 0$~~

This gives

$$a_0 (s^2 - n^2) x^s + a_1 \{(s+1)^2 - n^2\} x^{s+1} + \sum_{r=2}^{\infty} \left[a_r\{(s+r)^2 - n^2\} + a_{r-2}\right] x^{s+r} = \qquad (3.3)$$

Equate the coefficient of x^s to 0. This gives the indicial equation. It is $s^2 - n^2 = 0$, so that $s = \pm n$.

Equate the coefficients of x^{s+1} ~~and x^{s+r}~~ to 0. This gives the ~~recurrence relation~~ value of a_1. That is

$a_1 \{(s+1)^2 - n^2\} = 0$. But $s = \pm n$, so that $\qquad (3.4)$

$a_1 \{(\pm n + 1)^2 - n^2\} = 0$ or $a_1 \{n^2 \pm 2n + 1 - n^2\} = 0$

or $a_1 (1 \pm 2n) = 0$.

We assumed $n \geq 0$. If we use the root of the indicial equation $s = +n$, then $a_1 = 0$.

The recurrence relation is given by equating the coefficient of x^{s+r} to 0. This gives

$$a_r\{(s+r)^2 - n^2\} + a_{r-2} = 0 . \qquad \text{for } r \geq 2 \qquad (3.5)$$

For $s = +n$, this gives

$$a_r \{2nr + r^2\} + a_{r-2} = 0 \quad \text{or} \quad a_r = -\frac{a_{r-2}}{r(2n+r)} , \qquad (3.6)$$

R... It follows that for this case all $a_r = 0$ for r odd.

For n an even integer we get

$$a_2 = \frac{-a_0}{2(2n+2)}, \quad a_4 = \frac{-a_2}{4(2n+4)} = \frac{a_0}{2\cdot4(2n+2)(2n+4)}, \text{ etc.}$$

Therefore, one solution to Bessel's eqn., corresponding to ..., is

$$y_1 = a_0 x^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot4(2n+2)(2n+4)} - \cdots\right] \quad (3.7)$$

If we give the particular value $\frac{1}{2^n\Gamma(n+1)}$ to a_0, then the solution y_1 of Bessel's function is called $J_n(x)$ [Bessel function of first kind of order n]. It can be written as

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{n+2r}}{r!\,\Gamma(n+r+1)}. \quad (3.8)$$

Eq (3.8) defines the Bessel function $J_n(x)$ for all real values of n and for any finite value of x (positive or negative).

Bessel's diff. eqn. is a 2nd order diff. eqn. The general solution is the sum of two independent particular equations. [Or 2 arbitrary const. are required for the general sol.] One of them is $J_n(x)$. The second independent particular solution depends on the value of n.

Case I. Unequal roots of the indicial equation not differing by an integer (i.e. $2n$ not equal to zero or an integer)

Then using the second root of the indicial equation,

$$a_1(1-2n) = 0.$$

This can be satisfied only if $a_1 = 0$.

The previous analysis needs no modification, and yields the second particular solution

$$y_2 = J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{-n+2r}}{r!\,\Gamma(-n+r+1)} \quad (3.9)$$

The general solution for this case is (3.10)

$$y = A J_n(x) + B J_{-n}(x).$$

Case II. $2n$ equal to an integer.

$y_1 = J_n(x)$ is still a particular solution. For $s = -n$, however, the preceding analysis must be changed. Suppose that $n = 1/2$ ($s = -1/2$). Then $a_1(1-2n) = 0$ from the term in parenthesis, so a_1 need not be zero. If $a_1 \neq 0$, then $a_{2r+1} \neq 0$. This gives as a solution to Bessel's eqn.

$$y = a_0 x^{-1/2}\left[1 - \frac{x^2}{2(-1+2)} + \frac{x^4}{2\cdot 4(-1+2)(-1+4)} - \cdots\right]$$

$$+ a_1 x^{-1/2}\left[x - \frac{x^3}{3(-1+3)} + \frac{x^5}{3\cdot 5(-1+3)(-1+5)} + \cdots\right] \quad (3.11)$$

where a_0 and a_1 are arbitrary solution.

Now a solution to a 2nd order diff. eq. which contains 2 arbitrary constants is a general solution. If a_1 is set equal to $\frac{1}{(2)^{1/2}\,\Gamma(\frac{1}{2}+1)}$ the second series in (3.11) corresponds to $y_1 = J_{1/2}(x)$ as in (3.7). The first series in (3.11), with $a_0 = \frac{1}{(2)^{-1/2}\,\Gamma(-\frac{1}{2}+1)}$, is called $y_2 = J_{-1/2}(x)$. The general solution is then

$$y = A J_{1/2}(x) + B J_{-1/2}(x).$$

The series for $J_{1/2}(x)$ and $J_{-1/2}(x)$ converge for all finite values of x, except that $J_{-1/2}(x)$ does not converge at $x = 0$. These series may be summed and shown to equal

$$J_{1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\sin x, \quad J_{-1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\cos x,$$

using the fact that $\Gamma(1/2) = \sqrt{\pi}$.

When $n = 3/2$, using the second root of the indicial equation $s = -3/2$, we see $a_1(\cancel{2n}\ 1-2n) = 0$ requires that $a_1 = 0$. However, from (3.5),

$$a_3\left\{\left(-\tfrac{3}{2}+3\right)^2 - \left(\tfrac{3}{2}\right)^2\right\} + a_1 = 0$$

so a_3 is indeterminate, as will be a_5, a_7, etc.

2-6

Case III. $n = 0$

In this case the indicial equation becomes $s^2 = 0$ which has the repeated root $s = 0$, which yields the solution

$$(y)_{s=0} = J_0(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{2r}}{(r!)^2} \qquad (3.13)$$

This is not a general solution. To obtain one, we must add to (3.13) an independent solution.

The authors prove that an independent solution, for this case, is given by the series $\left(\frac{\partial y}{\partial s}\right)_{s=0}$ where $y = \sum_{r=0}^{\infty} a_r x^r$. The general solution to Bessel's eqn. in this case is a linear combination of $(y)_{s=0}$ and $\left(\frac{\partial y}{\partial s}\right)_{s=0}$

Case IV. n a positive integer

As an example, take $n = 1$. Bessel's eqn. then is

$$x^2 y'' + x y' + (x^2 - 1) y = 0 \qquad (3.18)$$

The indicial eqn. gives $s = \pm 1$, so from (3.4) $a_1 = 0$ and from (3.5), with $n = 1$,

$$a_r (s + r - 1)(s + r + 1) + a_{r-2} = 0 \qquad r \geq 2$$

or

$$a_r = \frac{-a_{r-2}}{(s+r-1)(s+r+1)} \qquad r \geq 2 .$$

Using this relation, and remembering that $a_1 = 0$, we get ~~the~~ a solution to Bessel's eqn. as

$$y = a_0 x^s \left[1 - \frac{x^2}{(s+1)(s+3)} + \frac{x^4}{(s+1)(s+3)^2(s+5)} - \cdots \right] \qquad (3.19)$$

The solution $y_1(x)$, obtained by putting $s = 1$ and $a_0 = \frac{1}{2\Gamma(2)}$, is

$$y_1 = J_1(x)$$ which is the same as eq (3.8) with $n = 1$.

For $s = -1$, the denominators in the terms of the series in (3.19) become 0, so the terms become infinite, so this solution is meaningless for $s = -1$. The authors show that a second independent solution then becomes infinite for $n = 1$ and $s = -1$.

solution is

$$y_2(x) = b\left[-J_1(x)\log_e x + (1/x)\left\{1 + \frac{x^2}{2} - \frac{x^4}{2^2\cdot 4}\left(\frac{2}{2}+\frac{1}{4}\right) + \frac{x^6}{2^2\cdot 4^2\cdot 6}\left(\frac{2}{2}+\frac{2}{4}+\frac{1}{6}\right) - \cdots\right\}\right] \qquad (3.21)$$

The general solution is then a linear combination of $y_1(x)$ & $y_2(x)$.

[Next go to pg 128, Irving and Mullineux].

In summary, we have shown that

$$J_m(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{m+2r}}{r!\,\Gamma(m+r+1)}$$

is a particular solution of the Bessel's diff eqn for all m. For ~~m = 0~~ ~~or an integer, however,~~ m not equal to an integer or zero, ~~the~~ a second independent solution is $J_{-m}(x)$. However, there is a second independent solution to Bessel's eqn. valid for all values of x. It is called $Y_m(x)$ — the Bessel function of the second kind of order m. A series expression for $Y_m(x)$ when m is an integer is given in eq. (3.4) on pg. 134. The series itself is seldom used, because the function $Y_m(x)$, just as $J_m(x)$ is tabulated.

Thus, for all m, a general solution of Bessel's eqn. is

$$y(x) = A\,J_m(x) + B\,Y_m(x).$$

The series definition of $J_n(x)$ can be used to tabulate it without difficulty for small values of x.

Looking at the expression

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

$+1) = \infty$
$\tfrac{1}{2}) = \sqrt{\pi}$
$\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}$
$-\tfrac{1}{2}) = -2\sqrt{\pi}$

we can see that $J_n(0)$ is finite for $n \geq 0$. For n negative, $J_n(0)$ is infinite for non-integral n (look at first term in series when $r = 0$ — numerator is infinite and denominator is unity). For n negative and integral, $J_n(0)$ is finite ($\frac{1}{\Gamma(-\text{integer})} = 0$).

We can get an approximate value of $J_n(x)$ for large x by going back to the diff. eqn. and letting $y = w/\sqrt{x}$. The diff. eqn. becomes

$$\frac{d^2w}{dx^2} + \{1 + (1-4n^2)/4x^2\}\, w = 0$$

and for large x (compared to n), it becomes approximately

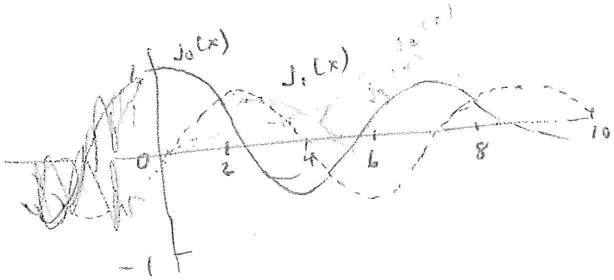
$$\frac{d^2w}{dx^2} + w = 0$$

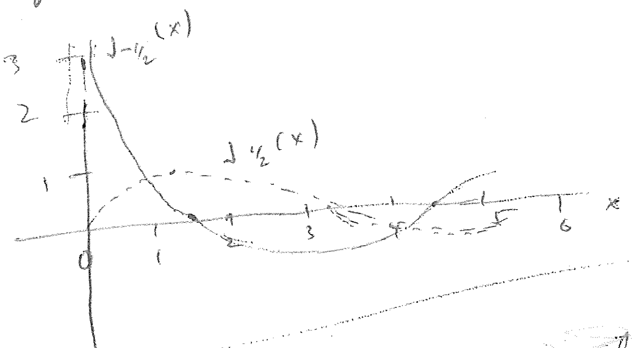
which has a solution $\cancel{y}$ $w = A\cos(x+B)$ or $y = \frac{A}{\sqrt{x}}\cos(x+B)$.

The solution $J_n(x)$ corresponds to particular values of A and B.

We can see that $J_n(x)$ is oscillatory for large x and approaches 0 as $x \to \infty$. It has an infinite number of zeros. A more complete asymptotic expression is eq. (2.6) of I + M, Fig 131.

Sketches of $J_n(x)$:





What are values for x ~~negative~~ ??

Recurrence Relations for $J_n(x)$.

(48) Consider a power series ~~In the~~ expression for

$$n J_n(x) + x \frac{d J_n(x)}{dx}$$, formed from the ~~power~~ series expression for $J_n(x)$. It is

Consider ~~substitute~~ the $(r+1)$'th term of the power series.

$$\frac{n(-1)^r (x/2)^{n+2r}}{r!\,\Gamma(n+r+1)} + \frac{x\,(-1)^r \left(\frac{1}{2}\right)(n+2r)\left(\frac{x}{2}\right)^{n+2r-1}}{r!\,\Gamma(n+r+1)}$$

$$= \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)} \{ n + (n+2r) \}$$

$$= \frac{(-1)^r (x/2)^{n+2r}\, 2(n+r)}{r!\,\Gamma(n+r+1)}$$

But $\Gamma(n+r+1) = (n+r)\,\Gamma(n+r)$

Therefore the last expression is equal to

$$= \frac{2(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r)} = x \frac{(-1)^r \left(\frac{x}{2}\right)^{n-1+2r}}{r!\,\Gamma(n-1+r+1)}$$

which is x times the $(r+1)$'th term in the series expansion of $J_{n-1}(x)$.

Thus, equating coefficients of x^{n-1+2r}, and then summing over all r,

$$n J_n(x) + x \frac{d J_n(x)}{dx} = x J_{n-1}(x). \qquad (2.1.1)$$

Next multiply (2.1.1) by x^{n-1}. This gives

$$n x^{n-1} J_n(x) + x^n \frac{d J_n(x)}{dx} = x^n J_{n-1}(x)$$

But the left hand side is just $\frac{d}{dx}\{x^n J_n(x)\} = n x^{n-1} J_n(x) + x^n \frac{d J_n(x)}{dx}$.

Thus $$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x) \qquad (2.1.2)$$

Other recurrence relations are

$$n J_n(x) - x \frac{dJ_n(x)}{dx} = x J_{n+1}(x) \qquad (2.1.3)$$

$$\frac{d}{dx}\left\{ \frac{J_n(x)}{x^n} \right\} = -\frac{J_{n+1}(x)}{x^n} \qquad (2.1.4)$$

and, from (2.1.3) with $n=0$

$$\frac{dJ_0(x)}{dx} = -J_1(x) \qquad (2.1.5)$$

Thus

$$\int_0^\infty J_1(x)\,dx = -\int_0^\infty \frac{dJ_0(x)}{dx}\,dx = -\left[J_0(x)\right]_0^\infty = -(0-1) = 1 \qquad (2.1.6)$$

Addition of (2.1.1) and (2.1.3) gives

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \qquad (2.1.7)$$

Subtraction of (2.1.3) from (2.1.1) gives

$$J_{n-1}(x) - J_{n+1}(x) = 2\frac{dJ_n(x)}{dx}. \qquad (2.1.8)$$

Eq. (2.1.7) tells us that we need only evaluate $J_0(x)$ and $J_1(x)$ by the series method. The ~~rest~~ Bessel functions of higher order can be obtained from them. For example,

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x).$$

(50)

Bessel's function of Second Kind of Order n, $Y_n(x)$

Write and eqns. on pg 133, eq (3.1), (3.3)

See fig. III-3 on pg 136 of I & M for a plot of $Y_0(x)$ and $Y_1(x)$.

→ A very important property is $Y_n(0) = -\infty$.

$Y_n(x)$ is oscillating and approaches 0 as $x \to \infty$.

Recurrence relations given as eqs. (3.1.1) to (3.1.10) on pp. 137 & 138 of I & M.

Modified Bessel Functions : $I_n(x)$ and $K_n(x)$

Consider the diff. eqn.

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+n^2)y = 0. \qquad (6.1)$$

This differs from Bessel's equation because the term in x^2 multiplying y is negative. It can be put in the form of Bessel's diff eqn by the substitution $x = -it$ which gives

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0$$

which has independent solutions $J_n(t)$ and $Y_n(t)$. Thus, ~~the~~ independent solutions of (6.1) are

$$y_1 = J_n(ix) \quad \text{and} \quad y_2 = Y_n(ix). \qquad (6.3)$$

The solutions (6.3) are not always real. Therefore, for convenience we define the real functions

$$I_n(x) = i^{-n} J_n(ix) \qquad (6.4)$$

$$= i^{-n}\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{x}{2}\right)^{n+2r}(i)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

$$= \sum^{\infty}\frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)} \qquad (6.5)$$

(51) $I_n(x)$ is called the modified Bessel fcn of first kind of order n. It is sketched in fig III-11 of I & M, pg 143.

For n not zero or an integer,

$$K_n(x) = \frac{\pi}{2}\left[\frac{I_{-n}(x) - I_n(x)}{\sin n\pi}\right] \qquad (6.7)$$

gives an independent solution. For $n = 0$ or any integer, see eq (6.11) on pg. 144.

Asymptotic values of $I_n(x)$ and $K_n(x)$ are given in eqs (6.6) and (6.12), respectively.

Recurrence relations for the modified Bessel functions are given as eqs. (6.1.1) to (6.1.18)

<u>Hankel Functions</u> - Bessel Functions of 3rd Kind

Useful in problems of cylindrical wave propagation.

$$H_n^{(1)}(x) \equiv J_n(x) + i\,Y_n(x) \qquad (6.3.1)$$

$$H_n^{(2)}(x) \equiv J_n(x) - i\,Y_n(x) \qquad (6.3.2)$$

These functions are complex for real x.

(52) Orthogonal Properties of Bessel's Functions

It can be verified by substitution that $J_n(kx)$ is a solution of

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - n^2)y = 0 .$$

Now consider the equations

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \left(\alpha^2 - \frac{n^2}{x^2}\right)u = 0 \qquad (8.1)$$

$$\text{and} \quad \frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} + \left(\beta^2 - \frac{n^2}{x^2}\right)v = 0 \qquad (8.2) \quad \text{with } \alpha \neq \beta$$

which have solutions

$u = J_n(\alpha x)$ and $v = J_n(\beta x)$.

Multiply (8.1) by xv, (8.2) by xu, subtract the second resulting eqn. from the first, and integrate with respect to x between 0 and a. This gives

$$\int_0^a \{x(u''v - uv'') + (uv' - vu')\}\,dx + (\alpha^2-\beta^2)\int_0^a xuv\,dx = 0$$

$$\text{or} \int_0^a \frac{d}{dx}\{x(u'v - uv')\}\,dx = (\beta^2-\alpha^2)\int_0^a xuv\,dx \qquad (8.3)$$

$$\text{or} \quad (\beta^2-\alpha^2)\int_0^a xuv\,dx = \left[x(u'v - uv')\right]_0^a .$$

hence $x(u'v' - u'v') = 0$

where $u' \equiv \dfrac{du}{dx} = \dfrac{d\,J_n(\alpha x)}{dx}$.

~~Consider the first term in the series expansion of $xu'v$ ($x \to 0$).~~

~~$\dfrac{d\,J_n(x)}{dx} = J_{n-1}(x) - \dfrac{n}{x}J_n(x)$~~

~~It is~~

~~$\dfrac{x\left(\frac{\alpha x}{2}\right)^n}{\Gamma(n+1)} \cdot \dfrac{\frac{n}{2}\left(\frac{\alpha x}{2}\right)^{n-1}}{\Gamma(n+1)}$~~

The series expansion of $x(u'v - uv')$ will have x^{2n+2} as the term with lowest power of x. Thus the lower limit, when $x = 0$, vanishes if $n > -1$.

wrong

(53) There are 3 ways in which the upper limit of the right-hand side of eq. (8-3) can vanish.

(1) First, let $\alpha = \xi_i$ and $\beta = \xi_j$ be roots of the equation

$$J_n(a\xi) = 0 \quad (8.4) \quad \text{with } i \neq j.$$

Then $u(a) \equiv \left[J_n(\xi_i x) \right]_{x=a} = 0$ and $v(a) \equiv \left[J_n(\xi_j x) \right]_{x=a} = 0$

and the r.h.s. of (8.3) is zero.

It follows that the left hand side of (8.3) is zero.

Because $\alpha^2 \neq \beta^2$ $(\xi_i \neq \xi_j)$, this can only be true if

$$\int_0^a x \, J_n(\xi_i x) \, J_n(\xi_j x) \, dx = 0 \quad \text{for } n > -1. \quad (8.5)$$

This tells us that the function $J_n(\xi x)$ with weighting function x is orthogonal on the interval $(0, a)$ if the ξ_i, ξ_j are roots of the equation $J_n(\xi a) = 0$.

If $i = j$, then it can be shown that

$$\int_0^a x \left[J_n(\xi_i x) \right]^2 dx = \left(\frac{a^2}{2}\right) \left[J_n'(\xi_i a) \right]^2 = \left(\frac{a^2}{2}\right) \left[J_{n-1}(\xi_i a) \right]^2$$

$$= \left(\frac{a^2}{2}\right) \left[J_{n+1}(\xi_i a) \right]^2 \quad (8.11)$$

$$= \left(\frac{a^2}{2}\right) \left[J_n^2(ka) - J_{n-1}(ka) J_{n+1}(ka) \right]$$

where $J_n'(\xi_i a) \equiv \left\{ \frac{d \, J_n(\xi_i x)}{d(\xi_i x)} \right\}_{x=a}$

(2) Second, let $\alpha = \xi_i$ and $\beta = \xi_j$ be roots of the equation

$$\xi \frac{d J_n(a\xi)}{d(a\xi)} + h \, J_n(a\xi) = 0 \quad (8.6)$$

where h is a known constant.

Now $\frac{d J_n(\xi x)}{dx} = \frac{d J_n(\xi x)}{d(\xi x)} \frac{d(\xi x)}{dx} = \xi \frac{d J_n(\xi x)}{d(\xi x)}$

(14)

Thus

$$\frac{d\, J_m(x\xi)}{d(x\xi)} = \frac{1}{\xi}\frac{d\, J_m(x\xi)}{dx}$$

so that (8.6) becomes

$$\xi_i \cdot \frac{1}{\xi_i}\frac{d\, J_m(x\xi_i)}{dx} + h\, J_m(x\xi_i) = 0 \qquad \text{at } x = a$$

and

$$\xi_j \frac{1}{\xi_j}\frac{d\, J_m(x\xi_j)}{dx} + h\, J_m(x\xi_j) = 0 \qquad \text{at } x = a.$$

Thus

$$[u'v - uv']_{x=a} = \left[-h\, J_m(a\xi_i)\, J_m(a\xi_j)\right] - \left[-J_m(a\xi_i)\, h\, J_m(a\xi_j)\right]$$

$$= 0$$

and the r.h.s. of (8.3) is zero at the upper limit if ξ_i and ξ_j are distinct roots of

$$\xi \frac{d\, J_m(a\xi)}{d(a\xi)} + h\, J_m(a\xi) = 0 . \qquad (8.6)$$

Because $i \neq j$,

$$\int_0^a x\, J_m(\xi_i x)\, J_m(\xi_j x)\, dx = 0 \qquad \text{when}$$

ξ_i and ξ_j are roots of (8.6).

If $\xi_i = \xi_j$, it can be shown that

$$\int_0^a x \left[J_m(\xi_i x)\right]^2 dx = \frac{a^2}{2\xi_i^2}\left[J_m(\xi_i a)\right]^2\left[h^2 + \left(\xi_i^2 - \frac{m^2}{a^2}\right)\right] \qquad (8.12)$$

(55)

3) Third, let $\alpha = \xi_i$ and $\beta = \xi_j$ be roots of the equation (8.6) with h set equal to zero (special case of #2). This gives

$$\left[\frac{d\,J_n(\xi_i x)}{dx}\right]_{x=a} = \left[\frac{d\,J_n(\xi_j x)}{dx}\right]_{x=a} = 0 \qquad (8.6')$$

and, if these equations are satisfied, $[u'v - v'u]_{x=a} = 0$.

~~Thus, if (8.6') is satisfied by ξ_i, ξ_j~~,

Thus, if ξ_i, ξ_j are the roots of $\left[\frac{d\,J_n(\xi x)}{dx}\right]_{x=a} = 0$,

then $$\int_0^a x\, J_n(\xi_i x)\, J_n(\xi_j x)\, dx = 0$$

and, if $\xi_i = \xi_j$

$$\int_0^a x\,[J_n(\xi_i x)]^2\, dx = \frac{a^2}{2\xi_i^2}\,[J_n(\xi_i a)]^2 \left(\xi_i^2 - \frac{n^2}{a^2}\right). \qquad (8.12')$$

<u>Expansion of $f(x)$ in Terms of $J_n(\xi_i x)$.</u>

Case I ~~Let ξ_i indicate the zeros of $J_n(\xi a) = 0$ in the interval 0 to ∞ ($i = 1, 2, 3$, etc.). Write~~

Let $$f(x) = \sum_{i=1}^{\infty} A_i\, J_n(\xi_i x). \qquad (8.1.1).$$

Our problem is to find the A_i.

Multiply both sides of (8.1.1) by $x\,J_n(\xi_j x)$ and integrate with respect to x from 0 to a. This gives

$$\int_0^a x\, f(x)\, J_n(\xi_j x)\, dx = \int_0^a \sum_{i=1}^{\infty} A_i\, x\, J_n(\xi_i x)\, J_n(\xi_j x)\, dx.$$

(56)

Integrate the series term by term. By the orthogonality property of the Bessel function, all integrals except $\int_0^a A_j\, x [J_n(\xi_j x)]^2\, dx$ are zero.

Case 1) If the ξ_j are roots of $J_n(a\xi_j) = 0$, then the integral equals

$$\frac{A_j a^2}{2} [J_n'(\xi_j a)]^2$$

and
$$A_i = \frac{2}{a^2} \frac{\int_0^a x f(x) J_n(\xi_i x)\, dx}{[J_n'(\xi_i a)]^2}. \qquad (8.1.2)$$

$$\text{Thus } f(x) = \frac{2}{a^2} \sum_{i=1}^{\infty} \left\{ \frac{\int_0^a x f(x) J_n(\xi_i x)\, dx}{[J_n'(\xi_i a)]^2} J_n(\xi_i x) \right\}. \qquad (8.1.3)$$

The function $\int_0^a x f(x) J_n(\xi_i x)\, dx \equiv g(\xi_i)$ is called the finite Hankel transform of $f(x)$. The inversion formula, to give $f(x)$ when $g(\xi_i)$ is known, is

$$f(x) = \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{g(\xi_i) J_n(\xi_i x)}{[J_n'(\xi_i a)]^2}.$$ This follows immediately from (8.1.3) and the definition of $g(\xi_i)$.

Case 2) If the ξ_j are roots of $h J_n(a\xi) + \xi J_n'(a\xi) = 0$,

$$A_i = \frac{2\xi_i^2}{a^2} \frac{\int_0^a x f(x) J_n(\xi_i x)\, dx}{[J_n(\xi_i a)]^2 [h^2 + (\xi_i^2 - \frac{n^2}{a^2})]}$$

and

$$f(x) = \frac{2}{a^2} \sum_{i=1}^{\infty} \left\{ \frac{\int_0^a x f(x) J_n(\xi_i x)\, dx}{[h^2 + (\xi_i^2 - \frac{n^2}{a^2})][J_n(\xi_i a)]^2} \cdot \xi_i^2 J_n(\xi_i x) \right\}$$

(8.1.4)

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Expressions for $J_n(x)$ in the form of definite integrals ($n=0$ or an integer) and expressions for integrals which contain $J_n(x)$ in the integrand are given on pp. 159–171 of F & M.

Note on pg. 159 that the function

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{i=-\infty}^{\infty} t^i J_i(x) \quad \text{for } t \neq 0 .$$

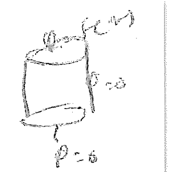
This function is called the generating function.

See Handbook of Mathematical Functions for similar equations involving integrals, roots, asymptotic values, etc. of $Y_n(x)$, $I_n(x)$, $K_n(x)$ as well as their tabulated values. There is no general orthogonality property for $I_n(x)$ and $K_n(x)$ because they have no real roots except at the origin.

(17)
~~34~~

Example: Laplace's eqn. in polar cylindrical coordinates

Find the steady-state ~~heat flow~~ temperature inside a circular cylinder of radius a and height h, with bottom and curved surfaces at temperature $\phi = 0$ and top surface at temperature $\phi = f(r)$. There are no sources in the cylinder.



Solution: Let the z axis be the axis of the cylinder. Then ϕ is not a function of the angle θ, so Laplace's eqn. reduces to

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

with particular solutions

$$\left.\begin{matrix} e^{pz} \\ e^{-pz} \end{matrix}\right\} \left.\begin{matrix} J_0(pr) \\ Y_0(pr) \end{matrix}\right\} \quad \text{or} \quad \left.\begin{matrix} \sinh \\ \cosh \end{matrix} pz\right\} \left.\begin{matrix} J_0(pr) \\ Y_0(pr) \end{matrix}\right\}.$$

The coefficient of $Y_0(pr)$ must be zero, because $Y_0(0) = -\infty$ which would mean that the temp. would be infinite on the axis of the cylinder. Also the coefficient of $\cosh pz$ must be zero, because this would not give $\phi = 0$ on $z = 0$.

The boundary condition $\phi = 0$ for $r = a$ gives

$$J_0(p_n a) = 0,$$

because otherwise, if the coefficient of $J_0(pa)$ were zero, ϕ would be zero everywhere inside the cylinder.

The $(p_n a)$'s are roots of the Bessel function of zero order, where $n = 1, 2, \ldots$ They form a discrete set of numbers. $\phi = \sum_{n=1}^{\infty} A_n \sinh p_n z \; J_0(p_n r)$

Finally the boundary condition $\phi = f(r)$ on $z = h$ gives

$$f(r) = \sum_{n=1}^{\infty} A_n J_0(p_n r) \sinh(p_n h)$$

$$f(r) = \sum_{n=1}^{\infty} K_n J_0(p_n r) \qquad \text{where } K_n = A_n \sinh(p_n h).$$

This last equation looks similar to a Fourier series, except we have a Bessel function of zero order instead of a trigonometric function.

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Therefore we must use a Fourier-Bessel series expansion. [see Churchill eq 1-3, pp 162-6]

For $0 \le r \le a$,
$$C_n = \frac{2\int_0^a r f(r)\, J_0(p_n r)\, dr}{a^2\{[J_0(p_n a)]^2 + [J_1(p_n a)]^2\}} = \frac{2\int_0^a r f(r)\, J_0(p_n r)\, dr}{a^2 [J_1(p_n a)]^2}$$

(Case 1)

because $J_0(p_n a) \equiv 0$.

Finally,
$$\phi = \sum A_n \cancel{\sinh(p_n h)} \sinh(p_n z)\, J_0(p_n r)$$

where
$$A_n = \frac{C_n}{\sinh(p_n h)} = \frac{2\int_0^a r f(r)\, J_0(p_n r)\, dr}{a^2 [J_1(p_n a)]^2 \sinh(p_n h)} .$$

65) Legendre's Equation — Series Solutions

$$(1-x^2)y'' - 2xy' + m(m+1)y = 0 \qquad (4.1)$$

Assume a solution $y = \sum_{r=0}^{\infty} a_r x^{s+r}$.

Substitute assumed solution into Legendre's diff. eqn. This gives

$$(1-x^2)\sum_{r=0}^{\infty}(s+r)(s+r-1)a_r x^{s+r-2} - 2x\sum_{r=0}^{\infty}(s+r)a_r x^{s+r-1}$$

$$+ m(m+1)\sum_{r=0}^{\infty} a_r x^{s+r} = 0$$

or

$$\sum_{r=0}^{\infty}(s+r)(s+r-1)a_r x^{s+r-2} - \sum_{r=0}^{\infty} a_r\left[(s+r)(s+r-1) + 2(s+r) - m(m+1)\right]x^{s+r} = 0$$

The last equation must be valid for all x, which happens only if the coefficient of every power of x is identically zero. Take the coefficient of the lowest power of x, i.e. for $r=0$ with the first sum the coefficient of x^{s-2}, and equate to zero. This gives

$s(s-1)a_0 = 0$, which is the indicial equation. The solutions of it are $s_1 = 0$, $s_2 = +1$.

The coeff. a_1 is indeterminate for $s=0$, as can be seen by letting $r=1$ in the first sum.

This gives

$$(s+1)(s)a_1 = 0$$

so when $s = s_1 = 0$, a_1 is indeterminate. From Theorem 2, pg. 78, the root $s_1 = 0$ determines the general solution of Legendre's eqn.

To obtain the recurrence relation, put $s=0$ and $r \geq 2$. Then, equating coefficients of like powers of x,

$$r(r-1)a_r - \left[(r-2)(r-3) + 2(r-2) - m(m+1)\right]a_{r-2} = 0$$

or

$$r(r-1)a_r - \left[r^2 - 3r + 2 - m^2 - m\right]a_{r-2} = 0$$

or

(4.6)

Therefore

$$a_2 = \frac{-m(1+m)\,a_0}{2(2-1)} = -\frac{m(m+1)}{2!}\,a_0$$

$$a_4 = \frac{(2-m)(3+m)\,a_2}{3\cdot 4} = \frac{m(m-2)(m+1)(m+3)}{4!}\,a_0 \quad \text{etc.}$$

$$a_3 = \frac{(1-m)(2+m)\,a_1}{3\cdot 2} = -\frac{(m-1)(m+2)\,a_1}{3!}$$

$$a_5 = \frac{(3-m)(4+m)\,a_3}{5\cdot 4} = \frac{(m-1)(m-3)(m+2)(m+4)\,a_1}{5!} \quad \text{etc.}$$

Then

$$y = a_0\left[1 - \frac{m(m+1)}{2!}x^2 + \frac{m(m-2)(m+1)(m+3)}{4!}x^4 - \cdots\right]$$

$$+ a_1\left[x - \frac{(m-1)(m+2)}{3!}x^3 + \frac{(m-1)(m-3)(m+2)(m+4)}{5!}x^5 - \cdots\right] \qquad (4.3)$$

or $y = a_0 u_m(x) + a_1 v_m(x)$

where the functions $u_m(x)$ and $v_m(x)$ are defined by the series in brackets in (4.3). Both series converge if $-1 < x < 1$.

If we would have used the root of the indicial equation $s_2 = +1$, we would get merely the series for $v_m(x)$.

For the particular case m equal to an integer, one of the series will terminate (will have only a finite number of terms). For m even, $u_m(x)$ terminates. For m odd, $v_m(x)$ terminates.

For $m = 1$,

$$y = a_0\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} \cdots\right) + a_1(x + 0 + 0 \cdots)$$

$$= a_0 u_1(x) + a_1 v_1(x) \qquad (4.4)$$

where $u_1(x)$ is the series in parentheses and $v_1(x) = x$.

(67) For $n = 2$,

$$y = a_0(1 - 3x^2 + 0 + 0 \cdots)$$
$$+ a_1\left(x - \frac{2}{3}x^3 - \frac{x^5}{5} - \frac{8x^7}{35} \cdots\right)$$
$$= a_0 u_2(x) + a_1 v_2(x)$$

where $u_2(x) = 1 - 3x^2$

$$v_2(x) = \left(x - \frac{2}{3}x^3 - \frac{x^5}{5} - \cdots\right)$$

The Legendre polynomial or function of the first kind, $P_n(x)$, is defined as

$$P_n(x) = \frac{u_n(x)}{u_n(1)} \quad \text{for } n = 0 \text{ or an even integer}$$

$$P_n(x) = \frac{v_n(x)}{v_n(1)} \quad \text{for } n \text{ an odd integer.}$$

$$P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!(n-2r)!(n-r)!} x^{n-2r}$$

$N = n/2$ for n even
$N = \frac{n-1}{2}$ for n odd

Thus, for example,

$$P_0(x) = \frac{1}{1} = 1$$

$$P_1(x) = \frac{x}{1} = x$$

$$P_2(x) = \frac{(1-3x^2)}{(1-3)} = \frac{3x^2-1}{2}$$

$$P_3(x) = \frac{(x - \frac{5}{3}x^3)}{(1 - \frac{5}{3})} = \frac{(\frac{5}{3}x^3 - x)}{\frac{2}{3}} = \frac{(5x^3 - 3x)}{2}$$

$$P_4(x) = \frac{(1 - \frac{4\cdot 5}{2}x^2 + \frac{4\cdot 2\cdot 5\cdot 7}{4!}x^4)}{a_0(1 - 10 + \frac{35}{3})}$$

$$= \frac{1 - 10x^2 + \frac{35}{3}x^4}{8/3} = \frac{3 - 30x^2 + 35x^4}{8}$$

Note that $P_n(x)$ is a polynomial of degree n. It is an even function for n even and odd for n odd. Thus P_n . . . (4.6)

(68)

The Legendre function of the second kind, $Q_n(x)$ is defined as

$$Q_n(x) \equiv -N_n(1)\, u_n(x) \quad \text{for } n \text{ odd}$$

$$Q_n(x) \equiv u_n(1)\, N_n(x) \quad \text{for } n \text{ even}.$$

For $n = 0$,

$$Q_0(x) = u_0(1)\, N_0(x) = 1\left[x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \cdots\right)$$

$$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \tanh^{-1} x \qquad \text{for } |x|<1$$

Prince 1969, pg 41 $\qquad$ #799

For $n = 1$,

$$Q_1(x) = -1\left[1 - x^2 + \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} \cdots\right) \qquad |x| < 1.$$

$$= \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

$$= P_1(x)\, Q_0(x) - 1$$

It can be shown in a similar manner that

$$Q_2(x) = P_2(x)\, Q_0(x) - \frac{3x}{2}$$

$$Q_3(x) = P_3(x)\, Q_0(x) - \frac{5x^2}{2} + \frac{2}{3}$$

$$Q_4(x) = P_4(x)\, Q_0(x) - \frac{35x^3}{8} + \frac{55x}{24}. \qquad (4.7)$$

$$Q_n(x) \doteq \frac{1}{2} P_n(x) \log\left(\frac{1+x}{1-x}\right) - \sum_{r=0}^{m} (-1)^{n-2r} \frac{(2n-4r-1)}{(2r+1)(n-r)} P_{n-2r-1}(x)$$

(11.1.10) Chap 3.

53

Legendre's equation

Read pp 69–174, pp 82–84.

Legendre's eqn. is, for n having any real positive value, and zero

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0.$$

The general solution is

$$y = a_0 u_n(x) + a_1 v_n(x)$$

where

$$u_n(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \dots$$

$$v_n(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \dots .$$

Both of these series will converge if $|x|<1$, i.e. $-1<x<1$.

For the special case of n being a positive integer, one of the series will terminate. For example, for $n=1$

$$y = a_0\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \dots\right) + a_1 x = a_0 u_1(x) + a_1 v_1(x)$$

For $n=2$,

$$y = a_0(1-3x^2) + a_1\left(x - \frac{2x^3}{3} - \frac{x^5}{5} - \frac{8x^7}{35} - \dots\right) = a_0 u_2(x) + a_1 v_2(x).$$

The Legendre polynomial (only for n integral) or function of the first kind, $P_n(x)$, is defined as $\frac{u_n(x)}{u_n(1)}$ for $n=0$ or an even integer; for n an odd integer $P_n(x)$ is defined as $\frac{v_n(x)}{v_n(1)}$. Thus for n real and integral, the Legendre polynomial or fn. of first kind can be represented by a terminating series. It is a polynomial of degree n, containing only even powers of x if n is even and only odd powers if n is odd. Thus $P_n(x)$ is an even function for n even, and an odd function for n odd. $P_n(-x) = (-1)^n P_n(x)$

The Legendre functions of the second kind, $Q_n(x)$ involves the non-terminating series solution. For n odd, $Q_n(x) \equiv -v_n(1)\, u_n(x)$. For n even or zero, $Q_n(x) \equiv u_n(1)\, v_n(x)$.

They have no zeros in the interval $x=-1$ to $x=+1$.

Sketch $P_n(x)$ [Jahnke and Emde fig 118] and $Q_n(x)$ [J & E, fig 110] Note that $Q_n(x)$ is infinite at $x=\pm 1$

Different expressions for $Q_n(x)$ depending on whether $|x|<1$ or >1

Generating Function for Legendre Polynomials (Irving and Mullineux, pg 178 ff

We have seen that a solution of the Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (11.1)$$

obtained by substitution of a power series is $P_n(x)$, $n = 0, 1, 2, \ldots$, where the $P_n(x)$ are terminating (non-infinite) series.

It is possible to define the Legendre polynomials in an entirely different manner. Consider a function of x and t defined by the relation

$$f(x,t) \equiv \frac{1}{\sqrt{1-2xt+t^2}} = \frac{1}{\sqrt{1-(2xt-t^2)}} \qquad (11\text{-}2)$$

Expand this function using the binomial theorem, assuming $|t|$ is small enough that $|2xt - t^2| < 1$. This gives

$$f(x,t) = 1 + \frac{\frac{1}{2}}{1!}(2xt-t^2) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}(2xt-t^2)^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}(2xt-t^2)^3 + \cdots$$

$$+ \frac{(\frac{1}{2})(\frac{3}{2})\cdots(\frac{2n-1}{2})}{n!}(2xt-t^2)^n + \cdots .$$

Rearrange the terms so as to obtain a power series in t (requires uniform convergence with regard to both x and t). This gives

$$f(x,t) = 1 + xt + \left(\frac{3x^2-1}{2}\right)t^2 + \left(\frac{5x^3-3x}{2}\right)t^3 + \cdots + \overset{P_n(x)}{A_n(x)}\, t^n + \cdots \qquad (11.3)$$

where

$$\overset{P_n(x)}{A_n(x)} = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!(n-2r)!(n-r)!} x^{n-2r} \qquad (11.5)$$

$N = \frac{n}{2}$ for n even and $\frac{n-1}{2}$ for n odd.

$\Rightarrow$ Thus $f(x,t) = \sum_{n=0}^{\infty} t^n \overset{P_n(x)}{A_n(x)}$.

~~By substitution one~~ can show that $A_n(x)$ satisfies Legendre's eqn (11.1) — see pp 56a' and 56a''

In fact, from our previous definition of $P_n(x)$, we can see that $A_n(x)$ corresponds to $P_n(x)$. Therefore

$$f(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x) \qquad (11.1.1)$$

we do not know that $A_n(x) = P_n(x)$, we can observe that $A_n(x)$ is a solution of Legendre's eqn, and show that $f(1,t) = 1 + t + t^2 + \cdots = \sum_{n=0}^{\infty} t^n$. Thus $A_n(1) = 1$. But we know from previous work that $P_n(1) = 1$. Thus we identify $A_n(x)$ with $P_n(x)$. $A_n(x)$ and $P_n(x)$ are both polynomials in x which satisfy Legendre's eqn. They have the same [illegible], therefore [illegible]

By substituting $x = 1$ into (11.2), we get

$$f(1,t) = \frac{1}{1-t} = 1 + t + t^2 + \cdots + t^n + \cdots \qquad \text{for } |t| < 1$$

Thus $A_n(1) = 1$ and $P_n(1) = 1$. Similarly, $A_n(-1) = (-1)^n$ and $P_n(-1) = (-1)^n$.

56a' — (64a)

Thus

$$f(x,t) = \sum_{n=0}^{\infty} A_m(x) \cdot t^m$$

We want to show that $A_m(x)$ satisfies Legendre's diff. eqn.

$$(1-x^2)\frac{d^2y}{dx^2} - 2xy\frac{dy}{dx} + n(n+1)y = 0.$$

To do this, let us form some identities by differentiating $f = \frac{1}{\sqrt{1-2xt+t^2}}$

$$\frac{\partial f}{\partial t} = \frac{-\frac{1}{2}(-2x+2t)}{(1-2xt+t^2)^{3/2}} = \frac{(x-t)\, f(x,t)}{1-2xt+t^2} \qquad (1) \qquad (1-2xt+t^2)\frac{\partial f}{\partial t} = (x-t) f(x,t)$$

$$\frac{\partial f}{\partial x} = \frac{-\frac{1}{2}(-2t)}{(1-2xt+t^2)^{3/2}} = \frac{\partial f}{\partial t}\left(\frac{t}{x-t}\right) \qquad (2) \qquad t\frac{\partial f}{\partial t} = (x-t)\frac{\partial f}{\partial x}$$

Substitute the series expression for $f(x,t)$ into (1), and equate coefficients of t^{m-1}. This gives, from

$$(1-2xt+t^2)\left[\cdots + A_{m-2}(m-2)t^{m-3} + A_{m-1}(m-1)t^{m-2} + A_m \cdot m \cdot t^{m-1} + \cdots\right]$$

$$= (x-t)\left[\cdots + A_{m-2}t^{m-2} + A_{m-1}t^{m-1} + \cdots\right],$$

the result

$$m A_m - 2x(m-1)A_{m-1} + (m-2)A_{m-2} = x A_{m-1} - A_{m-2}$$

or

$$m A_m - x(2m-1)A_{m-1} + (m-1)A_{m-2} = 0 \qquad (3)$$

In a similar way, substituting the series expression for $f(x,t)$ into (2) and equating coefficients of t^{m-1} gives

$$x\frac{dA_{m-1}(x)}{dx} - \frac{dA_{m-2}(x)}{dx} = (m-1)A_{m-1}(x) \qquad (4)$$

Differentiate (3) with respect to x. This gives

$$m\frac{dA_m}{dx} - (2m-1)A_{m-1} - x(2m-1)\frac{dA_{m-1}}{dx} + (m-1)\frac{dA_{m-2}}{dx} = 0.$$

Substituting from (4) for $\frac{dA_{n-2}}{dx}$,

$$n\frac{dA_n}{dx} - (2n-1)A_{n-1} - x(2n-1)\frac{dA_{n-1}}{dx} + (n-1)\left[x\frac{dA_{n-1}}{dx} - (n-1)A_{n-1}\right] = 0$$

or

$$n\frac{dA_n}{dx} + \left[-x(2n-1) + x(n-1)\right]\frac{dA_{n-1}}{dx} + \left[-(2n-1) - (n-1)^2\right]A_{n-1} = 0$$

or

$$\frac{dA_n}{dx} - x\frac{dA_{n-1}}{dx} = nA_{n-1} \qquad (5)$$

In (4), replace n by $(n+1)$. This gives

$$x\frac{dA_n}{dx} - \frac{dA_{n-1}}{dx} = nA_n \qquad (6).$$

delete

Multiply (6) by $-x$ and add to (5). This gives

$$-x^2\frac{dA_n}{dx} + x\frac{dA_{n-1}}{dx} - xnA_n + \frac{dA_n}{dx} - x\frac{dA_{n-1}}{dx} - nA_{n-1} = 0$$

or

$$(1-x^2)\frac{dA_n}{dx} = n(A_{n-1} + xA_n) \qquad (7).$$

Differentiate (7) with respect to x. This gives

$$-2x\frac{dA_n}{dx} + (1-x^2)\frac{d^2A_n}{dx^2} = n\cdot\left(\frac{dA_{n-1}}{dx} + x\frac{dA_n}{dx} + A_n\right)$$

Using (6), we get

$$(1-x^2)\frac{d^2A_n}{dx^2} - 2x\frac{dA_n}{dx} = n\left(\frac{dA_{n-1}}{dx} + \overset{-nA_n}{nA_m} - \frac{dA_{n-1}}{dx} + A_n\right)$$

$$= -A_n\, n(1+n)$$

or

$$(1-x^2)\frac{d^2A_n}{dx^2} - 2x\frac{dA_n}{dx} + n(n-1)A_n = 0$$

Thus $A_n(x)$ is a solution of Legendre's equation.

(70) Rodrigues' Formula (I. & M., pg 176)

To prove: $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

Using the Binomial expansion,

$$(x^2-1)^n = \sum_{r=0}^{\infty} \frac{n!}{(n-r)!\,r!} (x^2)^{n-r} (-1)^r .$$

The $(r+1)$'th term of this series is

$$(-1)^r \frac{n!}{(n-r)!\,r!} x^{2n-2r} .$$

Take the n'th derivative with respect to x, as in the term in Rodrigues' formula. This gives

$$\underbrace{(2n-2r)(2n-2r-1)(2n-2r-2)\cdots\cdots(2n-2r-\{n-1\})}_{n-2r+1} x^{n-2r} \frac{(-1)^r n!}{(n-r)!\,r!}$$

which equals

$$\frac{(2n-2r)!}{(n-2r)!} \frac{(-1)^r n!}{(n-r)!\,r!} x^{n-2r}$$

if $n-2r \geq 0$.
That is, if
$r \leq \frac{n}{2}$ for n even
$r \leq \frac{n-1}{2}$ for n odd.

~~Summing~~ The last expression was $\frac{d^n}{dx^n}$ of the $(r+1)$'th term in the series for $(x^2-1)^n$. Summing all such expressions, and multiplying by $\frac{1}{2^n n!}$, we get

$$\frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n!} \sum_{r=0}^{N} \frac{(2n-2r)!\,(-1)^r n!}{(n-2r)!\,(n-r)!\,r!} x^{n-2r}$$

$$= \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!\,(n-2r)!\,(n-r)!} x^{n-2r} \equiv P_n(x)$$

where $N = \frac{n}{2}$ for n even and $N = \frac{n-1}{2}$ for n odd.

[All the values of $r > N$ contribute nothing to the series, since each such term has a value of zero.]

$\frac{d^n}{dx^n}(x^{2n-2r}) = 0$ if $2r > n$ because $\frac{d^n}{dx^n}(x^{n-m}) = 0$

Start (11)

Some Recurrence Relations

We had previously

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x). \qquad (11.1.1)$$

Differentiate (11.1.1) with respect to t. This gives

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) \qquad (11.1.2)$$

or

$$\frac{x-t}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x) = \underbrace{(x-t)\sum_{n=0}^{\infty} t^n P_n(x)}_{\text{from (11.1.1)}} \qquad (11.1.3)$$

Equate coefficients of t^n in (11.1.3). This gives

$$(n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) = xP_n(x) - P_{n-1}(x)$$

or

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \qquad (11.1.4)$$

$\Rightarrow$ Eq. (11.1.4). is a recursion equation which gives $P_{n+1}(x)$ in terms of the lower order polynomials $P_n(x)$ and $P_{n-1}(x)$.

Next differentiate (11.1.1) with respect to x. This gives

$$\frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} t^n P_n'(x) \qquad (11.1.5)$$

From (11.1.2) and (11.1.5),

$$\frac{1}{(1-2xt+t^2)^{3/2}} = \frac{1}{(x-t)}\sum_{n=0}^{\infty} n t^{n-1} P_n(x) = \frac{1}{t}\sum_{n=0}^{\infty} t^n P_n'(x)$$

Thus

$$\sum_{n=0}^{\infty} n t^n P_n(x) = (x-t)\sum_{n=0}^{\infty} t^n P_n'(x).$$

Equate coefficients of t^n in last equation. This gives

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x) \qquad (11.1.6)$$

$\Rightarrow$ Gives $P_n'(x)$ in terms of $P_n(x)$ and $P_{n-1}'(x)$.

Next differentiate (11.1.4) with respect to x so that

$$(n+1)P'_{n+1}(x) - (2n+1)xP_n'(x) - (2n+1)P_n(x) + nP'_{n-1}(x) = 0 \qquad (11.1.7).$$

Use (11.1.6) to eliminate $P_n'(x)$ from (11.1.7). This gives

$$(n+1)P'_{n+1}(x) - (2n+1)\left[nP_n(x) + P'_{n-1}(x)\right] - (2n+1)P_n(x) + nP'_{n-1}(x) = 0$$

or

$$(n+1)P'_{n+1}(x) - (2n+1)(n+1)P_n(x) - (n+1)P'_{n-1}(x) = 0$$

or

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x) \qquad \text{for } n \geq 1 \qquad (11.1.8)$$

Integrating (11.1.8) with respect to x gives

$$(2n+1)\int P_n(x)\,dx = P_{n+1}(x) - P_{n-1}(x) + \text{constant} \qquad \text{for } n \geq 1 \qquad (11.1.9)$$

or

$$\int_x^1 P_n(x)\,dx = \frac{1}{2n+1}\left(P_{n-1}(x) - P_{n+1}(x)\right).$$

State Rodrigues' formula and those involving $Q_n(x)$ on next page.

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Some Relations Involving Legendre Functions (stated without proof)

Delete

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \qquad (11.9) \qquad \text{Rodrigues formula (valid for all } x\text{, including } x>1)$$

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \qquad (n=1,2,\dots) \qquad (11.1.4.)$$

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x) \qquad (11.1.6)$$

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x) \qquad (n=1,2,\dots) \qquad (11.1.8)$$

$$\int_x^1 P_n(x)\,dx = \frac{1}{2n+1}\left(P_{n-1}(x) - P_{n+1}(x)\right) \qquad (n=1,2,\dots) \qquad (11.1.9)$$

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$Q_n(x) = \frac{1}{2}P_n(x)\ln\left(\frac{1+x}{1-x}\right) - \sum_{r=0}^{M}(-1)^{n-2r}\frac{(2n-4r-1)}{(2r+1)(n-r)}P_{n-2r-1}(x) \qquad (11.1.10)$$

for $|x|<1$, $n \geq 1$, and $M = \begin{matrix}(n-1)/2 & \text{for } n \text{ odd} \\ (n-2)/2 & \text{for } n \text{ even}\end{matrix}$.

For $|x|>1$,

$$Q_n(x) = \frac{1}{2}P_n(x)\ln\left(\frac{1-x}{1+x}\right) - \sum_{r=0}^{M}(-1)^{n-2r}\frac{(2n-4r-1)}{(2r+1)(n-r)}P_{n-2r-1}(x).$$

Orthogonality Relations for the Legendre Polynomials

We have seen that the Legendre polynomials $P_n(x)$ satisfy the equation (11.1). Substituting $P_n(x)$ for y in that equation and rearranging terms, we get

$$\frac{d}{dx}\left\{(1-x^2)P_n'(x)\right\} + n(n+1)P_n(x) = 0$$

and, ~~for $m \neq n$~~ also write,

$$\frac{d}{dx}\left\{(1-x^2)P_m'(x)\right\} + m(m+1)P_m(x) = 0 \text{. Consider } n \neq m.$$

Multiply these equations by $P_m(x)$ and $P_n(x)$, respectively, subtract, and integrate with respect to x for x from -1 to $+1$. This gives

$$\int_{-1}^{1}\Big[P_m(x)\frac{d}{dx}\left\{(1-x^2)P_n'(x)\right\} - P_n(x)\frac{d}{dx}\left\{(1-x^2)P_m'(x)\right\}$$
$$+ P_m(x)P_n(x)\left\{n(n+1) - m(m+1)\right\}\Big]\,dx = 0.$$

Now
$n(n+1) - m(m+1) = (n-m)(n+m+1)$. Thus

$$(n-m)(n+m+1)\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \int_{-1}^{1}\left[P_n(x)\frac{d}{dx}\{(1-x^2)P_m'(x)\} - P_m(x)\frac{d}{dx}\{(1-x^2)P_n'(x)\}\right]dx$$

~~Now the terms in the integrand on the right hand side are identical except for a difference of sign.~~ Thus their sum is zero and their integral is zero. Therefore $(n-m)(n+m+1)\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = 0$

and
$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = 0 \qquad \text{for } n \neq m \qquad (11.2.1)$$

{ If $n = m$, the left hand side of the equation above could be zero because the coefficient of the integral were zero. Thus the integral would not have to equal zero.

Eq. (11.2.1) tells us that $P_n(x)$ and $P_m(x)$ are orthogonal in the interval $(-1, 1)$. [The weighting function is 1.]

If $m = n$, it can be shown that

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1} \qquad (11.2.2)$$

~~Have students read the 5 exercises in pp 179–183~~

Proof: Square both sides of $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x)$ (11.1.1)
and integrate with respect to x from -1 to $+1$. This gives

$$\int_{-1}^{1}\frac{dx}{(1-2xt+t^2)} = \int_{-1}^{1}\left[P_0(x) + t\,P_1(x) + t^2 P_2(x) + \cdots + t^n P_n(x) + \cdots\right]^2 dx$$

because the integral of all the cross product terms is zero on account of orthogonality

$$= \int_{-1}^{1}\sum_{n=0}^{\infty} t^{2n}[P_n(x)]^2\,dx = \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1}[P_n(x)]^2\,dx.$$

Now
$$\int_{-1}^{1}\frac{dx}{(1+t^2-2xt)} = \frac{1}{(-2t)}\ln\left(1+t^2-2xt\right)\Big]_{-1}^{1} = -\frac{1}{2t}\ln(1-t)^2 + \frac{1}{2t}\ln(1+t)^2$$

$$= \frac{2}{2t}\ln\left(\frac{1+t}{1-t}\right) = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)$$

The expression $\frac{1}{t} \ln\left(\frac{1+t}{1-t}\right)$ can be expanded in a power series. It ~~gives~~ is

$$\frac{1}{t} \ln\left(\frac{1+t}{1-t}\right) = 2 \sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1} .$$

Therefore

$$2 \sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1} = \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} [P_n(x)]^2 \, dx \qquad (11.2.1)$$

Equate coefficients of t^{2n}. This gives

$$\int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1} . \qquad (11.2.2)$$

$$\int_{\pi}^{0} [P_n(\cos\theta)]^2 (-\sin\theta \, d\theta) = \int_{0}^{\pi} \sin\theta \, [P_n(\cos\theta)]^2 \, d\theta = \frac{2}{2n+1}$$

~~We shall show that~~ A given function $f(x)$ can be developed in a series of Legendre polynomials in the interval $(-1, 1)$. We shall determine the coefficients in this series.

Write

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + \cdots + a_n P_n(x) + \cdots = \sum_{n=0}^{\infty} a_n P_n(x)$$

Multiply both sides by $P_n(x)$ and integrate with respect to x from -1 to $+1$. This gives, using the orthogonality property of the Legendre polynomials,

$$\int_{-1}^{1} f(x) P_n(x) \, dx = a_n \int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2 a_n}{2n+1} .$$

Thus

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \, dx .$$

$$= \frac{2n+1}{2} \int_{0}^{\pi} f(\theta) \sin\theta \, P_n(\cos\theta) \, d\theta$$

Exercise 1

a) The polynomial $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$

where $a_0, a_1, \ldots, a_n$ are given constants, can be expressed in the form

$$f(x) = b_n P_n(x) + b_{n-1} P_{n-1}(x) + \cdots + b_0 P_0(x) \qquad (11.2.5.)$$

Proof:

We had before that

$$P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!\,(n-2r)!\,(n-r)!} x^{n-2r} \qquad (11.5)$$

where $N = n/2$ for n even and $N = (n-1)/2$ for n odd.

The right hand side is a power series in x. Write it as

$$P_n(x) = A_0 x^n + A_1 x^{n-1} + \cdots + A_N . \qquad (11.5')$$

From (11.5),

$$A_0 = \frac{(2n)!}{2^n (n!)^2} .$$

Now $f(x) - a_0 x^n$ is a polynomial of degree $n-1$. Call it $\phi(x)$.

Thus

$$f(x) - a_0 \left[\frac{P_n(x) - A_1 x^{n-1} - A_2 x^{n-2} \cdots - A_n}{A_0} \right] = \phi(x)$$

where the term in brackets is the expression for x^n from (11.5').

We can write the last equation as

$f(x) = b_n P_n(x) + \phi_1(x)$ where $\phi_1(x)$ is a polynomial of degree $n-1$ ~~and~~

and

$$b_n = \frac{a_0}{A_0} = \frac{a_0 2^n (n!)^2}{(2n)!} .$$

56 ab (75 h)

Proceeding in a similar manner for $\phi_1(x)$ (cite), we establish (11.2.5)

b) Prove that

$$\int_{-1}^{1} x^m P_n(x)\,dx = \begin{cases} 0 & \text{for } m \le (n-1) \\ \dfrac{2^{n+1}(n!)^2}{(2n+1)!} & m = n \end{cases} \qquad (11.2.6)$$

Calling $f(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_m$, we have $a_0 = 1$, $n = m$, $a_i = 0$ for $i \ne m$

From (11.2.5),

$$x^m = b_m P_m(x) + b_{m-1} P_{m-1}(x) + \dots + b_0$$

$$\text{where} \quad b_m = \frac{2^m (m!)^2}{(2m)!}$$

Thus

$$\int_{-1}^{1} x^m P_n(x)\,dx = \int_{-1}^{1} \left[\frac{2^m (m!)^2}{(2m)!} P_m(x) + b_{m-1} P_{m-1}(x) + \dots + b_0 \right] P_n(x)\,dx$$

If $m \le n-1$, then we use the fact that $\int_{-1}^{1} P_m(x) P_n(x)\,dx = 0$ for $n \ne m$

and the first of (11.2.6) follows.

If $m = n$, then

$$\int_{-1}^{1} x^n P_n(x)\,dx = \int_{-1}^{1} \frac{2^n (n!)^2}{(2n)!} [P_n(x)]^2\,dx = \frac{2^n (n!)^2}{(2n)!} \cdot \frac{2}{2n+1} = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

which is the second of (11.2.6).

$\phi_1(x) = B_1 x^{n-1} + B_2 x^{n-2} + \dots + B_n$

~~56 dq~~ 75C

Exercise 2. Prove that

$$I = \int_{-1}^{1} x^m P_n(x)\,dx = \left\{m!\,\Gamma\left(\frac{m-n+1}{2}\right)\right\} \Big/ \left\{2^n (m-n)!\,\Gamma\left(\frac{m+n+3}{2}\right)\right\}$$

$$= 2\,\frac{m(m-1)(m-2)\cdots\cdots(m-n+2)}{(m+n+1)(m+n-1)\cdots\cdots(m-n+3)} \quad \text{for } m \geq n$$

when $(m-n)$ is even or zero

$I = \int_{-1}^{1} x^m P_n(x)\,dx = 0$ for $(m-n)$ an odd integer or for $m < n$.

Proof: We have already established in ex. 1 that $I = 0$ for $m < n$.

To discuss the cases for $m \geq n$, rewrite the integrand using Rodrigues' formula.

Thus

$$I = \frac{1}{2^n n!} \int_{-1}^{1} x^m \frac{d^n}{dx^n}(x^2-1)^n\,dx.$$

Integrate by parts. This gives

$$I = \frac{1}{2^n n!}\left[x^m \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n - \int \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, m x^{m-1}\,dx\right]_{-1}^{1} {}^{*}$$

[After taking $(n-1)$ derivatives there will remain (x^2-1), thus $x=\pm1$ in all members of the derivative] The integrated part is zero at both the upper and lower limits. Thus

$$I = \frac{-m}{2^n n!} \int_{-1}^{1} x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx.$$

Integrate by parts $(n-1)$ more times. This gives

$$I = \frac{(-1)^n}{2^n n!}\left[m(m-1)(m-2)\cdots\cdots\{m-(n-1)\}\right] \int_{-1}^{1} x^{m-n}(x^2-1)^n\,dx \quad \text{for } m \geq n.$$

[For $m < n$ the term in brackets would be zero.]

~~It can be shown that~~ Now

$$\int_{-1}^{1} x^{m-n}(x^2-1)^n\,dx = \begin{cases} 0 & \text{for } (m-n) \text{ an odd integer} \\ 2\int_{0}^{1} x^{m-n}(x^2-1)^n\,dx & \text{for } (m-n) \text{ an even integer.} \end{cases}$$

because $(x^2-1)^n$ is an even function, and $\int_{-c}^{c}(\text{odd function of } x)\,dx = 0$

$\int_{-c}^{c}(\text{even function of } x)\,dx = 2\int_{0}^{c}(\text{even fun})\,dx$

56-88 (15A)

For the case $(m-n)$ is even and $m>n$, make the substitution $t = x^2$.

This gives

$$2\int_0^1 x^{m-n}(x^2-1)^n\,dx = 2\int_0^1 t^{(m-n)/2}(t-1)^n\,\frac{1}{2}\frac{dt}{\sqrt{t}}$$

$$= (-1)^n \int_0^1 (1-t)^n t^{(m-n-1)/2}\,dt$$

$$= (-1)^n \frac{\Gamma(n+1)\,\Gamma[(m-n+1)/2]}{\Gamma[(m+n+3)/2]}$$

Thus, for $(m-n)$ even and $m>n$,

$$I = \int_{-1}^{1} x^m P_n(x)\,dx = \frac{m(m-1)\cdots(m-n+1)}{2^n\, n!}\;\frac{\Gamma(n+1)\,\Gamma[(m-n+1)/2]}{\Gamma[(m+n+3)/2]}$$

$$= \frac{m(m-1)\cdots(m-n+1)\,(m-n)!}{2^n\, n!\;\;(m-n)!}\cdot\frac{n!\;\Gamma[(m-n+1)/2]}{\Gamma[(m+n+3)/2]}$$

$$= \frac{m!}{2^n(m-n)!}\;\frac{\Gamma[(m-n+1)/2]}{\Gamma[(m+n+3)/2]}$$

56 dd (75 e)

Exercise 3. Show that $\int_{-1}^{1} e^{iyx} P_n(x)\,dx = i^n \sqrt{\frac{2\pi}{y}}\, J_{n+\frac{1}{2}}(y)$.

Proof:

Expand e^{iyx} in a power series.

$$e^{iyx} = 1 + (iyx) + \frac{(iyx)^2}{2!} + \dots = \sum_{m=0}^{\infty} \frac{(iyx)^m}{m!}$$

Thus using the results of the preceding exercise,

$$\int_{-1}^{1} e^{iyx} P_n(x)\,dx = \sum_{m=0}^{\infty} \frac{(iy)^m}{m!} \int_{-1}^{1} x^m P_n(x)\,dx$$

$$= \sum_{m} (iy)^m\, \Gamma\left(\frac{m-n+1}{2}\right) \Big/ \left\{ 2^n (m-n)!\, \Gamma\left(\frac{m+n+3}{2}\right) \right\}$$

where $m = n, n+2, n+4, \dots, \infty$.

Let $2r = m-n$, or $m = n+2r$. Substitution gives

$$\int_{-1}^{1} e^{iyx} P_n(x)\,dx = \sum_{r=0}^{\infty} (-1)^r i^n y^{n+2r}\, \Gamma\left(\frac{2r+1}{2}\right) \Big/ \left\{ 2^n (2r)!\, \Gamma(n+\tfrac{1}{2}+r+1) \right\}$$

It can be shown that $\dfrac{\Gamma\left(\frac{2r+1}{2}\right)}{(2r)!} = \dfrac{\sqrt{\pi}}{2^{2r}\, r!}$.

Therefore

$$\int_{-1}^{1} e^{iyx} P_n(x)\,dx = \sum_{r=0}^{\infty} \frac{(-1)^r i^n y^{n+2r} \left(\frac{\sqrt{\pi}}{2^{2r}\, r!}\right)}{2^n\, \Gamma(n+\frac{1}{2}+r+1)} = \sqrt{\pi}\, i^n \sum_{r=0} \frac{(y/2)^{n+2r} \cdot (-1)}{r!\, \Gamma(n+\frac{1}{2}+r+1)}$$

$$= \sqrt{\frac{2\pi}{y}}\; i^n \sum_{r=0}^{\infty} \frac{(y/2)^{n+2r+\frac{1}{2}} (-1)^r}{r!\, \Gamma(n+\frac{1}{2}+r+1)}$$

But, from the definition of the Bessel function (Irv. & Mull, pg 128, eq. (2.1)),

$$J_{n+\frac{1}{2}}(y) = \sum_{r=0}^{\infty} (-1)^r \frac{(y/2)^{n+\frac{1}{2}+2r}}{r!\, \Gamma(n+\frac{1}{2}+r+1)}.$$

Thus $\displaystyle\int_{-1}^{1} e^{iyx} P_n(x)\,dx = \sqrt{\frac{2\pi}{y}}\; i^n J_{n+\frac{1}{2}}(y)$.

56 ff (75 ft)

Exercise 4. Prove that

$$e^{ikz} \equiv e^{ikr\cos\theta} = \sum_{n=0}^{\infty} (2n+1)\, i^n \sqrt{\frac{\pi}{2kr}}\; P_n(\cos\theta)\; J_{n+1/2}(kr)$$

$$= \sum_{n=0}^{\infty} (2n+1)\, i^n\, P_n(\cos\theta)\, {}_{*}j_n(kr)$$

where the spherical Bessel function ${}_{*}j_n(kr) \equiv \sqrt{\frac{\pi}{2kr}}\; J_{n+1/2}(kr)$.

Proof:

Because $e^{ikr\cos\theta}$ is an axially symmetric function that remains finite on the symmetry axis ($\theta = 0$ or $\theta = \pi$), it may be expanded in terms of Legendre polynomials $P_n(\cos\theta)$. Thus

$$e^{ikr\cos\theta} = \sum_{n=0}^{\infty} f_n(r)\, P_n(\cos\theta) \quad , \text{ where the } f_n(r) \text{ are to be determined}$$

Multiply both sides of the equation by $\sin\theta\, P_m(\cos\theta)$ and integrate with respect to θ from 0 to π. This gives

$$\int_0^{\pi} e^{ikr\cos\theta}\, P_m(\cos\theta) \sin\theta\, d\theta = \int_0^{\pi} f_m(r) \left[P_m(\cos\theta)\right]^2 \sin\theta\, d\theta$$

$$= \frac{2 f_m(r)}{2m+1} \qquad (m = 0, 1, 2, \dots)$$

Thus

$$f_m(r) = \frac{2m+1}{2} \int_{-1}^{1} P_m(\mu)\, e^{ikr\mu}\, d\mu$$

$$= \left(\frac{2m+1}{2}\right) i^m \sqrt{\frac{2\pi}{kr}}\; J_{m+1/2}(kr) \qquad \text{from exercise 3.}$$

$$= (2m+1)\, i^m\, {}_{*}j_m(kr).$$

Thus

$$e^{ikr\cos\theta} = \sum_{n=0}^{\infty} i^m (2m+1)\, {}_{*}j_m(kr)\, P_m(\cos\theta).$$

– See note on next page about spherical Bessel functions →

56 dg (757)

Note on Spherical Bessel Functions (See Kraut "Fundamentals of Mathematical Physics, pg 279)

Spherical Bessel functions are solutions of the diff. eqn

$$\frac{d^2y}{dr^2} + \frac{2}{r}\frac{dy}{dr} + \left[k^2 - \frac{p(p+1)}{r^2}\right] y = 0.$$

Make a change in variables. Let $y = \frac{z(r)}{\sqrt{r}}$. The diff. eqn becomes

$$\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} + \left[k^2 - \frac{(p+\frac{1}{2})^2}{r^2}\right] z = 0 .$$

This is Bessel's eqn (2.2), with $x = kr$ and $n = p + \frac{1}{2}$. Therefore the general solution is

$$y = \frac{z}{\sqrt{r}} = \frac{A}{\sqrt{r}} J_{p+\frac{1}{2}}(kr) + \frac{B}{\sqrt{r}} Y_{p+\frac{1}{2}}(kr)$$

where A and B are arbitrary constants.

The spherical Bessel functions are defined by

$$* j_p(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{p+\frac{1}{2}}(kr)$$

and the spherical Neumann functions by

$$* n_p(kr) = \sqrt{\frac{\pi}{2kr}}\, (-1)^{p+1} J_{-p-\frac{1}{2}}(kr) .$$

Thus the solution to the spherical Bessel diff. eqn is

$$y = C_* j_p(kr) + D_* n_p(kr)$$

where C and D are arbitrary constants.

These functions are useful in writing solutions to Helmholtz's equation in spherical coordinates.

~~56~~ d h (75 h)

Exercise 5. Express the function $f(\theta)$ given by

$$f(\theta) = \begin{cases} 0 & \text{for } \alpha \le \theta \\ 1 & \text{for } 0 < \theta < \alpha \end{cases} \qquad \text{where } |\cos\alpha| \le 1$$

in terms of the Legendre polynomials $P_n(\cos\theta)$.

Solution:

Let $f(\theta) = \sum_{m=0}^{\infty} a_m P_m(\cos\theta)$ where the a_m's are to be determined.

From orthogonality,

$$\int_0^{\pi} f(\theta) P_m(\cos\theta) \sin\theta \, d\theta = a_m \int_0^{\pi} \left[P_m(\cos\theta)\right]^2 \sin\theta \, d\theta = \frac{2a_m}{2m+1}.$$

Thus

$$a_m = \frac{2m+1}{2} \int_0^{\alpha} P_m(\cos\theta) \sin\theta \, d\theta,$$

$$= \frac{1}{2}\left[P_{m+1}(\cos\theta) - P_{m-1}(\cos\theta)\right]_0^{\alpha} \quad \text{for } m \ge 1 \quad \text{by means of (11.1.9)}$$

Therefore

$$a_m = \frac{1}{2}\Big[P_{m+1}(\cos\alpha) - P_{m-1}(\cos\alpha) - \underbrace{P_{m+1}(1)}_{=1} + \underbrace{P_{m-1}(1)}_{=+1}\Big]$$

$$= \frac{1}{2}\left[P_{m+1}(\cos\alpha) - P_{m-1}(\cos\alpha)\right].$$

~~Finally~~, We also need a_0.

~~$f(\theta) =$~~ $$a_0 = \frac{1}{2}\int_0^{\alpha} \sin\theta \, d\theta = \frac{1-\cos\alpha}{2}$$

Finally,

$$f(\theta) = \frac{1-\cos\alpha}{2} + \frac{1}{2}\sum_{m=1}^{\infty}\left[P_{m+1}(\cos\alpha) - P_{m-1}(\cos\alpha)\right] P_m(\cos\theta).$$

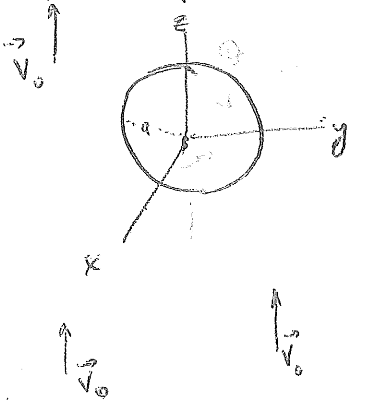
More generally, as can be seen from the development above, if $f(x)$ satisfies the Dirichlet conditions in the interval $(-1,1)$,

$$\sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x) \int_{-1}^{1} f(x) P_n(x)\, dx = \begin{cases} f(-1_+) & \text{for } x = -1 \\ \frac{1}{2}\left[f(x+) + f(x-)\right] & \text{for } -1 < x < 1 \\ f(1_-) & \text{for } x = 1 \end{cases}.$$

36 (72)

Example: Laplace's eqn. in polar spherical coordinates (Irving + Mullin; pg. 184)

A rigid sphere is placed in a stream of fluid which in the undisturbed state is uniform with a velocity $\vec{V}_0$. Determine the velocity at any point in the fluid after the sphere has been placed in it and steady-state conditions prevail. (incompressible ideal fluid)



Sol.: The velocity $\vec{V} = -\nabla\phi$.

The diff. eqn. is $\nabla^2\phi = 0$ for $r > a$.

The bdy conditions are: 1) $\frac{\partial\phi}{\partial r} = 0$ on $r = a$

2) $\phi \to -V_0 z = -V_0 r\cos\theta = -V_0 r P_1(\cos\theta)$ as $r \to \infty$

where we use $P_1(\cos\theta) = \cos\theta$

a) The velocity pot. ϕ does not depend on the angle φ. Therefore our solution will contain the Legendre, rather than the associated Legendre, ~~polynomials~~ functions.

However, b) the velocity potential ϕ is finite along the z-axis ($\theta = 0$ or π). However, the Legendre functions of the second kind, namely $Q_n(\cos\theta)$ are infinite for $\theta = 0$ or π. Therefore they cannot appear in the solution to this problem.

We have $\phi = \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} P_n(\cos\theta)$ as particular solutions.

This can be written as

$$\phi = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta).$$

From the 2nd boundary condition, all the $A_n = 0$ except for $n = 1$, and

$$\lim_{r\to\infty}\phi = -\lim_{r\to\infty}\left[V_0 r P_1(\cos\theta)\right] = +\lim_{r\to\infty}\left[A_1 r P_1(\cos\theta)\right]$$

Thus $A_1 = -V_0$ and $\phi = -V_0 r P_1(\cos\theta) + \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta)$

$\phi = -V_0\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)P_1(\cos\theta)$

From the 1st bdy. cond.,

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = 0 = -V_0 P_1(\cos\theta) + \sum_{n=0}^{\infty} -(n+1) B_n a^{-(n+2)} P_n(\cos\theta)$$

We can equate coefficients of Legendre functions of the same order because of their orthogonality. Thus $B_n = 0$ except for $n = 1$, and

$$-V_0 = 2a^{-3}B_1 \quad\text{or}\quad B_1 = \frac{-V_0 a^3}{2}.$$

37 (72)

Thus the velocity pot. is

$$\phi = -V_0\, r\, P_1(\cos\theta) - V_0 \frac{a^3}{2r^2} P_1(\cos\theta)$$

$$= -V_0\, P_1(\cos\theta)\left[r + \frac{a^3}{2r^2} \right]$$

$$= -V_0 \left(r + \frac{a^3}{2r^2} \right) \cos\theta \; .$$

The velocity components are:

$$V_r = -\frac{\partial \phi}{\partial r} = +V_0 \left(1 - \frac{a^3}{r^3} \right) \cos\theta$$

$$V_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{V_0}{r} \left(r + \frac{a^3}{2r^2} \right) \sin\theta = -V_0 \left(1 + \frac{a^3}{2r^3} \right) \sin\theta \; ,$$

Another example: ~~Consider~~ Find the electrostatic ~~field~~ potential inside ~~within~~ a uniform spherical surface on whose surface $\phi = f(\theta)$.

Sol. : Because we have axial symmetry, the solutions take the form

$$\phi = \left. \begin{matrix} r^n \\ r^{-(n+1)} \end{matrix} \right\} \left. \begin{matrix} P_n(\cos\theta) \\ Q_n(\cos\theta) \end{matrix} \right\}$$

The potential must be finite at $\theta = 0, \pi$. Thus the coefficients of $Q_n(\cos\theta)$ must be zero. The potential must be finite at $r = 0$. Therefore the coefficients of $r^{-(n+1)}$ must be zero. This gives

$$\phi = \sum_{n=0}^{\infty} A_n \left(\frac{r}{a} \right)^n P_n(\cos\theta)$$

Now ~~use~~ the boundary condition, $\phi = f(\theta) = F(\cos\theta)$ at $r = a$

$\int_{-1}^{1} P_n(\mu) P_m(\mu)\, d\mu = 0$ if $n \neq m$
$= \frac{2}{2n+1}$ if $n = m$.

Then

$$F(\cos\theta) = F(\mu) = \sum_{n=0}^{\infty} A_n P_n(\mu)$$

where the A_n in the Legendre's series above are given by the eqn.

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} F(\mu) P_n(\mu)\, d\mu$$

We will prove this latter.

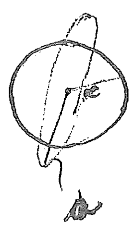
See Churchill pg 191, eq (1) & (2)

See over

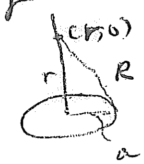
Snedden "Partial Diff Equations" pp 162-163

We shall consider a problem similar to the preceding one, but ~~which~~ which will illustrate some caution we must exhibit.

Consider a grounded spherical conductor of radius c surrounded by a circular wire of radius a which carries a uniform electric charge. Call the linear charge density e. (The center of the sphere and the circular wire coincide.) Find the electrostatic potential.



First find the potential due to the wire alone at any point in space. ~~Do~~ this by determining the potential ϕ_w at a point on the axis, and put the resultant expression in series form. Compare the coefficients of this series with those for $\phi_w(r,\theta)$ for any value of ~~θ~~ and/or r.



$$\phi_w(r,0) = \int_C \frac{e\,ds}{R} = \frac{2\pi e a}{\sqrt{a^2+r^2}} .$$

Expand $\frac{1}{\sqrt{a^2+r^2}}$ by the binomial expansion. This gives:

1) for $r \le a$,

$$\frac{1}{a\sqrt{1+(\frac{r}{a})^2}} = \frac{1}{a}\left\{1 - \frac{1}{2}\left(\frac{r}{a}\right)^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left[\left(\frac{r}{a}\right)^2\right]^2 - \cdots\right\}$$

$$= \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{2})_n}{n!}\left(\frac{+r^2}{a^2}\right)^n$$

where $(\frac{1}{2})_n = \frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)(\frac{1}{2}+3)\cdots\cdots(\frac{1}{2}+n-1)$.

2) for $r \ge a$,

$$\frac{1}{r\sqrt{1+(\frac{a}{r})^2}} = \frac{1}{r}\left\{1 - \frac{1}{2}\left(\frac{a}{r}\right)^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left[\left(\frac{a}{r}\right)^2\right]^2 - \cdots\right\} = \frac{1}{r}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!}(-1)^n\left(\frac{a^2}{r^2}\right)^n$$

For $r \le a$,

$$\phi_w(r,0) = 2\pi e\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{2})_n}{n!}\left(\frac{+r^2}{a^2}\right)^n = 2\pi e\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!}(-1)^n\left(\frac{r}{a}\right)^{2n}.$$

For $r \ge a$,

$$\phi_w(r,0) = 2\pi e\left\{\frac{a}{r} - \frac{1}{2}\left(\frac{a}{r}\right)^3 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(\frac{a}{r}\right)^5 + \cdots\right\} = 2\pi e\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(-1)^n}{n!}\left(\frac{a}{r}\right)^{2n+1}.$$

In general, ~~for~~

$$\phi_w(r,\theta) = 2\pi e\sum_{n=0}^{\infty}\left[A_n r^n P_n(\cos\theta) + B_n\frac{1}{r^{n+1}}P_n(\cos\theta)\right]$$

The potential must be finite for $r \le a$. Therefore all the B_n's are zero for this region. ~~and~~ Comparing coefficients in the series expansion for $\varphi_w(r,0)$ and $\varphi_w(r,\theta)$, we ~~$A_0 = \frac{(\frac{1}{2})_0}{0!}(-1)^0$~~ get

$$A_{2m} = \frac{(\frac{1}{2})_m (-1)^m}{m!\, a^{2m}}$$

and

$$\varphi_w(r,\theta) = 2\pi e \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (-1)^m r^{2m}}{m!\, a^{2m}} P_{2m}(\cos\theta) \qquad \text{for } r \le a.$$

The potential must approach zero as $r \to \infty$. Therefore, it can be shown that the A_n's, with the exception of A_0, $= 0$ for $r \ge a$. Comparing coefficients in the series expansion for $\varphi_w(r,\theta)$ and $\varphi_w(r,\theta)$, we get

$$\varphi_w(r,\theta) = 2\pi e \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (-1)^m a^{2m+1}}{m!\, r^{2m+1}} P_{2m}(\cos\theta) \qquad \text{for } r \ge a.$$

Now consider the conducting, grounded sphere placed inside the electrified wire. Because the conductor is grounded the potential on its surface and within it is zero. Let us call φ_1 the potential in the region $c \le r \le a$ and φ_2 the potential in the region $r \ge a$. Then the boundary conditions are:

A) $\varphi_1 = 0$ for $r = c$

B) $\varphi_1 = \varphi_2$ for $r = a$

C) $\dfrac{\partial \varphi_1}{\partial r} = \dfrac{\partial \varphi_2}{\partial r}$ for $r = a$ except at the points in the electrified wire.

The presence of the conducting sphere within the electrified wire modifies the expansion for the potential due to the wire. We may think of these modifications as additional potential functions. Because they are independent of ψ, we may write:

see note over

1) for $c \le r \le a$, $\varphi_1 = 2\pi e \sum_{m=0}^{\infty} \left\{ (-1)^m \frac{(\frac{1}{2})_m}{m!} \left(\frac{r}{a}\right)^{2m} + A_m \left(\frac{r}{a}\right)^{2m} + B_m \left(\frac{c}{r}\right)^{2m+1} \right\} P_{2m}(\cos\theta)$

2) for $r \ge a$, $\varphi_2 = 2\pi e \sum_{m=0}^{\infty} \left\{ (-1)^m \frac{(\frac{1}{2})_m}{m!} \left(\frac{a}{r}\right)^{2m+1} + C_m \left(\frac{a}{r}\right)^{2m+1} \right\} P_{2m}(\cos\theta)$

There are no terms in $\left(\frac{r}{a}\right)^{2n}$ in (2) because ϕ must equal zero as $r = \infty$.

Boundary condition (A) yields

$$\phi_1 = 0 = (-1)^n \frac{\left(\frac{1}{2}\right)_n}{n!}\left(\frac{c}{a}\right)^{2n} + A_n\left(\frac{c}{a}\right)^{2n} + B_n = 0$$

Boundary condition (B) yields

$$(\phi_1)_{r=a} = (\phi_2)_{r=a} = \;;\; A_n + B_n\left(\frac{c}{a}\right)^{2n+1} - C_n = 0 .$$

Boundary condition (C) presents some difficulties. Although the series expansions for ϕ_1 and ϕ_2 in eqs (1) and (2) are convergent at $r = a$, they are not uniformly convergent ~~at~~ if $r = a$. Therefore we cannot interchange the order of differentiation and summation as we might expect to do for the condition $\left(\frac{\partial \phi_1}{\partial r}\right)_{r=a} = \left(\frac{\partial \phi_2}{\partial r}\right)_{r=a} = 0$.

Let us look at this problem physically. The terms which give the trouble are those directly related to the potential of the wire, because at points on the wire $\frac{\partial \phi}{\partial r}$ is discontinuous. Even though the wire only represents one circle ~~curve~~ on the surface of the sphere $r = a$, our expressions for ϕ_1 and ϕ_2 cannot be differentiated termwise and then summed at any point on that surface. However, there is no difficulty in equating the derivatives of the potential due to the ~~che~~ the sphere alone at $r = a$. That is

1') for $c \le r \le a$, $\phi_1' = 2\pi e \sum_{n=0}^{\infty} \left\{ A_n\left(\frac{r}{a}\right)^{2n} + B_n\left(\frac{c}{r}\right)^{2n+1} \right\} P_{2n}(\cos\theta)$

and 2') for $r \ge a$, $\phi_2' = 2\pi e \sum_{n=0}^{\infty} C_n\left(\frac{a}{r}\right)^{2n+1} P_{2n}(\cos\theta)$

are uniformly convergent at $r = a$. Thus we can modify our boundary condition C to be

c') $\left(\frac{\partial \phi_1'}{\partial r}\right)_{r=a} = \left(\frac{\partial \phi_2'}{\partial r}\right)_{r=a}$. This gives

$$\left[\frac{2n}{a} A_n \left(\frac{r}{a}\right)^{2n-1} + (2n+1) B_n \left(\frac{c}{r}\right)^{2n} \left(-\frac{c}{r^2}\right) - (2n+1) C_n \left(\frac{a}{r}\right)^{2n} \left(-\frac{a}{r^2}\right) \right]_{r=a} = 0$$

or $\frac{2n}{a} A_n - (2n+1) B_n \frac{1}{a} \left(\frac{c}{a}\right)^{2n+1} + \frac{1}{a}(2n+1) C_n = 0$.

The last equation reduces to:

$$\cancel{\frac{2n}{a}\left[A_n - B_n \left(\frac{c}{a}\right)^{2n+1} - \frac{1}{2n} B_n \left(\frac{c}{a}\right)^{2n+1}\right]}$$

$$2n A_n - (2n+1) B_n \left(\frac{c}{a}\right)^{2n+1} + (2n+1) C_n = 0 \qquad \text{(from (C))}$$

$$\left(\frac{c}{a}\right)^{2n} A_n + B_n = -(-1)^n \frac{(\frac{1}{2})_n}{n!} \left(\frac{c}{a}\right)^{2n} \qquad \text{(from (A))}$$

$$A_n + \left(\frac{c}{a}\right)^{2n+1} B_n - C_n = 0 \qquad \text{(from B)}$$

Thus

$$A_n = \frac{\begin{vmatrix} 0 & -(2n+1)\left(\frac{c}{a}\right)^{2n+1} & 2n+1 \\ -(-1)^n \frac{(\frac{1}{2})_n}{n!} \left(\frac{c}{a}\right)^{2n} & 1 & 0 \\ 0 & \left(\frac{c}{a}\right)^{2n+1} & -1 \end{vmatrix}}{\begin{vmatrix} 2n & -(2n+1)\left(\frac{c}{a}\right)^{2n+1} & 2n+1 \\ \left(\frac{c}{a}\right)^{2n} & 1 & 0 \\ 1 & \left(\frac{c}{a}\right)^{2n+1} & -1 \end{vmatrix}} = 0$$

$$B_n = \begin{vmatrix} 2n & 0 & 2n+1 \\ \left(\frac{c}{a}\right)^{2n} & -(-1)^n \frac{(\frac{1}{2})_n}{n!} \left(\frac{c}{a}\right)^{2n} & 0 \\ 1 & 0 & -1 \end{vmatrix} \Big/ D$$

$$= \frac{-\frac{(-1)^n (\frac{1}{2})_n}{n!} \left(\frac{c}{a}\right)^{2n} \left[-2n - (2n+1)\right]}{-2n + (2n+1)\left(\frac{c}{a}\right)^{4n+1} - (2n+1) - (2n+1)\left(\frac{c}{a}\right)^{4n+1}}$$

$$= \frac{+(-1)^n (\frac{1}{2})_n (4n+1)}{n!} \left(\frac{c}{a}\right)^{2n} \Big/ -(4n+1)$$

$$= \frac{-(-1)^n (\frac{1}{2})_n}{n!} \left(\frac{c}{a}\right)^{2n}$$

$$C_n = \begin{vmatrix} 2n & -(2n+1)\left(\frac{c}{a}\right)^{2n+1} & 0 \\ \left(\frac{c}{a}\right)^{2n} & 1 & -(-1)^n \frac{(\frac{1}{2})_n}{n!} \left(\frac{c}{a}\right)^{2n} \\ 1 & \left(\frac{c}{a}\right)^{2n+1} & 0 \end{vmatrix} \Big/ D$$

$$(-1)^n \left(\tfrac{1}{2}\right)_n \left(\tfrac{c}{a}\right)^{4n+1}$$

Thus, finally

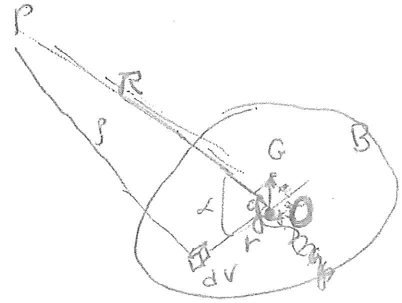
$$\varphi_1 = 2\pi e \sum_{n=0}^{\infty} \left\{ (-1)^n \frac{(\frac{1}{2})_n}{n!} \left(\frac{r}{a}\right)^{2n} - \frac{(-1)^n (\frac{1}{2})_n}{n!} \left(\frac{c}{a}\right)^{2n} \left(\frac{c}{r}\right)^{2n+1} \right\} P_{2n}(\cos\theta)$$

$$= 2\pi e \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2})_n}{n!} \left\{ \frac{r^{2n}}{a^{2n}} - \frac{c^{4n+1}}{a^{2n} r^{2n+1}} \right\} P_{2n}(\cos\theta)$$

$$\varphi_2 = 2\pi e \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2})_n}{n!} \left(\frac{a}{r}\right)^{2n+1} \left\{ 1 + \left(\frac{c}{a}\right)^{4n+1} \right\} P_{2n}(\cos\theta).$$

The Potential of a Body at a Distant Point

MacMillan (pg 81 ff)



Let B be any body, O the origin of coordinates (need not be in the body) and G the centroid of B. We want to find the potential at some distant point P.

Let σ be volume density, and g the projection of $\vec{OG}$ onto the line $\vec{OP}$.

The potential owing to B at P is

$$f(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

if $x = \cos\alpha$, $t = \frac{r}{R}$, then this

$$\phi = \int_B \frac{\sigma\, dV}{\rho}$$

If the angle between r and R is called α, then $\rho = (R^2 - 2rR\cos\alpha + r^2)^{1/2}$ and $\frac{1}{\rho} = \frac{1}{R}\left(1 - 2\frac{r}{R}\cos\alpha + \frac{r^2}{R^2}\right)^{-1/2}$. Expanding by the binomial theorem and rearranging terms, it can be shown that

$$\frac{1}{\rho} = \frac{1}{R} + \frac{r}{R^2}\cdot\cos\alpha + \frac{r^2}{R^3}\cdot\frac{1}{2}(3\cos^2\alpha - 1) + \frac{r^3}{R^4}\cdot\frac{1}{2}(5\cos^3\alpha - 3\cos\alpha)$$

$$+ \frac{r^4}{R^5}\cdot\frac{1}{8}(35\cos^4\alpha - 30\cos^2\alpha + 3) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{r^n}{R^{n+1}} P_n(\cos\alpha).$$

Δ follows.

Substitute into the expression for the potential. This gives

$$\phi = \frac{1}{R}\int_B \sigma\, dV + \frac{1}{R^2}\int_B \sigma r \cos\alpha\, dV + \frac{1}{2R^3}\int_B \sigma(3r^2\cos^2\alpha - r^2)\, dV$$

$$+ \frac{1}{2R^4}\int_B \sigma(5r^3\cos^3\alpha - 3r^3\cos\alpha)\, dV + \cdots$$

Now $\frac{1}{R}\int_B \sigma\, dV = \frac{M}{R}$

$\frac{1}{R^2}\int_B \sigma r\cos\alpha\, dV = \frac{1}{R^2}\cdot Mg$ where g is the projection of the distance from O to the centroid G on the line OP [The integral serves as the definition of the R component of the distance from the origin to the centroid.]

For these formulas to be valid, P must lie outside the smallest sphere of center O which completely contains the body B in its interior.