LECTURE NOTES

ON

SEISMOLOGICAL INSTRUMENTATION

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Seismological Instrumentation

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C_B	capacitance in Willmore bridge circuit	90
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G	galvanometer constant (amperes/radian)	59
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I	length of indicator arm	47
"	current	58
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K_c	moment of inertia about axis through center of mass	37
K_1	moment of inertia of galvanometer coil	59
L	length of spring under load	7
"	length of simple pendulum	11
"	length of conductor	58
"	length of strain meter	98
L_c	distance, axis of rotation to center of coil	92
"	inductance of seismometer coil	90
L_o	initial length of spring	7
M	mass	5
"	moment	62
M_o	amplitude of driving moment	72

Symbols	Definitions	
N	number of turns in coil	58
P	initial spring tension	41
Q^2	$RR_g + RS + R_gS$	64
R	resistance	59
"	resistance of seismometer branch of circuit (called R_c on p. 87)	63
$\vec{R}$	position vector to coil element	61
R_B, R_R, R_D, R_S	resistances in Willmore bridge circuit	90
R_g "	internal resistance of galvanometer, resistance of galvanometer branch of circuit	59
R_s	resistance of shunt when determining R_g	69
R_x	external critical damping resistance	61
S	velocity sensitivity of seismograph (a_m/x_m)	33
"	resistance of shunt	63
S_p	peak value of S	34
T	period of periodic motion	6
T_d	damped natural period	18
T_e	period of sinusoidal earth motion	30
T_n	undamped natural period	8
"	seismometer natural period	30
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U	dynamic response factor, seismometer	69
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V	a_m/x_m, magnification (displacement sensitivity) of seismograph	29
"	apparent surface velocity	39
$\vec{V}$	velocity vector	57
V_o	a/z, static magnification, direct recording seismograph	29
$Y(s)$	transfer function of linear differential equation	85
Z	mechanical impedance	73
a	distance from pivot to line of action of spring, hinged system	9
"	seismograph trace displacement	29
a_I	peak trace amplitude due to a current step	88
a_m	peak trace amplitude due to weight snatch	88
c	coefficient of viscous damping	16
c_c	critical coefficient of viscous damping	17

Symbols	Definitions	
d	moment arm of test weight	50
"	tilt arm	53
"	mechanical viscous damping coefficient of galvanometer	60
$d\vec{s}$	vector element of coil winding	61
$\vec{e}$	electric field vector	58
e_c	voltage across seismometer coil in bridge calibration	91
f	frequency of periodic motion	6
f_d	damped natural frequency	18
f_n	undamped natural frequency	8
$f(t)$	driving force	23
g	acceleration of gravity	8
"	galvanometer current sensitivity (amperes/mm at 1 meter)	59
b	distance from mass to node, simple pendulum	36
b_1, b_2	length of portions of spring	41
i	angle of inclination, horizontal pendulum	13
"	current in galvanometer coil	59
i_c	current in seismometer coil in bridge calibration	91
k	proportionality factor relating restoring force to displacement in linear system	5
"	spring constant	7
"	transfer factor, electromagnetic seismograph	65
$\vec{k}$	unit vector parallel to axis of rotation	61
m	mass of test weight	50
"	slope of line	93
m_s	mass of spring	9
p	distance from pivot to upper support point of spring	41
"	pitch of screw	53
q	distance from pivot to lower point of attachment of spring	41
r	distance from pivot to center of mass of hinged system	10
s	transform variable, Laplace transform	24
"	reading on scale, seismometer deflection	53
t	time variable	5
"	reading on auxiliary scale to determine tilt	53
t_m	time of maximum in oscillating motion	18
v_o	initial velocity	6

Symbols	Definitions	
x	displacement of mass in general translatory system	5
"	absolute displacement of frame	28
"	coordinate along strain meter rod	95
x_m	amplitude of steady-state response	24
x_o	initial displacement	6
x_p	peak amplitude of steady-state response	26
x_r	resonant amplitude of steady-state response	25
x_{st}	F_o/k	25
$x_s(t)$	steady-state response	24
y	absolute displacement of mass, frame in motion	28
"	change in distance between strain meter piers	96
z	$y - x$, relative displacement of mass	28
$\mathcal{Y}(s)$	function of s in Laplace transform of system response	77
ℓ	reduced pendulum length	13
"	length of air gap, variable reluctance transducer	83
$\mathcal{M}$	initial spring moment	51
$\mathcal{R}$	reluctance	83
Δ	$L - L_o$	7
"	total damping coefficient of seismometer	65
Γ	electrodynamic constant of seismometer, rotating system (called Γ_R, page 87)	62
Γ_T	electrodynamic constant of seismometer, translating system	62
Λ	apparent wave length along the surface	39
Φ	magnetic flux	57
α	angle between p and q	41
"	angle between p and spring	45
"	phase angle, electromagnetic seismograph	69
β	angle between q and spring	45
"	angle between ξ and strain meter rod	96
γ	electrodynamic constant of galvanometer (newton-meters/ampere)	58
δ	logarithmic decrement, $\ln \epsilon$	19
"	total damping coefficient, galvanometer	65
"	seismometer phase angle	69
δ'	one-half the logarithmic decrement	19
δ_1	galvanometer phase angle	69

Symbols	Definitions	
ϵ	damping ratio, viscous damping	22
$\vec{\epsilon}$	unit vector	58
h	c/c_c, damping factor	17
h'	part of damping factor due to current in circuit	68
h_1	galvanometer damping factor	60
h_0	open circuit damping factor	60
θ	angular displacement of inertial member in general vibrating system	5
"	angular displacement of galvanometer	59
μ	magnetic permeability	83
ξ	horizontal ground displacement	95
ρ	$\sqrt{\omega_n/\omega_{n1}}$	76
σ^2	$\sigma_s \sigma_g$, the coupling factor	66
σ_g	galvanometer coupling factor	65
σ_s	seismometer coupling factor	65
τ	proportionality factor relating restoring moment to angular displacement	5
"	torsion constant	11
"	dimensionless time variable, $\omega' t$	76
τ_1	torsion constant of galvanometer suspension	59
ϕ	phase angle	6
"	angular displacement of seismometer	61
ψ	angle of tilt of seismometer frame	40
ω	$\sqrt{k/M}$, $\sqrt{\tau/K}$	5
"	angular frequency	6
ω'	$\sqrt{\omega_n \omega_{n1}}$	76
ω_d	damped natural angular frequency	18
ω_e	angular frequency of sinusoidal driving force	24
"	angular frequency of sinusoidal ground motion	29
ω_n	undamped natural angular frequency	8
ω_{n1}	undamped natural angular frequency of galvanometer	60
ω_p	peak frequency of steady-state response	26

SEISMOLOGICAL INSTRUMENTATION

Part I

Theory of the Seismometer and Direct Recording Seismograph

1. The fundamental problem of seismometry

The purpose of this course is to present the techniques used in recording ground vibrations. The vibrations to be recorded are encountered in the study of earthquakes, microseisms, explosion-generated waves (prospecting seismology, nuclear test detection, quarry and mine blasting), and noise generated by industrial operations (machinery, etc.). The theory and design of the instruments, their installation, adjustment, and calibration will be considered. The emphasis will be placed on fundamental principles rather than on some of the exotic techniques that have been developed in modern instrumentation practice. Once the fundamentals are understood, the applications of the most recent technological advances to the problem are readily grasped.

The problem may be broken down into two parts: detection of the ground motion and recording. The detector is a device that responds to the motion of the ground and produces a signal suitable for recording. This signal contains the information about the ground motion (amplitude, waveform, etc.) that the seismologist wishes to study. The recorder is a device that accepts the signal and preserves the information contained in it in a permanent form, so that it is available for subsequent analysis and interpretation. The seismic recorder always includes a means of marking the time of arrival of the events, either absolute time, as in observatory instruments, or time relative to the occurrence of the seismic event, as in prospecting instruments.

Before going into detail concerning the solution of the problem, it is appropriate to consider briefly the properties of the motion to be measured. A complete discussion of this question involves the generation and propagation of seismic waves, and is covered in other places. Here all that is needed is enough insight to provide a basis for the selection of a suitable instrument for the particular seismological problem under consideration, and to indicate the scope of the instrumentation problem. Ground motion is described in the terminology of vibration analysis, and the definitions of a few basic terms will be useful.

Definitions:

a) Vibration is the motion of a material body under the action of fluctuating forces. The forces may change with time in direction or in magnitude and direction. Oscillation is often used as a synonym for vibration, but has the connotation of back-and-forth motion about a rest position. All oscillations are vibrations, but not all vibrations are oscillatory.

b) Periodic motion is motion which is repeated exactly during successive equal time intervals. True periodic motion has no beginning or end. Occasionally the notion of periodicity is extended (inexactly but usefully) to include motions which pass through the rest position of the oscillating body at equal time intervals, even though the motion is not repetitive.
c) Period of a periodic motion is the shortest increment of time at which the motion repeats itself.
d) Cycle is the complete sequence of positions that a body in periodic motion occupies during one period.
e) Frequency is the number of cycles completed in a unit time in a periodic motion. The frequency is the reciprocal of the period.
f) Amplitude of a vibration is the largest value of the displacement during the motion. This term is usually, but not necessarily, used in connection with periodic, or more particularly, sinusoidal, motion.
g) Number of degrees of freedom of a mechanical system is the number of independent coordinates required to completely describe the position of the parts of the system at all times.

The ground motions encountered in seismological observations are almost never periodic, although they are sometimes approximately so. Nevertheless, it is customary to speak of the period or frequency of ground motion. The implication is that the signal may be approximated by a short section of a periodic motion, or, from another viewpoint, if the spectrum of the signal is calculated, most of the energy will be found in the neighborhood of a predominant frequency.

The characteristics of ground motions that are encountered in the entire field of seismology are approximately as follows: frequencies from about 100 cps (or even higher) to about 0.00025 cps (1 cycle per hour); amplitudes of the order of a few millimicrons to a few millimeters (rarely a few centimeters in the case of the surface waves from the largest earthquakes); and direction of the ground motion is completely arbitrary. Thus we need instruments capable of covering approximately six decades in frequency, and with a dynamic range of some 120 db. No single instrument has been developed that can cover this entire range, and we resort to several instruments with more limited ranges to achieve the necessary flexibility.

A complete description of the motion of the earth's surface as a seismic wave impinges on it requires the specification of translations along three non-coplanar directions, and the specification of rotations about axes in these directions. As demonstrated by Wiechert in 1903, the rotations associated with a seismic wave are very small quantities compared to the translations (proportional to the difference of the space derivatives of the displacements in the directions normal to the displacements). Therefore, seismic instruments are designed to measure the displacements or translations in three orthogonal directions and the rotations are neglected. In recent years several attempts have been made to build rotation-recording instruments, but no success has been reported.

The measurement of motion is one of the oldest problems in experimental physics. Why, then, does the measurement of ground motion present a special problem? The reason is that motion must always be measured with respect to some frame of reference; motion only has meaning with respect to some reference. In ordinary laboratory measurements, the laboratory itself (floor, walls, table top, etc.) provides a "fixed" frame with respect to which motions may be observed. The problem in seismometry is that when a seismic wave arrives, the entire local environment, fixed as it is to the earth's surface, moves. Our laboratory building or observatory takes part in the motion we wish to observe. Therefore, the problem of seismometry comes down to the establishment of a reference that does not take part in the seismic motion. It is quite alright if this reference takes part in the general motions of the earth, e.g. rotation and revolution around the sun. It is only necessary that it remains at rest relative to the position of the earth's surface as it was before the seismic wave arrived.

The ideal seismometric instrument would be provided by an object that hovered over the earth's surface at a point fixed with respect to a coordinate system that rotated with the earth as a whole. From such a space platform, one could measure the motion of particles on the earth's surface as they vibrated under the influence of seismic energy. Such an ideal instrument, completely decoupled from the earth, is not practicable because of gravity. Therefore, we approach this ideal as closely as we can by supporting an object against the force of gravity, but coupling to the earth as loosely as possible. Because a compromise is necessary, in that the object to be used as a reference is coupled to the earth, the instrument will not be a perfect one. However, because the coupling is loose, there will be relative motion between the object and the earth. We can detect and record this relative motion, and knowing the characteristics of the coupling, we can recover the true motion of the earth more or less accurately. All seismographs with one important exception are based on the principle just outlined. We shall begin our study by considering the properties of the motion of a body that is loosely coupled to a supporting framework.

Several terms have been invented to describe seismic instruments. A <u>seismoscope</u> is a device that gives visual evidence that an earthquake has occurred. It does not provide a permanent record of ground motion as a function of time. A <u>seismometer</u> is a detector of seismic motion. It provides a signal containing quantitative information about the motion. A <u>seismograph</u> is a complete instrument for recording ground motion. It consists of a seismometer and recorder, including timer, with any associated filters and amplifiers.

2. Some general references

No adequate textbook on seismological instrumentation exists. Useful information is scattered through the literature, and summaries may be found in textbooks on seismology or exploration geophysics. For the most part, the seismologist must turn to the journals for material on his instruments.

The following list of references is not intended to be a complete bibliography on the subject, but rather a listing of key publications in which the most important ideas are presented. Some are primary sources, others are review papers. Additional references on specific topics will be given when those topics are discussed.

Basic vibration theory:

Thomson, W.T., <u>Vibration Theory and Applications</u>, 1965.
Myklestad, N.O., <u>Fundamentals of Vibration Analysis</u>, 1956.

Many other books on mechanical vibrations.
Summary articles on seismic instruments:
Benioff, H., "Earthquake Seismographs and Associated Instruments," in Advances in Geophysics, volume 2, 1955, pp. 219-275.
Coulomb, J., "Seismometrie," in Handbuch der Physik, volume 47 (Geophysics I), 1956, pp. 24-74 (In French).
Willmore, P.L., "The Detection of Earth Movements," in Methods and Techniques in Geophysics, volume 1, 1960, pp. 230-276.
Nomokonov, V.P. and D.K. Ganguli, Theory of Seismic Prospecting Instruments, Indian Institute of Technology, 1960.
General Instrument Theory:
Byerly, P., "Theory of the Hinged Seismometer with Support in General Motion," Bull. Seis. Soc. Amer., 42: 251-262, 1952.
Eaton, J., "Theory of the Electromagnetic Seismograph," Bull. Seis. Soc. Amer., 47: 37-76, 1957.
Sohon, F.W., Seismometry, Part II of Theoretical Seismology by Macelwane and Sohon, 1932.
Historical sketches and brief general treatments:
Macelwane, J.B., S.J., When the Earth Quakes, 1947, Chps. 9 & 10.
Richter, C.F., Elementary Seismology, 1958, Ch. 15.

3. Stability of equilibrium, restoring force, and linear systems

We are ultimately concerned with the motion of a mass that is loosely coupled to a supporting frame when the frame is moved. We assume that before the motion of the frame occurs the mass is in equilibrium, i.e. the sums of all the forces and all the moments acting on it are zero. Suppose that the frame remains at rest, and let us investigate the effect of displacing the mass by a small amount from its equilibrium position. We further assume that the mass is so constrained that it has only one degree of freedom.

When the mass is displaced, one of three things must happen: the mass will tend to move back to its original position, it will tend to continue moving in the direction of the displacement, or it still be in equilibrium after the displacement and will stay in the new position. In the first case, the original state of equilibrium is said to be stable, in the second case, unstable, and in the third case, neutral. Stable equilibrium implies that after the displacement the sum of the forces (sum of the moments) acting on the body is no longer zero, but an unbalanced force (moment) has been called into play that is directed toward the original position of the mass. This force (moment) is called a restoring force (restoring moment). If the equilibrium is unstable, the small displacement will result in an unbalanced force (moment) that is directed away from the original position. Such a force (moment) may be thought of as a negative restoring force, or an "unstabilizing" force. In neutral equilibrium, obviously, the sum of the forces and moments acting after the displacement is still zero. This does not imply that the forces acting on the body are all unchanged, but that if the displacement results in the action of a restoring force, it also calls forth an equal and opposite negative restoring force.

From another viewpoint, a state of stable equilibrium is a state of minimum potential energy of the system, so that any small displacement increases the potential energy. Unstable equilibrium is a state of maximum potential energy.

Concentrating on the case of stable equilibrium, we now examine the way in which the magnitude of the restoring force (moment) depends on the size of the displacement. The restoring force might be independent of the size of the displacement, it might be proportional to the size of the displacement (linear dependence) or it might vary in a non-linear manner with the displacement. The second case, that of a linear dependence of the restoring force on the displacement, is an important special case. A system consisting of an inertial member and an element that produces a restoring force with this characteristic is called a linear system (more properly, an undamped linear system as we have not yet taken into account mechanisms by which energy is taken out of the system).

In general, a mechanical system consists of one or more inertial elements, one or more elements that produce restoring forces (or negative restoring forces), and one or more elements that absorb energy from the system, called damping elements. We shall analyze the vibrations of such systems, limiting ourselves to the case of one degree of freedom. When the vibrations take place under the action of forces that come from within the system (restoring forces, damping forces), they are called free vibrations. When the vibrations are the result of forces acting from outside the system, they are called forced vibrations. Our goal is to investigate the forced vibrations of damped, linear, one degree-of-freedom systems that occur when the frame that supports the system is moving. The characteristics of these forced vibrations for any system depend on the properties of the free vibrations of the system. We shall therefore study free vibrations first.

4. The kinematics of undamped linear vibrations

Suppose a mass, M, is the inertial element in a stable linear system. Let x be the displacement of M in the direction of the one degree of freedom, with the positive sense chosen arbitrarily. Since the restoring force in a linear system is proportional to the displacement, we can write it as

$$R.F. = -kx,$$

where k is the proportionality factor (restoring force per unit displacement). The minus sign indicates that the force is a restoring force (force is directed oppositely to x). The factor k depends on the physical properties of the system, and must be determined in each new situation.

By Newton's second law of motion

$$M\ddot{x} = -kx,$$

where dots indicate differentiation with respect to time.
Dividing by M

$$\ddot{x} = -k/M\, x$$
$$= -\omega^2 x, \qquad (4\text{-}1)$$

where $\omega^2 = k/M$. The physical properties of the system are involved only in ω. The free motion of any stable one degree-of-freedom system in which the displacement is a translation is expressed by this differential equation.

It is easy to generalize this to motion in which the displacement is a rotation about some fixed axis. Let K be the moment of inertia about this axis and Θ the angular displacement. Then the condition for linearity

$$\text{restoring moment} = -\tau\Theta,$$

where τ is again a proportionality factor, the restoring moment per unit angular deflection. Then, the equation of motion is

$$K\ddot{\theta} = -\tau\theta$$

$$\ddot{\theta} = -(\tau/K)\theta$$

$$= -\omega^2\theta \qquad (4\text{-}2)$$

where $\omega^2 = \tau/K$. Again, the physical properties of the particular system are accounted for by ω. Equations (4-1) and (4-2) are mathematically identical. We need only find solutions for one and we have also solved the other. We choose to work with (4-1) for definiteness. In the following discussion, mass may be replaced by moment of inertia, k by τ, and translation by rotation.

The kinematics of the problem are expressed completely by (4-1). The general solution to this equation can be written as

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \qquad (4\text{-}3a)$$

$$= A\cos\omega t + B\sin\omega t \qquad (4\text{-}3b)$$

$$= C\sin(\omega t + \phi) \qquad (4\text{-}3c)$$

Exercise: solve (4-1) to obtain (4-3a). Express A, B in (4-3b) in terms of C_1, C_2. Express C, ϕ in (4-3c) in terms of A, B.

Exercise: Find A and B in (4-3b) if $x(0) = x_o$ and $\dot{x}(0) = v_o$. Use the Laplace transform technique to solve (4-1) with these same initial conditions.

Considering the solution in the form (4-3c), we realize that because the sine function is periodic, the motion is periodic. This is a very basic kind of periodic motion. A motion described by (4-1) or (4-3) is called a simple harmonic motion. The free vibrations of any stable undamped, linear, one degree-of-freedom system is simple harmonic motion.

We can discover the value of the period from the fact that the motion repeats each time the argument of the sine function increases by 2π, but not for a smaller increment. Thus

$$\sin(\omega t + \phi) = \sin(\omega t + \phi + 2\pi)$$

$$= \sin\left[\omega(t + 2\pi/\omega) + \phi\right]$$

so that the motion repeats after a time increment of $2\pi/\omega$. This is by definition the period

$$T = 2\pi/\omega \qquad (4\text{-}4a)$$

then, the frequency

$$f = \omega/2\pi \qquad (4\text{-}4b)$$

Going back to the original physical parameters of the system, for translational vibration

$$T = 2\pi\sqrt{M/K}; \quad f = 1/2\pi\sqrt{k/M}$$

and for rotational vibration

$$T = 2\pi\sqrt{K/\tau}; \quad f = 1/2\pi\sqrt{\tau/K}$$

The quantity ω, which contains all the information about the physical properties of the system needed to determine the frequency of free oscillations, is called the angular frequency of the system. ω is expressed in radians per second, f in cycles per second, and T in seconds per cycle.

The amplitude of the motion in (4-3c) is given by C, and ϕ is the phase angle. The motion passes through a given phase point (zero crossing, maximum, minimum) at a time ϕ/ω before the sine function $\sin \omega t$ goes through the same point. x leads the function $\sin \omega t$ by the angle ϕ, or the time ϕ/ω. If only one motion is involved in a situation, ϕ can be made zero without loss of generality. In many problems, including seismometry, two or more motions must be compared. Any one of these can be considered to start at t=0, and the phases of the others compared to it. If $\phi > 0$, x(t) leads $\sin \omega t$, if $\phi < 0$, x(t) lags $\sin \omega t$.

Exercise: Find the velocity and acceleration in the simple harmonic motion given by (4-3c). Show that the velocity leads the displacement by $\pi/2$ and the acceleration is out of phase (leads or lags) by π. Show this by plotting $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ on a single graph, with t as the abscissae.

In studying the dynamics of undamped linear vibration, it is necessary only to write the equation of motion in the form (4-1), using the laws of motion. We then can immediately write down the frequency of free oscillations of the system, knowing that the motion is given by (4-3).

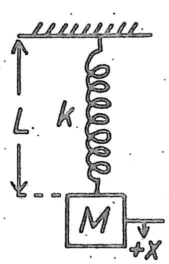
5. The natural frequencies of some systems useful in seismometry

We can now be more specific about the loosely coupled mass which is to form the basis of our ground motion detector. We require that it be the inertial member of a stable, linear, one degree-of-freedom system. We shall see that the response of such a system to ground motions depends on the frequency of its free vibrations. We shall now find this frequency for a variety of systems that have proven useful in seismometry. In these systems the restoring force or moment is provided either by an elastic element, e.g. a spring, or by gravity.

5.1. Systems with elastic restoring elements

5.11. Simple mass-spring system

The prototype of all systems to be considered can be taken as the simple mass-spring combination shown in the figure. A mass, M, is supported by a close-coil spring. Amplitudes will be assumed small enough that the strains in the spring are within the elastic limit of the material. Then the spring will exert a force proportional to its elongation,



i.e. $F = -k(L - L_0)$
$= -k\Delta$,

where L is the length of the spring with mass attached, in the equilibrium position, L_0 is the initial length of the spring before the load was applied (assuming the spring is not pre-stressed so that the coils close on themselves), and $\Delta = L - L_0$, is the elongation. The proportionality factor k is called the <u>spring constant</u> or <u>spring stiffness</u>. It is numerically equal to the force required to produce a unit elongation (dynes per cm., pounds per inch, newtons per meter).

For a helical spring wound of wire with a circular cross-section,

$$k = \mu d^4/8nD^3$$

where μ = rigidity modulus of the material
d = diameter of the wire
D = average diameter of the helix
n = number of turns

Assuming the number of turns per unit length is constant, the stiffness of the spring is inversely proportional to its length.

If the mass of the spring is negligible compared to the mass M, the condition for equilibrium is

$$Mg - k\Delta = 0 \qquad (5.1\text{-}1)$$

Imagine the mass is constrained so that it can move only vertically, choose positive displacement downward, and give the mass a displacement x. After the displacement, the sum of the forces acting on the spring is

$$Mg - k(\Delta + x),$$

so that equation of motion is

$$M\ddot{x} = Mg - k(\Delta + x)$$

Using the equilibrium condition (5.1-1)

$$M\ddot{x} = - kx,$$

or

$$\ddot{x} = - k/M\ x \qquad (5.1\text{-}2)$$

Equation (5.1-2) is identical to (4-1), with $\omega^2 = k/M$, and k now having a definite physical significance. We immediately know the character of the free oscillations. The motion is simple harmonic, with the period or frequency given by (4-4).

Because we shall deal with several different kinds of periods or frequencies, we shall put the subscript "n" on the ones related to undamped, free vibrations, and call these the undamped natural period or natural frequency. For a mass supported by a massless spring, the undamped natural angular frequency is

$$\omega_n = \sqrt{k/M}, \qquad (5.1\text{-}3a)$$

the undamped natural frequency is

$$f_n = 1/2\pi\sqrt{k/M} \qquad (5.1\text{-}3b)$$

and the undamped natural period is

$$T_n = 2\pi\sqrt{M/k} \qquad (5.1\text{-}3c)$$

We note from the equilibrium condition (5.1-1) that $k/M = g/\Delta$, so that we can write

$$T_n = 2\pi\sqrt{\Delta/g} \qquad (5.1\text{-}4)$$

and similar expressions for ω_n and f_n.

We also note that gravity does not appear in the equation of motion (5.1-2). Generally, if the effect of gravity is not changed by the deflection, it will drop out when the equilibrium condition is introduced. Here we get the same equation if we neglect gravity and the elongation Δ and call the spring force $-kx$.

Exactly the same result would have been obtained if instead of using Newton's second law of motion, we had used the conservation of energy. By writing the equation stating that the time rate of change of the sum of the kinetic and potential energy is zero in this undamped system, we would obtain (5.1-2). See Thomson, pp. 9-13.

We can easily take the effect of the mass of the spring into account. By assuming the displacement of any point on the spring is proportional to its distance from the top, and using the energy method (Thomson, p. 14), we can find the frequency to be

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{M + 1/3\, m_s}} ,$$

where m_s is the mass of the spring.

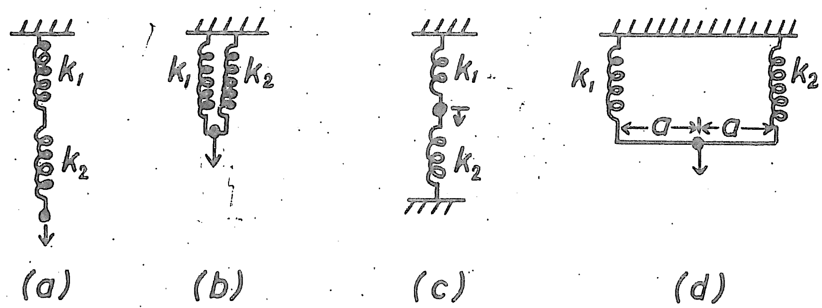
The restoring element may consist of several springs in combination. In this case an effective spring constant may be found. The effective spring constant is the constant of a single spring that would give the same deflections for a unit force acting at the position where the mass is attached as the actual combination of springs.

Exercise: Show that the effective constant of two springs in series (Figure a) is given by $1/k_{eff} = 1/k_1 + 1/k_2$.

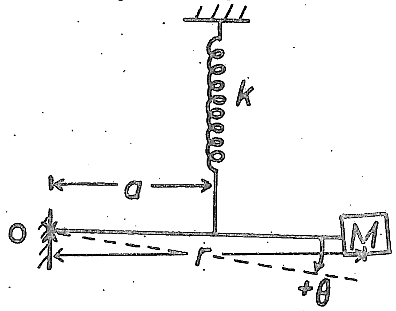
Show that the effective constant of two springs in a close parallel combination (Figure b) is $k_{eff} = k_1 + k_2$.

Show that the effective constant of two springs on a tandem arrangement, Figure c, is the same as the close parallel combination.

Show that the effective constant of the two springs in Figure d is $k_{eff} = (4k_1k_2)/(k_1 + k_2)$.



5.12. Spring-supported hinged boom.



A massless boom of length r, with a point mass M at one end is supported in a horizontal position by a spring with spring constant k. The pivot at O is perfectly flexible.

With the frame at rest, we can take moments about O. In equilibrium, $Mgr - k\Delta a = 0$.

When displaced through a small angle Θ (positive downward), the sum of the moments is, to first order terms in Θ (an exact analysis will be given later)

$$Mgr - k(\Delta + a\Theta)a = - ka^2\Theta, \text{ using the equilibrium condition.}$$

So the equation of motion is

$$Mr^2\ddot{\Theta} = - ka^2\Theta.$$

This is of the form leading to equation (4-2), with $K = Mr^2$ and $\tau = ka^2$. By inspection,

$$\omega_n^2 = ka^2/Mr^2$$

$$f_n = 1/2\pi \sqrt{ka^2/Mr^2}$$

$$T_n = 2\pi \sqrt{Mr^2/ka^2} \qquad (5.1\text{-}5)$$

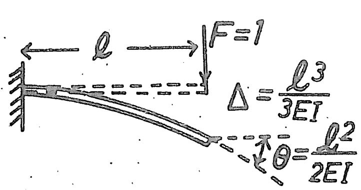
If instead of the idealized system, we have a boom with total mass M and moment of inertia K about a horizontal axis through O, with the center of mass a distance r from O, the equation of motion is

$$K\ddot{\Theta} = - ka^2\Theta$$

and

$$\omega_n^2 = ka^2/K$$

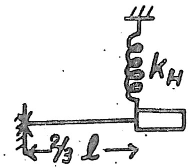
If the pivot at O is not perfectly flexible, but consists, for example, of a thin metal strip, an additional restoring torque is exerted by this hinge. The spring constant of such a strip, clamped at one end, is $3EI/\ell^3 = k_H$ where EI is the flexural stiffness, and ℓ is the length. The flexure of the hinge is such that the tangent to the hinge at the point to which the boom is fastened makes an angle $\ell^2/2EI$ under unit force. If we think of the hinge as a spring of stiffness k_H acting at the end of a pivoted rod, the length of the rod to give the same angular deflection at the free end is



Flexural hinge

$$\Delta/\Theta = (\ell^3/3EI)\cdot(3EI/\ell^2) = (2/3)\ell.$$

We can, therefore, replace the hinge by an extension of the rigid boom of $(2/3)\ell$ (a small length compared to the length of the boom), with a spring having a constant $3EI/\ell^3$ acting, and a flexible pivot. The torsion constant of this equivalent spring (analogous to ka^2 for the main spring) is



Equivalent Spring

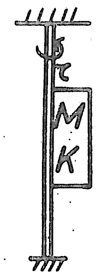
$$k_H(2/3\ell)^2 = (3EI/\ell^3)\cdot(4/3\ \ell^2) = 4EI/3\ell$$

The new value of $\omega_n^{\ 2}$ is

$$\frac{ka^2 + \frac{4EI}{3\ell}}{K}.$$

The effective pivot point is one-third the length of the hinge from the point where it enters the frame.

5.13. Torsion pendulum, vertical fiber



A mass M is fastened to a taut, vertical metal fiber. The moment of inertia of the mass about the axis of the fiber is K. Let the torsion constant of the fiber, the moment per unit angular deflection, be τ.
The equation of motion is $K\ddot{\theta} = -\tau\theta$

$$\omega_n^{\ 2} = \tau/K$$

$$f_n = 1/2\pi\sqrt{\tau/K} \quad (5.1\text{-}6)$$

The case in which the fiber is not vertical will be discussed later. The torsion constant of a fiber is given by

$$\tau = \frac{\mu I_p}{\ell}$$

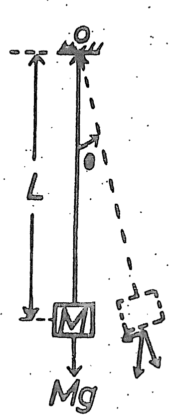
where μ is the rigidity modulus, I_p is the polar moment of inertia of the cross-section of the fiber, and ℓ is the length from the point of attachment to the frame to the point of attachment to the rotating mass. For a circular fiber this become

$\mu\pi d^4/32$, where d is the diameter.

In the above pendulum, we equivalently have two fibers, one from above and one from below, acting in parallel, so the total torsion constant is the sum of the two.

5.2. Systems with gravity restoring force

5.21. The simple pendulum



The simple pendulum is defined as a point mass M suspended from a frictionless pivot by a massless, inflexible rod of length L, constrained to swing in a plane.

When the pendulum is displaced, the force of gravity exerts a moment about O equal to $-(Mg\sin\theta)L$.

The equation of motion is

$$ML^2 \ddot{\Theta} = - MgL \sin\Theta.$$

This is, in general, not a linear system. However for values of Θ small enough that Θ^3 is negligible compared to Θ, $\sin \Theta \simeq \Theta$, and

$$ML^2 \ddot{\Theta} = - MgL \Theta.$$

To this approximation, a simple pendulum is a linear system. Then

$$\omega_n^2 = g/L$$

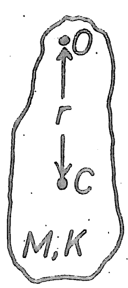
$$f_n = (1/2\pi)\sqrt{g/L} \quad , \quad T_n = 2\pi\sqrt{L/g} \qquad (5.2\text{-}1)$$

Comparison with (5.1-4) shows that the frequency of a mass-spring combination for which the initial elongation is Δ is the same as the frequency of a simple pendulum for which $L = \Delta = g/4\pi f_n^2 = gT_n^2/4\pi^2$

For any system of the kind under discussion, with period T_n, there is a simple pendulum of length $L = gT_n^2/4\pi^2$ that has the same period. This pendulum is called the equivalent simple pendulum, and L is the equivalent pendulum length.

5.22. The physical or compound pendulum

We shall define a physical pendulum to be any distributed mass that is suspended from a frictionless, perfectly flexible support, so that it is free to rotate about a horizontal axis. The mass is M, and the moment of inertia about O is K. The center of mass C is a distance r from O.



As for the simple pendulum, when the physical pendulum is displaced through a small angle Θ, a restoring moment, about the suspension point O, equal to $- Mgr\Theta$ acts. The equation of motion is

$$K \ddot{\Theta} = - Mgr\Theta$$

$$\omega_n^2 = Mgr/K.$$

$$T_n = 2\pi \sqrt{K/Mgr} \qquad (5.2\text{-}2)$$

The equivalent pendulum length corresponding to this period is K/Mr. It will be seen that this length has considerable significance in seismometer theory. It is the distance from the point of suspension O to another point on the line through the center of mass known as the center of oscillation or the center of percussion. One of the properties of the center of oscillation is that if the pendulum is suspended from an axis through this point, parallel to the original axis, the period is the same as the original period.

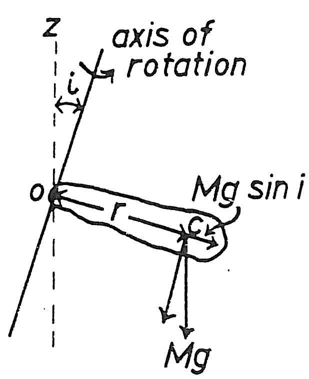
Exercise: Prove that the period of a compound pendulum is unchanged if the pendulum is suspended from the center of oscillation. (Hint: Use the parallel axis theorem for moments of inertia.)

The distance from the point of suspension to the center of oscillation is a property of the physical dimensions of the pendulum and the choice of O only. This distance, K/Mr, does not change even if the pendulum bar is suspended in a different manner, such as from an elastic hinge, or about an axis that is not horizontal. These changes will change the natural period of the system, and therefore the equivalent pendulum length, but not this length. For this reason we give this invariant length a special name, the reduced pendulum length, ℓ. The reduced pendulum length is the distance from the point of suspension to the center of oscillation. It is equal to the equivalent pendulum length when the bar oscillates about a horizontal axis through the same suspension point under the influence of gravity only. This suggests a method for determining ℓ for any pivoted distributed mass.

In seismology it is necessary to develop systems with long natural periods. Periods of 15 seconds to 30 seconds are common. The length of a simple pendulum with T_n = 30 sec is 225 meters. Clearly a better way is needed to achieve a long period.

5.23. The horizontal pendulum

One solution of the problem of achieving a long natural period with reasonable dimensions is to use the idea of the physical pendulum, but to arrange the oscillating boom so that only a small fraction of gravity produces the restoring moment. This can be done by mounting the pendulum so that the axis of rotation is almost vertical and the plane of oscillation is almost horizontal. Such a system is called a horizontal pendulum.



Let the angle between the axis of rotation and the vertical, called the angle of inclination, be i. Resolve the weight Mg, acting at the center of mass into components parallel to the axis of rotation and perpendicular to this direction. The system is in equilibrium when the component Mg sin i is directed through O, so it has no moment. This occurs when the center of mass is in the plane determined by the vertical and the axis of rotation, called the neutral plane. The moment due to Mg cos i is taken up by the supporting structures.

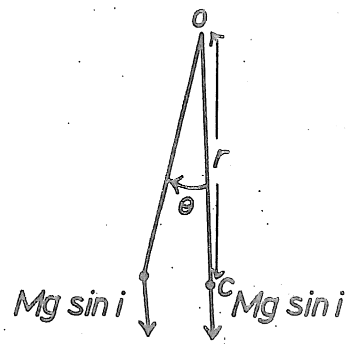
The system is now displaced through an angle θ about the axis of rotation. The weight is once again resolved into orthogonal components, one parallel to the axis of rotation. Since the two components of the vector Mg must be coplanar with it, the component Mg sin i is still parallel to the plane determined by the axis of rotation and the vertical, as in the figure.

(View along the axis of rotation) The restoring moment for small displacements is $-(Mg \cdot \sin i)\, r\, \theta$, so that the equation of motion is

$$K \ddot{\theta} = -(Mgr \cdot \sin i)\, \theta$$

$$\omega_n^2 = (Mgr \cdot \sin i)/K$$

$$T_n = 2\pi \sqrt{K/(Mgr \cdot \sin i)} \qquad (5.2\text{-}3)$$



By making i small, ω_n can be made small, or T_n large, without changing K, M, or r. When $i = 90^o$, the result for the physical pendulum, previously derived, follows. The effect of this arrangement is to replace g by g sin i.

If $i = 0^o$, ω_n becomes zero, and this period becomes infinite. This corresponds to neutral equilibrium. The displacement does not result in a restoring moment.

If i is negative, the period becomes imaginary. Physically this corresponds to unstable equilibrium, and the pendulum will flop over if given any small displacement.

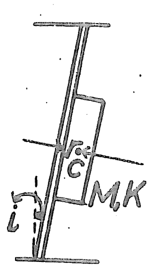
The equivalent pendulum length is $K/(Mr \sin i) = \ell/\sin i$. Thus, by making i small, we have the equivalent of a very long pendulum. In seismometry, the inclination is usually very small, so that to a good approximation sin i = i, then

$$T_n = 2\pi \sqrt{K/Mgri} .$$

5.3. Systems with combined elastic and gravity restoring force

In any pendulum, an elastic restoring element can be added by using metal strips for hinge material or adding auxiliary springs. The restoring moments due to the elastic elements and gravity are simply added.

5.31. Torsion pendulum, fiber not vertical



If the torsion fiber in 5.13 is not vertical, there will be a gravity restoring force. Using results from 5.13 and 5.23, the equation of motion is

$$K \ddot{\theta} = - (Mgr \cdot \sin i + \tau)\, \theta$$

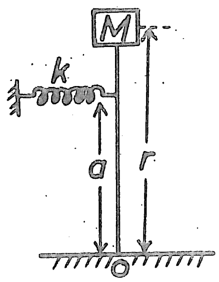
$$\omega_n^2 = (Mgr \cdot \sin i + \tau)/K$$

$$T_n = 2\pi \sqrt{K/(Mgr \cdot \sin i + \tau)}$$

The period is shortened if i is positive (inclined toward the mass). In this case i can be made negative, so that gravity acts as a negative restoring force. The condition for stability (real period) is that $(Mgr \sin i + \tau) > 0$.

An extreme case is that for which $i = -90^o$, so that the center of mass is vertically above the point of suspension. This leads to the inverted pendulum.

5.32. Inverted pendulum



The elastic restoring element can be schematically represented by a spring acting at a distance a from the frictionless, perfectly flexible pivot. In actual instruments, there is usually an elastic member at O, the action of which is the same as discussed in 5 12. The equation of motion is (small Θ)

$$K \ddot{\Theta} = - ka^2\Theta + Mgr\Theta \quad (\sin i = - 1)$$

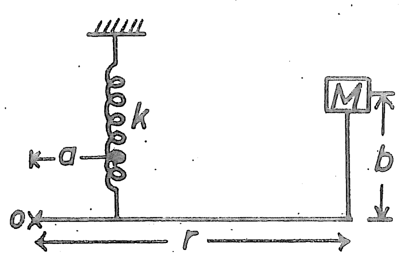
$$= - (ka^2 - Mgr)\ \Theta$$

$$\omega_n^2 = (ka^2 - Mgr)/K$$

$$T_n = 2\pi\sqrt{K/(ka^2 - Mgr)}$$

By playing ka^2 against Mr, a long period can be achieved with small dimensions.

Exercise: What is the stability condition for the inverted pendulum in the example above?



Exercise: Write the equation of motion for small oscillations and find the natural period and stability condition for the system in the figure.

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6. Free oscillations with damping

In any real mechanical system, energy is lost from the system because of various dissipative actions. Whereas the free oscillations discussed in Section 5 would ideally go on forever, in fact they stop after a while. Further, we shall see that it is desirable to introduce energy dissipation into seismometers in order to obtain the most favorable response to ground motion.

Any element that absorbs the energy of free vibrations of a system is called a damping element. Before proceeding to investigate the effects of damping, we shall calculate the work done in one cycle by a force that varies harmonically with the same frequency as the motion, but is out of phase with it. (Refer: Thomson, p. 68 ff).

Let the displacement be $x = x_o \sin\omega t$

and the force be $F = F_o \sin(\omega t + \phi)$

The work done in one cycle is $W = \int F dx$, integrated over one cycle

$$= \int_0^T F(dx/dt)dt$$

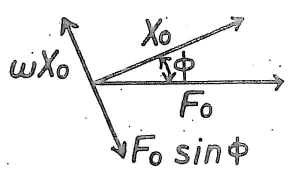
$$= \int_0^{2\pi/\omega} F_o \sin(\omega t + \phi)\omega x_o(\cos\omega t)dt$$

$$= x_o F_o \int_0^{2\pi/\omega} [\cos\omega t \sin\omega t \cos\phi + \cos^2\omega t \sin\phi]\omega dt$$

$$= X_o F_o \left\{ \cos\phi \left[\sin^2\omega t\right]_0^{2\pi/\omega} + \sin\phi\left[\omega t/2 + (1/2)\sin\omega t\cos\omega t\right]_0^{2\pi/\omega} \right\}$$

$$= \pi F_o x_o \sin\phi$$

Thus the work done is positive (energy added to the system) if F leads x, and is negative (energy dissipated) if F lags x. We shall return to the former case when we study forced oscillations. The component of F that absorbs energy is $F_o \sin\phi$, where ϕ is now specifically a lag. This is the component that lags x by $\pi/2$. Because the velocity leads the displacement by $\pi/2$, the dissipative component of F_o is the component that is directly opposed to the velocity, out of phase by π



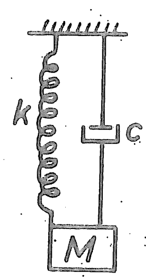
A force in phase with x_o, or out of phase by π, like the spring force, does not change the total energy in the system after one cycle.

6.1. Viscous damping

The damping force must oppose the velocity, but the form of that force has not been otherwise specified. One important special form is that in which the damping force is directly proportional to the velocity:

$$F_d = - c\dot{x} .$$

This kind of damping is called viscous damping, and c is the coefficient of viscous damping. The name derives from the fact that a viscous fluid resists the motion of a body through it according to this law when the speed is low. Several mechanisms for achieving viscous damping in seismographs will be discussed later.



We take as the prototype of a viscously damped system the same mass-spring combination examined originally, with a damping element, symbolized by a dash-pot, added. The force per unit velocity excited by the damping element is c(dynes/cm/sec., etc.). The equation of motion for free oscillations becomes

$$M\ddot{x} = - kx - c\dot{x} \qquad (6\text{-}1)$$

The solution of this equation is:

$$x(t) = e^{-\frac{c}{2M}t}\left[Ae^{\sqrt{\left(\frac{c}{2M}\right)^2 - \frac{k}{M}}\,t} + Be^{-\sqrt{\left(\frac{c}{2M}\right)^2 - \frac{k}{M}}\,t}\right] \qquad (6\text{-}2)$$

The nature of the free oscillations depends on whether the radical in the exponents is real, zero, or imaginary. This, in turn depends on the value of c relative to k and M. The value of c for which the radical is zero is

$$c_c = 2\sqrt{kM}$$

This is called the critical damping coefficient. If the actual value of c is less than this, the system is underdamped, if it is equal to this, the system is critically damped, and if it is greater than this, the system is overdamped. The numerical value of c tells us less about the system than the ratio of c to c_c,

$$c/c_c = \hbar \text{ , the damping factor.}$$

In terms of the damping factor, the system is underdamped if $\hbar < 1$, critically damped if $\hbar = 1$, and overdamped if $\hbar > 1$. In terms of the parameters of the system, $\hbar = c/(2\sqrt{kM}) = c/(2M\sqrt{k/M}) = c/(2M\omega_n)$ where ω_n is the undamped natural angular frequency We can rewrite the original equation of motion as

$$\ddot{x} + (c/M)\dot{x} + (k/M)x = 0$$

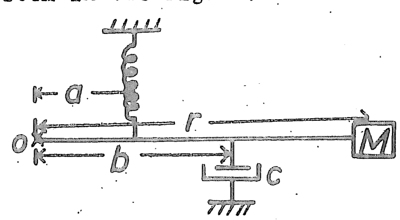
$$\ddot{x} + 2\hbar\omega_n\dot{x} + \omega_n^2 x = 0 \qquad (6\text{-}3)$$

and the solution as

$$x(t) = e^{-\hbar\omega_n t}\left[Ae^{\sqrt{\hbar^2 - 1}\,\omega_n t} + Be^{-\sqrt{\hbar^2 - 1}\,\omega_n t}\right] \qquad (6\text{-}4)$$

This form of the equation and solution has the advantage that it enables us to abstract from the particular mass-spring-dashpot system to any viscously damped, linear, one degree-of-freedom system. For each system ω_n is determined as before, and $\hbar$ is the ratio of the actual damping coefficient to the critical damping coefficient. However, the critical damping coefficient will not have the form found here, but must be determined for each system. For rotational motion, we must replace x by Θ, k by the equivalent τ, M by K, and c becomes the damping moment per unit angular velocity.

Exercise: Write the equation of motion and find the critical damping coefficient for the system in the figure. The boom is massless.



Exercise: Use the Laplace transform technique to solve (6-3) with initial conditions $x(0) = x_o$, $\dot{x}(0) = v_o$.

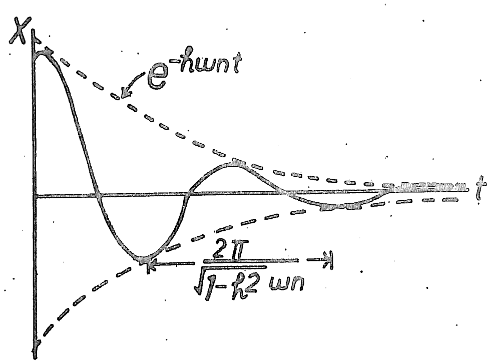
6.11. Underdamped system, $\zeta < 1$

For an underdamped system, the solution (6-4) becomes

$$x(t) = e^{-\zeta\omega_n t}\left[Ae^{i\sqrt{1-\zeta^2}\,\omega_n t} + Be^{-i\sqrt{1-\zeta^2}\,\omega_n t}\right]$$

$$= e^{-\zeta\omega_n t}\, x_m \sin(\sqrt{1-\zeta^2}\,\omega_n t + \phi). \qquad (6\text{-}4a)$$

This is the product of a sinusoidal motion with amplitude x_m and angular frequency $\sqrt{1-\zeta^2}\,\omega_n$ and an exponentially decaying function. The motion is as shown in the figure.



For $\phi = 0$, the motion is zero at $t = n\pi/(\sqrt{1-\zeta^2}\,\omega_n)$, $n = 0, 1, 2, \ldots$, it is equal to $\pm e^{-\zeta\omega_n t} x_m$ (tangent to the decay curve) at $t = \frac{(2n+1)\pi}{2\sqrt{1-\zeta^2}\,\omega_n}$ and has its extrema at

$$t_m = \frac{1}{\sqrt{1-\zeta^2}\,\omega_n} \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Exercise: Prove all the above statements as to the properties of the motion.

Because the zero crossings and extrema occur at equally spaced intervals, the motion is pseudo-periodic. It is not periodic because it never repeats, but phase points do repeat periodically. We extend the notion of period and frequency to include this case. The damped natural angular frequency is $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

The damped natural frequency is $f_d = (1/2\pi)\sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-\zeta^2}\, f_n$

and the damped natural period is $T_d = 2\pi/\omega_d = T_n/\sqrt{1-\zeta^2}$ (6-5)

The damped period of underdamped motion is longer than the undamped period of the same system.

The amount of viscous damping in an underdamped system can be determined from observations of the decay of free vibrations. In principle, the period could be used, but it is not sufficiently sensitive to small changes in ζ to give accurate results. Because the decay is exponential, the ratio of successive maxima (or maximum to following minimum) is constant. Substituting the expression for t_m in (6-4a) and using the periodicity of the tangent function, we find for the ratio of successive maxima:

$$\frac{x_m}{x_{m+1}} = \exp(2\pi h/\sqrt{1 - h^2})$$

This ratio is called the damping ratio. The natural logarithmic of the damping ratio is called the logarithmic decrement.

$$\delta = \ln(x_m/x_{m+1}) = 2\pi h/\sqrt{1 - h^2} \qquad (6\text{-}6)$$

Thus, h can be determined from observation of the damping ratio by

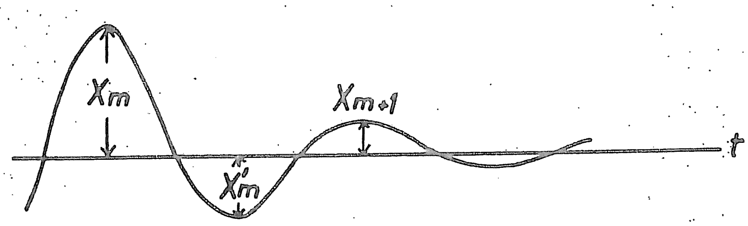
$$h = \frac{\delta/2\pi}{1 + (\delta/2\pi)^2} \qquad (6\text{-}7)$$

This definition of the damping ratio follows the usage in engineering mechanics, where h is usually small, and $x_m:x_{m+1}$ is only slightly greater than 1. In seismometry, on the other hand, h is usually 0.5 or greater, so that the ratio of successive turning points on the same side of zero is greater than about 38:1. This means the second maximum is usually too small to read accurately. It has become customary, therefore, in seismometry, to use the ratio of successive turning points on opposite side of the zero position. This ratio is, of course, the square root of the ratio defined above.

$$\frac{x_m}{x'_m} = \exp(\pi h/\sqrt{1 - h^2})$$

$$\delta' = \ln(x_m/x'_m) = \pi h/\sqrt{1 - h^2} = \delta/2$$

and $$h = \frac{\delta'/\pi}{\sqrt{1 + (\delta'/\pi)^2}} \qquad (6\text{-}7a)$$



The results for underdamped free oscillations are summarized (different notation) in Figure 5, p. 70, of Eaton.

Exercise: Show that $\delta = \frac{1}{n}\ln\frac{x_0}{x_n}$, where x_n is the amplitude after n completes oscillations, starting with amplitude x_0. This is a useful result for treating light damping.

6.12. Critically damped system, $h = 1$

For critical damping, the general solution, corresponding to (6-4a) is

$$x(t) = e^{-\omega_n t}\left[C_1 + C_2 t\right] \qquad (6\text{-}8)$$

This motion is not oscillatory. Critical damping is the smallest amount of damping the results in non-oscillatory free-vibration.

Exercise:

1. Use the Laplace transform technique to solve (6-3) for $h = 1$, and $x(0) = x_o$, $\dot{x}(0) = v_o$.

2. Show that for $x(0) = 0$, $\dot{x}(0) = v_o$, $x(t) = v_o t e^{-\omega_n t}$. Show that this motion has a single maximum at $t_m = \frac{1}{\omega_n}$, and $x_m = V_o/e\,\omega_n$

6.13. Overdamped system, $h > 1$

In this case, the solution (6-4) is used as it stands. Substituting hyperbolic functions for the exponential functions.

$$x(t) = e^{-h\omega_n t}\left[C_1 \cosh \sqrt{h^2 - 1}\,\omega_n t + C_2 \sinh \sqrt{h^2 - 1}\,\omega_n t\right] \qquad (6\text{-}9)$$

Neither cosh nor sinh is oscillatory, so $x(t)$ is not oscillatory. We further note that $\sqrt{h^2 - 1} < h$, so both exponential terms in (6-4) decay with time. Overdamped free vibration has a single maximum and decays toward zero monotonically after reaching the maximum.

Overdamping is often used in modern seismographs. The modern approach to calibration, however, is to calibrate the system as a whole and not measure the intrinsic constants, such as the damping factor, separately. If it is desired to measure h directly for an overdamped system, a method suggested by Eaton is convenient. The system is set in motion by giving an impulse, and the time required for the displacement to return to 1/10 the maximum value is determined.

From (6-9), assuming $x(0) = 0$, $\dot{x}(0) \neq 0$,

$$\left(\frac{x_{max}}{\frac{1}{10}x_{max}}\right) = 10 = \frac{e^{-h\omega_n t_m}\,\sinh \sqrt{h^2 - 1}\,\omega_n t_m}{e^{-h\omega_n (t_m + \bar{t})}\,\sinh \sqrt{h^2 - 1}\,\omega_n (t_m + \bar{t})},$$

where $\bar{t}$ is the time after t_m for the displacement to come down to $\frac{1}{10}x_{max}$.

The time t_m is $\dfrac{1}{\sqrt{h^2 - 1}\,\omega_n}\tanh^{-1}\dfrac{\sqrt{h^2 - 1}}{h}$

Eaton introduces this and solves the resulting equation for t by successive approximations. The result for $1 < h < 3$ is presented in his Figure 6, p. 71.

6.2. Coulomb damping (dry friction)

Any mechanical system in which parts rub against each other will be damped by the action of friction or Coulomb damping. This kind of damping is considered undesirable in seismograph systems and is always minimized by careful design and construction. It is important in some old instruments that use pivots and in any instrument that records with a stylus, such as smoked paper or pen-and-ink recorders.

The force of friction depends on the roughness of the surfaces that are sliding over one another and the normal force that is pressing them together. The force is always opposed to the velocity. In an oscillating system, the force of friction is opposed to the restoring force during the two quarter-cycles during which the velocity is increasing, and acts in the same direction as the restoring force during the two quarter-cycles in which the velocity is decreasing. This action can be expressed by two equations, using a mass-spring system to represent any system, F is the force of friction:

$$M\ddot{x} = -kx + F, \text{ x decreasing (going from maximum to minimum)}$$

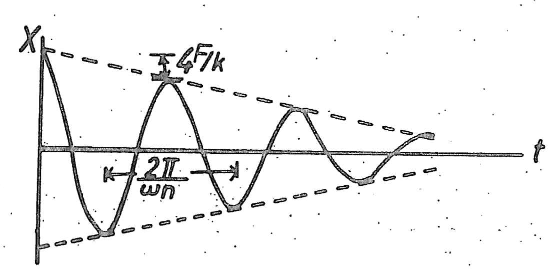
$$M\ddot{x} = -kx - F, \text{ x increasing (going from minimum to maximum).}$$

This is non-linear vibration, since F is independent of x.

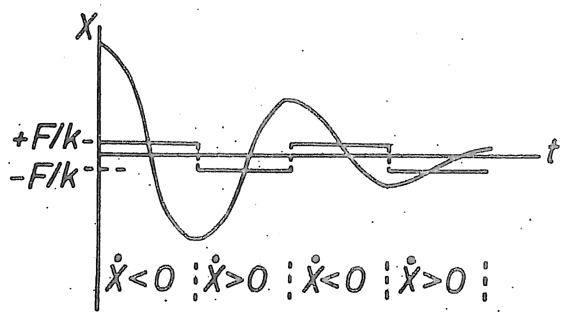
With initial conditions $x(0) = x_o$, $\dot{x}(0) = 0$, the motion during the first half cycle is

$$x(t) = (x_o - \frac{F}{k}) \cos \omega_n t + \frac{F}{k}$$

After one half cycle $x(\frac{\pi}{\omega_n}) = -x_o + 2\frac{F}{k}$, so that the amplitude has decreased by 2F/k in one-half cycle. The amplitude after a complete cycle, using the second of the two equations, is x_o - 4F/k. Therefore, the decrease in amplitude during each cycle is 4F/k, a linear decrease. The period (again, in the extended sense) is unchanged by Coulomb damping. The amount of friction can be obtained experimentally from the observed decay of the motion.



The motion does not go on forever. When the amplitude has become small enough that kx < F at a turning point, the system will stick. Note that the effect of friction is the same as the effect of shifting the zero position in the direction opposed to the velocity by the amount F/k during



each half cycle and taking simple harmonic motion with respect to this shifted zero. This is apparent if we write the equation of motion

$$M\ddot{x} = - k(x \pm F/k).$$

Having determined F/k from the decay curve, we can write the acceleration due to friction from

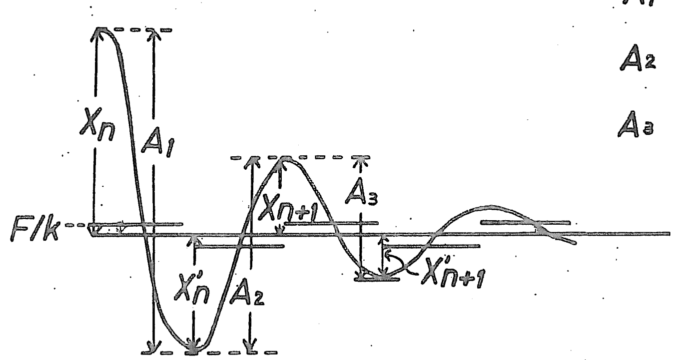
$$\frac{F}{M} = \frac{F}{k}\frac{k}{M} = \frac{F}{k}\omega_n^2 = \frac{4\pi^2}{T_n^2}\frac{F}{k}.$$

6.3. Determining the amount of viscous damping and friction when both are present

If the viscous damping can be removed, the amount of friction present can be determined separately. If this is not practical, the two can be separated by considering the effect of friction to be a zero shift opposed to the velocity.

Assume the viscous damping is light. Let x_n, x_{n+1} be successive maxima, and x_n', x_{n+1}' be the corresponding following maxima.

Then the successive peak-to-peak amplitudes are



$$A_1 = X_n + X_n'$$

$$A_2 = X_n' + X_{n+1}$$

$$A_3 = X_{n+1} + X_{n+1}'$$

Now, taking free, viscously damped motion about the shifted zeroes

$$\frac{x_n - F/k}{x_n' + F/k} = \epsilon^{\frac{1}{2}} \quad ; \quad \frac{x_n' - F/k}{x_{n+1} + F/k} = \epsilon^{\frac{1}{2}} \quad ; \text{ etc}$$

where ϵ is the damping ratio associated with the viscous damping. From the expressions for $\epsilon^{\frac{1}{2}}$,

$$x_n - F/k = \epsilon^{\frac{1}{2}}(x_n' + F/k) \qquad (1)$$

$$x'_n - F/k = \epsilon^{\frac{1}{2}}(x_{n+1} + F/k) \qquad (2)$$

$$x_{n+1} - F/k = \epsilon^{\frac{1}{2}}(x'_{n+1} + F/k) \qquad (3)$$

Adding (1) and (2) $\quad x_n + x'_n - 2F/k = \epsilon^{\frac{1}{2}}(x_{n+1} + x'_n + 2F/k)$

or $\quad A_1 - 2F/k = \epsilon^{\frac{1}{2}}A_3 + 2(F/k)\epsilon^{\frac{1}{2}}$

From (2) and (3) $\quad A_2 - 2F/k = \epsilon^{\frac{1}{2}}A_3 + 2(F/k)\epsilon^{\frac{1}{2}}$

$$A_1 - A_2 = \epsilon^{\frac{1}{2}}(A_2 - A_3)$$

$$\epsilon = \left(\frac{A_1 - A_2}{A_2 - A_3}\right)^2$$

Since $\delta = \ln \epsilon$, h can be determined as before.
Knowing ϵ , F/k can be calculated from $\quad \frac{F}{k} = \frac{A_1 - \epsilon^{\frac{1}{2}}A_2}{2(1 + \epsilon^{\frac{1}{2}})}$

If F is negligibly small, $A_1 = \epsilon^{\frac{1}{2}}A_2$ or $A_1/A_2 = \epsilon^{\frac{1}{2}}$

Thus the damping ratio can be calculated using peak-to-peak amplitudes. This offers the advantage that the zero position need not be known.

6.4. Solid damping

Even in modern instruments in which friction is minimized, there are still losses due to hysteresis in the elastic elements, such as coil springs, torsion fibers, and flat springs used as hinges. The energy absorbed by this mechanism is proportional to the maximum stress in the element during each cycle. Because this effect is small compared to the loss due to viscous damping, it is usually not separately analyzed, but is lumped together with the viscous damping. In other words, the energy accounted for by this solid damping in one cycle can also be accounted for by some amount of viscous damping, and is so treated.

7. Forced motion of a viscously damped, linear, one degree-of-freedom system

7.1. Solution of the equation for a harmonic driving force acting on the mass

If a force f(t) acts on the mass in the direction of the one degree of freedom of the system, the equation of motion becomes

$$M\ddot{x} = - kx - c\dot{x} + f(t) \qquad (7-1)$$

or $$\ddot{x} + 2h\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{M} \qquad (7\text{-}2)$$

This equation is generalized in the usual way for rotation, for which $f(t)$ becomes a driving moment rather than a force.

The complementary function of this differential equation has already been found, and is given by equation (6-4). In physical terminology, this term, which represents the free oscillations of this system, is called the transient part of this solution, because it decays with time, more or less rapidly, depending on h.

The particular integral represents the response of the system that is dependent on the properties of $f(t)$. It is found by standard methods of solving differential equations. If we apply the Laplace transform to this equation, assuming $x(0) = \dot{x}(0) = 0$, we get

$$s^2X(s) + 2h\omega_n sX(s) + \omega_n^2 X(s) = F(s) \quad \text{, where}$$

$$X(s) = \mathcal{L}[x(t)] \quad \text{and} \quad F(s) = \mathcal{L}\left[\frac{f(t)}{M}\right] \text{, assumed to exist.}$$

Then $$X(s) = \frac{F(s)}{(s - s_1)(s - s_2)} \text{, where } s_1, s_2 = (-h \pm \sqrt{h^2 - 1})\omega_n$$

If $f(t)$ is specified, $X(s)$ is known, and $x(t)$ may be found by finding the inverse Laplace transform. The factors involving s_1 and s_2 contribute the complementary function.

The special case $f(t) = F_o \sin \omega_e t$ will be treated in considerable detail. The result will be knowledge of the harmonic response of the system, a result that gives valuable insight to the behavior for more general inputs. The angular frequency of the driving force is ω_e, where subscript refers to "earth", since our ultimate interest is in the case in which earth motion drives the system. For this input

$$F(s) = \frac{F_o}{M} \frac{\omega_e}{s^2 + \omega_e^2}$$

By the Laplace transform or other standard method, the particular integral is found to be

$$X_s(t) = \frac{F_o}{M}\left[\frac{\omega_n^2 - \omega_e^2}{(\omega_n^2 - \omega_e^2)^2 + (2h\omega_e\omega_n)^2} \sin\omega_e t + \frac{-2h\omega_n\omega_e}{(\omega_n^2 - \omega_e^2)^2 + (2h\omega_e\omega_n)^2} \cos\omega_e t\right]$$

(7-3)

The subscript "s" stands for "steady-state", as this is the motion that persists after the transient part of the solution has decayed. The result can be rewritten in the form $x_m \sin(\omega_e t + \phi)$.

where $$x_m = \frac{F_o/M}{\left[(\omega_n^2 - \omega_e^2)^2 + (2h\omega_e\omega_n)^2\right]^{\frac{1}{2}}} \quad \text{(always considered positive)}$$

and

$$\phi = \tan^{-1} \frac{-2\zeta\omega_e\omega_n}{\omega_n^2 - \omega_e^2}$$

Dividing numerator and denominator by ω_n^2, and using $M\omega_n^2 = k$, we obtain for the amplitude and phase of the steady-state motion

$$x_m = \frac{F_o/k}{\left[(1 - (\frac{\omega_e}{\omega_n})^2)^2 + (2\zeta\frac{\omega_e}{\omega_n})^2\right]^{\frac{1}{2}}} \qquad (7\text{-}4a)$$

$$\phi = \tan^{-1} \frac{-2\zeta(\omega_e/\omega_n)}{1 - (\omega_e/\omega_n)^2}$$

We can interpret the minus sign in ϕ by considering the result of the introduction to Section 6. There it was found that a force adds energy to the system only if it leads the displacement. We realize that the driving force must add energy or the motion would stop. Therefore, the displacement x must lag behind f, so ϕ is an angle of lag. We therefore rewrite x as $x_m \sin(\omega_e t - \phi)$, where now

$$\phi = \tan^{-1} \frac{2\zeta(\omega_e/\omega_n)}{1 - (\omega_e/\omega_n)^2} \qquad (7\text{-}4b)$$

and we have used $\tan(-\phi) = -\tan\phi$.

We conclude that the steady-state response to a sinusoidal driving force is a sinusoidal motion that lags the driving force by the angle given by (7-4b) and has an amplitude given by (7-4a).

Exercise: Prove (7-3) by finding the inverse Laplace transform.

7.2. The amplitude and phase response

X_m, given by (7-4a) as a function of ω_e/ω_n, is the harmonic amplitude response function of the system. If we interpret F_o/k as the displacement a constant force F_o would produce, denoting this by x_{st}, the static amplitude response can be written in dimensionless form

$$\frac{x_m}{x_{st}} = \frac{1}{\left[(1 - (\frac{\omega_e}{\omega_n})^2)^2 + (2\zeta\frac{\omega_e}{\omega_n})^2\right]^{\frac{1}{2}}} \qquad (7\text{-}5)$$

$x_m/x_{st} = 1$ for $\omega_e/\omega_n = 0$ and approaches zero as ω_e/ω_n becomes very large, for all values of ζ. At resonance, $\omega_e = \omega_n$, or $\omega_e/\omega_n = 1$, the resonant response is

$$x_r/x_{st} = 1/2\zeta \qquad (7\text{-}6)$$

This resonant response increases without limit as the damping factor goes to zero.

Exercise: Solve the equation of forced motion (7-2) for $\zeta = 0$ and $f(t) = F_o \sin \omega_n t$. Note the manner in which the response of an undamped system goes to infinity at resonance.

The peak response occurs at

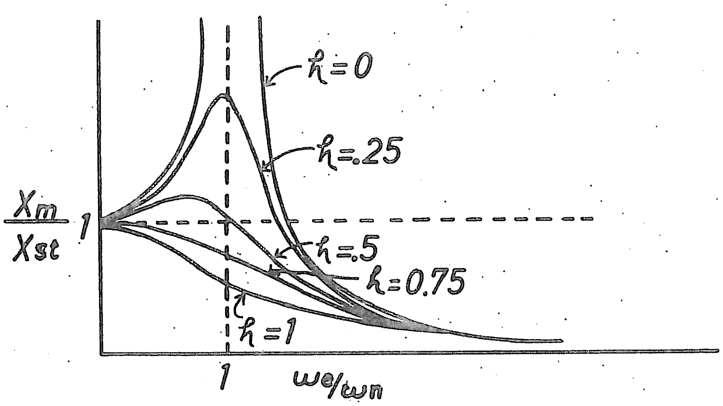
$$\omega_p/\omega_n = \sqrt{1 - 2\zeta^2} \qquad (7\text{-}7)$$

Thus, the peak response does not occur at either the undamped or damped natural frequency of the system. The value of the peak response is

$$x_p/x_{st} = 1/(2\zeta\sqrt{1-\zeta^2}) \qquad (7\text{-}8)$$

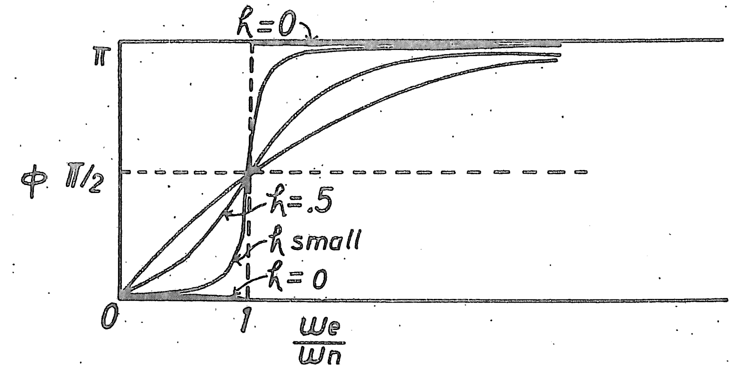
Exercise: Prove (7-7) and (7-8) by differentiating (7-5) with respect to (ω_e/ω_n).

The peak frequency, ω_p, is real only for $\zeta \leq \sqrt{2}/2$. For larger values of the damping factor, the largest response occurs at $\omega_e = 0$, and the response decrease monotonically as the driving frequency increases. All of these conclusions are included in the figure.



For the undamped case, $\frac{x_m}{x_{st}} = \left| \frac{1}{1 - (\frac{\omega_e}{\omega_n})^2} \right|$, and the change in sign for $\omega_e/\omega_n > 1$ is taken care of by the phase angle.

The phase angle by which the displacement lags the driving force, given by (7-4b), is always zero at $\omega_e = 0$ (D.C. limit), $\pi/2$ at resonance, and approaches π at high driving frequencies. In the case of zero damping, the displacement is in phase for frequencies below resonance, and out of phase by π for frequencies above resonance.

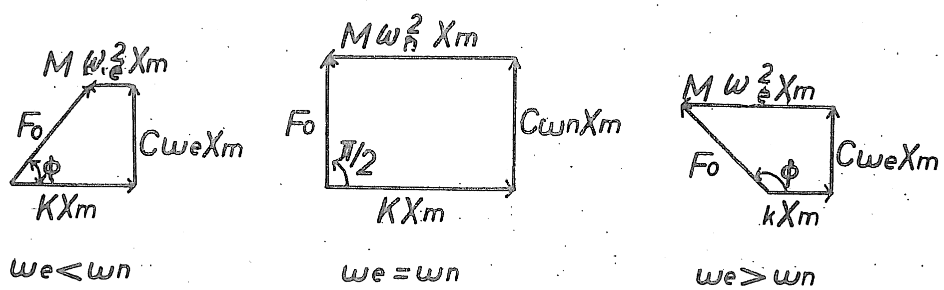


The lag of $\pi/2$ at resonance is accounted for by the fact that at this frequency the driving force has only to supply the energy required to overcome damping, so the terms $c\dot{x}$ and $f(t)$ are equal and opposite in phase. This holds because the inertial term $m\ddot{x}$ and the spring force kx are equal, a relation that holds only at resonance, where $m\ddot{x} = -m\omega_e^2 x = -m\omega_n^2 x = -kx$. For lower frequencies $m\ddot{x} < kx$, and the angle of lag is less than $\pi/2$, while $m\ddot{x} < kx$ above resonance, and the angle is greater than $\pi/2$. At very high frequencies, the inertial term completely dominates because of the factor ω_e^2, and the driving force is practically in phase with the inertial term, or out of phase with the displacement by almost π.

$$M\ddot{x} + c\dot{x} + kx = F_o \sin \omega_e t$$

$$x = x_m \sin (\omega_e t + \phi)$$

$$M\omega_e^2 x_m \sin(\omega_e t + \phi + \pi) + c\omega_e x_m \sin(\omega_e t + \phi + \pi/2) + kx_m \sin(\omega_e t + \phi) = F_o \sin\omega_e t$$

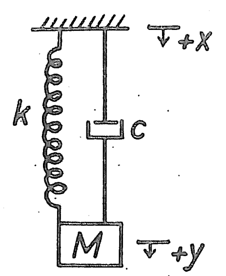


We note that the effect of appreciable damping, (h near 1) in the system is twofold: 1) the free oscillations, represented by the transient part of the solution, decay quickly; and 2) the amplitude response curve is flattened, so that the response is almost the same over a fairly large range of the frequency. Both of these effects are desirable in a seismometer. We also note that for damping of this same order, the phase angle is almost a linear function of frequency for the same range of frequencies for which the amplitude response is flat. This is also a desirable characteristic of the system for use in seismometry, as will be shown.

7.3. The response of the system to motion of the frame

The essential problem of seismometry, the response of the system to motion of the frame, can now be treated, using the concepts developed in the previous sections. We shall assume for the present that the motion of the frame is a translation in the direction of the one degree of freedom of the system. The force that drives the mass is transmitted through the coupling between the mass and frame, i.e. the spring, pivot, hinges, etc., and the damping device. In the ideal seismometer with zero or perfectly flexible coupling, no force would be transmitted and the mass would not move at all.

We shall again use the mass-spring-dashpot as the prototype system, and generalize from it quickly. The following change in notation is introduced: x(t) is now the displacement of the frame in the direction of the one degree of freedom of the system, measured with respect to a set of coordinates fixed in the earth and not taking part in local motions; y(t) is the displacement of the mass with respect to this same coordinate system (the "absolute" displacement of the mass), and $z(t) \equiv y(t) - x(t)$ is the displacement of the mass relative to the frame.



The spring force on the mass is $- k (y - x) = - kz$.
The force exerted by the damping element is $- c (\dot{y} - \dot{x}) = - c\dot{z}$.
The equation of motion is

$$M\ddot{y} = - kz - c\dot{z} \qquad (7\text{-}9)$$

Equation (7-9) could be solved for the absolute displacement, y. We are more interested in the nature of the relative motion, z, so we rewrite the equation, subtracting $M\ddot{x}$ from both sides:

$$M\ddot{y} - M\ddot{x} = - kz - c\dot{z} - M\ddot{x}$$

or

$$M\ddot{z} + c\dot{z} + kz = - M\ddot{x}$$

The equation of relative motion is identical to equation (7-1) for forced motion, with $f(t) = - M\ddot{x}$. Thus, the <u>relative motion</u> of the mass with respect to the moving frame is the same as the motion that would result if the frame were at rest and a force equal to $- M\ddot{x}$ were applied directly to the mass. Dividing by M, we obtain the fundamental equation of the seismometer:

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = - \ddot{x} \qquad (7\text{-}10)$$

In this form, the equation is generalized for any system of the type that has been discussed. However, if the system consists of a pivoted mass, the point of the mass to which z corresponds must be determined. This will be done subsequently.

Let $x(t)$, the ground displacement, be given by $X_o \sin \omega_e t$. Then $-\ddot{x} = \omega_e^2 X_o \sin \omega_e t$, and the problem becomes the same as that treated in Section 7-1, with $F_o = M\omega_e^2 X_o$. The free oscillations damp out as before. With reference to the steady-state solution, given by (7-4a),

$$\frac{F_o}{k} = \frac{M}{k}\omega_e^2 X_o = \left(\frac{\omega_e}{\omega_n}\right)^2 X_o$$

The solution becomes

$$z(t) = \frac{X_o \left(\frac{\omega_e}{\omega_n}\right)^2 \sin(\omega_e t - \phi)}{\left[\left(1 - \left(\frac{\omega_e}{\omega_n}\right)^2\right)^2 + \left(2h\frac{\omega_e}{\omega_n}\right)^2\right]^{\frac{1}{2}}} \qquad (7\text{-}11)$$

The relative motion z, or one of its derivatives, $\dot{z}$ or $\ddot{z}$, is the quantity that will be recorded by the seismograph. The behavior of z determines the response characteristics of the instrument. For the present we shall consider only the case in which z itself is recorded. Then (7-11) gives the response. Because X_o, and therefore z, is small in most cases, it is necessary to magnify z before recording it, in order to produce a useable record. Call the factor by which z is magnified V_o, so that the trace amplitude, $a(t) = V_o z(t)$. V_o is a constant of the instrument called the static, or geometric, magnification. The manner in which V_o is achieved will be discussed below when particular instruments are considered. We shall use the term static magnification only in the case in which the trace amplitude is a constant multiple of the relative displacement z.

The equation for the trace, called the indicator equation, is then

$$a(t) = V_o z(t) = \frac{V_o \left(\frac{\omega_e}{\omega_n}\right)^2 X_o \sin(\omega_e t - \phi)}{\left[\left(1 - \left(\frac{\omega_e}{\omega_n}\right)^2\right)^2 + \left(2h\frac{\omega_e}{\omega_n}\right)^2\right]^{\frac{1}{2}}}$$

The actual magnification of ground displacement by the instrument is a/x, where a and x must refer to the same phase point in a cycle, not the same time. If we use the trace amplitude a_m and ground amplitude X_o, the magnification, or displacement sensitivity, is

$$V \equiv \frac{a_m}{X_o} = \frac{V_o \left(\frac{\omega_e}{\omega_n}\right)^2}{\left[\left(1 - \left(\frac{\omega_e}{\omega_n}\right)^2\right)^2 + \left(2h\frac{\omega_e}{\omega_n}\right)^2\right]^{\frac{1}{2}}}$$

Thus for the simply-conceived seismograph, the amplitude response can be written as a function of frequency as

$$\frac{V}{V_o} = \frac{\left(\frac{\omega_e}{\omega_n}\right)^2}{\left[\left(1 - \left(\frac{\omega_e}{\omega_n}\right)^2\right)^2 + \left(2h\frac{\omega_e}{\omega_n}\right)^2\right]^{\frac{1}{2}}} \qquad (7\text{-}12)$$

Because frequencies less than 1 cps are common in seismology, it is often convenient to rewrite this in terms of the period ratio $T_e/T_n = \omega_n/\omega_e$

$$\frac{V}{V_o} = \frac{1}{\left[\left(1 - \left(\frac{T_e}{T_n}\right)^2\right)^2 + \left(2\xi\frac{T_e}{T_n}\right)^2\right]^{\frac{1}{2}}} \qquad (7\text{-}13)$$

The response function (7-12) is not of the same form as x_m/x_{st} in (7-5) because of the factor $(\omega_e/\omega_n)^2$. This results from the frequency dependence of F_o in the present problem. With F_o depending on ω_e, it is no longer convenient to use it as a reference in writing the response in dimensionless form.

The important properties of the amplitude response, from (7-12), are

$V = 0$ at $\frac{\omega_e}{\omega_n} = 0$ (no relative motion at very low earth frequencies).

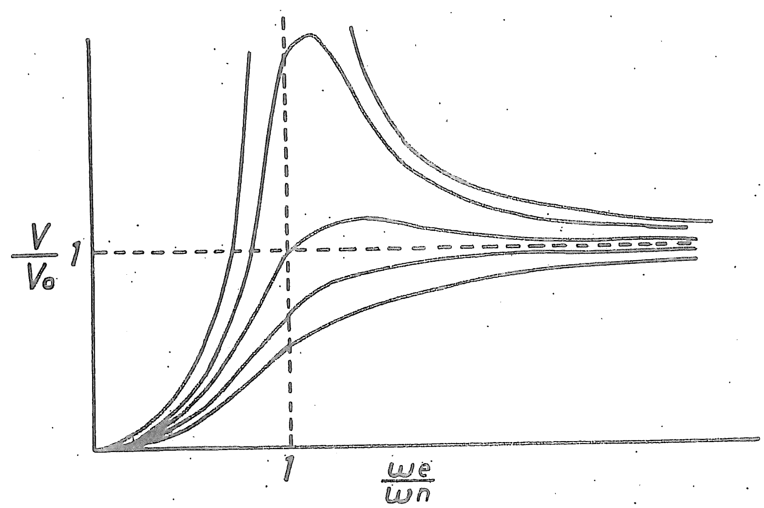
$V \rightarrow V_o$ as $\frac{\omega_e}{\omega_n} \rightarrow \infty$ (no absolute motion at very high earth frequencies).

$\frac{V}{V_o} = \frac{1}{2\xi}$ at $\frac{\omega_e}{\omega_n} = 1$ (resonant response).

$\frac{V_e}{V_o} = \frac{1}{2\xi\sqrt{1-\xi^2}}$ at $\frac{\omega_e}{\omega_n} = \frac{\omega_p}{\omega_n} = \frac{1}{\sqrt{1-2\xi^2}}$ (peak response)

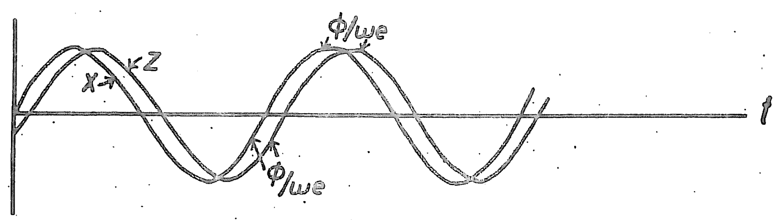
Exercise: Prove that the above relations are true.

The results for the magnification, or amplitude response, can be summarized in the figure.

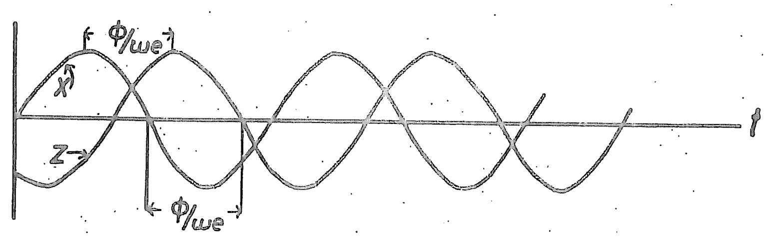


The peak magnification occurs at a frequency above resonance, and a maximum in the sense $\frac{d}{d(\omega_e/\omega_n)} \frac{V}{V_o} = 0$ exists only for $\zeta \leq \frac{\sqrt{2}}{2}$. For all values of damping V approaches V_o at high frequencies. Thus V_o, originally defined as the factor by which z is magnified in the record, turns out to be the magnification of the system for very high earth frequencies.

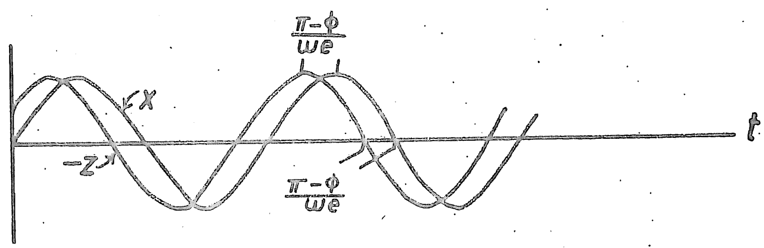
The phase response is $\phi = \tan^{-1} \frac{2\zeta(\omega_e/\omega_n)}{1 - (\omega_e/\omega_n)^2}$. This is the angle by which the relative motion z lags the ground motion x. This angle behaves with variations in ω_e/ω_n, with ζ as a parameter, in the manner shown above. However, when discussing the phase response of a seismograph, in which z has been magnified and recorded, one must be careful. In making the mechanical and optical connections from the mass to the record, one can make the indicator move in either sense for a given sense of z. It is then customary to specify which direction the trace moves (up or down) for a given direction of ground motion. This refers, however, to very rapidly applied ground motion, so fast that the mass doesn't move. Thus, it is customary to take the high frequency limit as the reference. Since z is approximately 180° behind x at this limit, by using this convention, we essentially reverse z. With this convention, the trace displacement leads the ground displacement by the angle $\pi - \phi$. The time by which a given phase point on the record leads the corresponding phase point in the ground motion is $(\pi - \phi)/\omega_e$.



Low frequency, φ is small.



High frequency, φ almost π.



High frequency, z inverted.

(Ground moves up, record moves up. Record leads ground by this convention.)

7.4. Displacement meters and accelerometers

Consider the special case $\frac{\omega_e}{\omega_n} >> 1$. Then

$$\frac{V}{V_o} \approx 1 \ , \ \phi \approx \pi$$

In this case $a(t) \approx - V_o x(t)$, and the record is a magnified version of the ground displacement. The minus sign is taken care by the convention introduced above. By choosing ω_n to be small compared to the ground frequencies to be recorded, an instrument which produces a record of displacement multiplied by a constant is obtained. Thus, a large values of T_n, corresponding to loose coupling to the frame (small k) is desirable. Such an instrument is called a <u>displacement meter</u>, but this term should be used with care. It is apparent that a given system acts as a displacement meter only for ground frequencies that are high enough.

The same result can be obtained by inspection of the equation of motion (7-10), for ω_n very small. Then

$$\ddot{z} \approx - \ddot{x}$$

and $$z \approx - x.$$

Since $a = V_o z$, the above result follows.

For the special case $\frac{\omega_e}{\omega_n} << 1$, $\frac{V}{V_o} \approx \left(\frac{\omega_e}{\omega_n}\right)^2$, a very small number. However, this gives $a(t) = (V_o/\omega_n^2)\,\omega_e^2 x(t)$, with a phase shift $\phi \approx 2\zeta\frac{\omega_e}{\omega_n}$ ($\tan\alpha \approx \alpha$ for small values). Thus, the record is proportional to the ground acceleration. The instrument can be called an <u>accelerometer</u> for ground frequencies that are low enough relative to the <u>natural frequency</u> of the instrument. Note, however, that this effect is achieved by making ω_n large, so that V_o must be large to give a usable record. The linear dependence of phase on frequency results in a record free of phase distortion.

Exercise: Prove that if the phase shift is a linear function of frequency, no phase distortion appears in the record.

Again the same result could have been obtained by inspection of the equation of motion. Under the condition that ω_n is very large, only the third term is significant.

$$\omega_n^2 z = - \ddot{x}$$

$$z = - \ddot{x}/\omega_n^2$$

$$a = V_o z \approx - V_o/\omega_n^2 \ddot{x} ,$$

and the trace is proportional to acceleration.

7.5. Velocity and acceleration sensitivity

The magnification has been defined as $V = a_m/x_o$, the ratio of trace amplitude to ground amplitude. One may also define the velocity sensitivity, the ratio of trace amplitude to ground velocity.

$$S = a_m/(\dot{x})_m = a_m/\omega_e x_o = V/\omega_e \qquad (7\text{-}14)$$

for sinusoidal earth motions.

The acceleration sensitivity is defined in a similar way.

$$E = a_m/(\ddot{x})_m = a_m/\omega_e^2 x_o = V/\omega_e^2 \qquad (7\text{-}15)$$

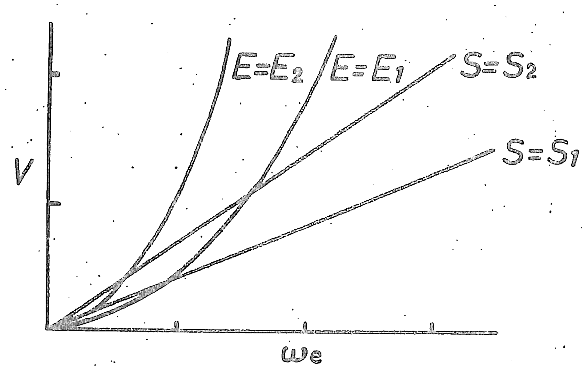
These can be written in dimensionless form as follows:

$$\frac{V}{V_o} = \frac{z_m}{x_o} = \frac{(\frac{\omega_e}{\omega_n})^2}{\left[(1-(\frac{\omega_e}{\omega_n})^2)^2\right] + (2h\frac{\omega_e}{\omega_n})^2 \,\Big]^{\frac{1}{2}}} \qquad (7\text{-}12)$$

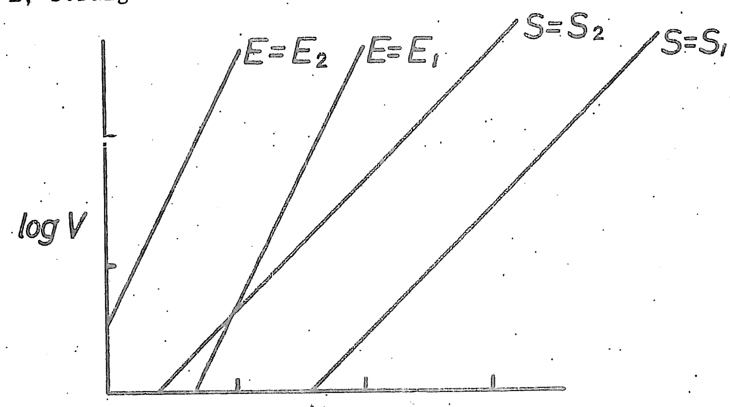
$$\frac{\omega_n S}{V_o} = \frac{\omega_n z_m}{(\dot{x})_m} = (\frac{1}{\omega_e/\omega_n}) \frac{V}{V_o} \qquad (7\text{-}14a)$$

$$\frac{\omega_n^2 E}{V_o} = \frac{\omega_n^2 z_m}{(\ddot{x})_m} = \frac{1}{(\omega_e/\omega_n)^2} \frac{V}{V_o} \qquad (7\text{-}15a)$$

It is convenient to plot response curves as the logarithm of the response vs. the logarithm of (ω_e/ω_n). On a linear plot, with ω_e, V as axes, curves of constant S are given by $V = \omega_e S$, from (7-14). These are straight lines with slope S. Similarly, curves of constant E are given by $V = \omega_e^2 E$, parabolas.

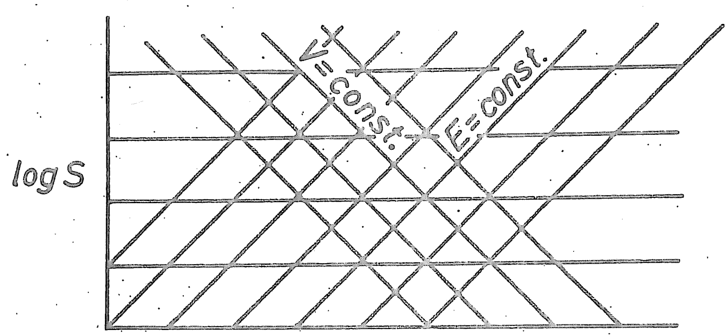


On a log-log plot, log V = log ω_e + log S, and curves of constant S are straight lines with slope +1. At log ω_e = 0, ω_e = 1, V = S. Similarly curves of constant acceleration sensitivity are given by log V = 2 log ω_e + log E, straight lines with slope +2.



The log-log plot of the response offers several advantages in addition to the above simplification. A constant ratio of frequency or magnification becomes a constant length along the axes. It becomes convenient to express the shape of the response curve by the slope of straight line segments that approximate the curve. Suppose, for example, that over some range of frequency, the magnification is doubled if the frequency is doubled. The slope of this portion of the curve is +1, corresponding to 6db/octave. Thus, for the velocity sensitivity to remain constant, the magnification must increase 6db/octave.

There is also an advantage to plotting the response as S vs. ω_e, rather than V vs. ω_e. On such a plot, curves of constant V are straight lines with slope -1 and curves of constant acceleration are straight lines of slope +1.



One advantage of using logarithmic plot of velocity sensitivity is that the harmonic response curve is then always symmetric about $\omega_e/\omega_n = 1$ (See Willmore, 1960, Figure 2). The peak velocity sensitivity, $S_p = V_o/2\,\omega_n$ (from (7-14a)), always occurs at resonance. For small values of ω_e/ω_n, $\log S = \log V_o/\omega_n + \log(\omega_e/\omega_n)$, so that on a plot of $\log \omega_n S/V_o$ vs. $\log \omega_e/\omega_n$, the response has a slope of +1 (+6db/octave). For large values of ω_e/ω_n, $\log S \longrightarrow \log V_o/\omega_n - \log(\omega_e/\omega_n)$, giving a slope of -1 (-6db/octave). With $\omega_n S_p/V_o = 1/2$ fixing the peak, and the slopes for large and small values of ω_e/ω_n known, an approximate response curve is easily drawn.

Exercise: Prove that the velocity sensitivity has the properties given above. Sketch approximate velocity sensitivity curves ($\omega_n S/V_o$) for h = 5, 1, 2.

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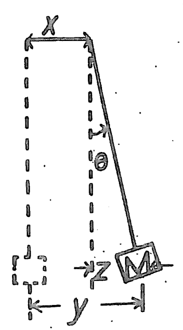
8. Response of pivoted systems to motion of the frame

The theory of the seismometer developed in the preceding section can be applied to any linear system with the prescribed properties. With regard to pivoted systems, which are commonly employed in seismology, it is only necessary to determine the point in the boom or mass to which the relative displacement z applies.

We shall assume that the motion of the ground is again parallel to the direction of the one degree of freedom of the system.

8.1. The simple pendulum



The transition from a translating mass on a spring to a pivoted system can be made by considering the response of a simple pendulum. Because all the mass is at a point, we can use the same notation as in the mass-spring system.

For small displacement Θ, we can consider that the relative motion of the mass is horizontal. Then the absolute displacement of the mass is $x + L\Theta$. The force acting on the mass depends only on the relative displacement $z = L\Theta$ (corresponding to $-k(y - x)$ for the mass-spring system).

The equation of motion becomes

$$M\ddot{y} = -Mg\Theta = -Mg\,z/L$$
$$M\ddot{z} = -Mg/L\,z - M\ddot{x}$$
$$\ddot{z} = -g/L\,z - \ddot{x} \qquad (8\text{-}1)$$
or
$$\ddot{z} + \omega_n^2 z = -\ddot{x}.$$

This is identical to (7-10), with $h = 0$. Damping is easily added and appears as in (7-10). The mass of the simple pendulum has the same response for horizontal motion as the mass in the mass-spring system has for vertical motion, and all of the remarks concerning the amplitude and phase response are applicable. For example, to build an instrument which will act as a horizontal displacement meter, ω_n must be small, or L large.

Additional insight into the behavior of the pendulum can be obtained by considering the location along the original line of the pendulum at which the absolute displacement is zero, i.e., the position of a node. The absolute amplitude of the mass (zero damping) is

$$y = z + x = \frac{(\omega_e/\omega_n)^2}{1 - (\omega_e/\omega_n)^2} x + x = \frac{1}{1 - (\omega_e/\omega_n)^2} x$$

Let y' be the absolute displacement of any point located at a distance b above the mass. Then

$$\frac{y' - x}{y - x} = \frac{L - b}{L}$$

$$y' = y \left(\frac{L - b}{L}\right) + x \left(1 - \frac{L - b}{L}\right)$$

$$= 0 \text{ at a node.}$$

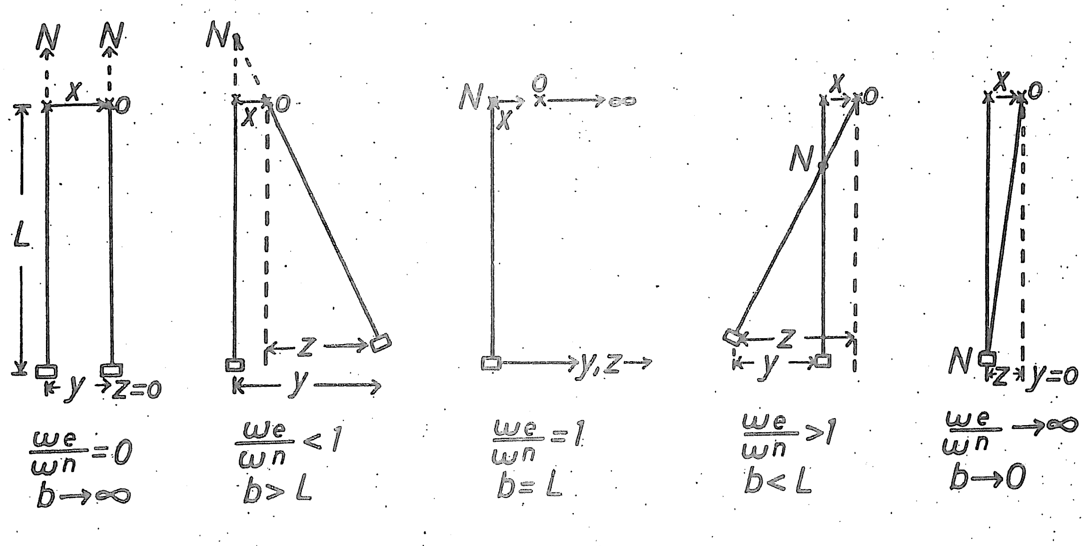
Substituting for y

$$\left[\frac{1}{1 - (\omega_e/\omega_n)^2} \cdot \frac{L - b}{L} + \frac{b}{L}\right] x = 0$$

$$L - b(\omega_e/\omega_n)^2 = 0$$

$$b = (\omega_e/\omega_n)^{-2} L.$$

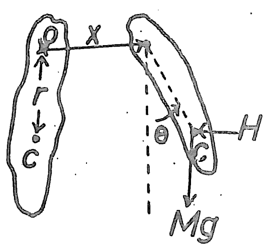
The position of the mass is found by drawing a line through the node, N, and the pivot O. For $(\omega_e/\omega_n) \longrightarrow 0$, $b \longrightarrow \infty$, and the pendulum moves parallel to itself ($z = 0$). As $(\omega_e/\omega_n) \longrightarrow 1$, $b \longrightarrow L$ from above. z is in phase with x. At $(\omega_e/\omega_n) = 1$, $b = L$. The pivot is a node, and the amplitude goes to infinity (this violates the assumption of small but is still a useful notion to illustrate behavior at resonance). For (ω_e/ω_n) greater than 1, $b < L$, and the node is a point on the string. z is now out of phase with x by π (only for zero damping, of course). Finally, as $(\omega_e/\omega_n) \longrightarrow \infty$, $b \longrightarrow 0$, and the node is in the mass. $/z/ = x$, and the relative motion is π out of phase.



Exercise: For what frequency ratio is the node at the midpoint of the string?

8.2. Pendulum with distributed mass

The response of a physical pendulum can be determined in a similar fashion. Some care must be exercised in taking moments because the pivot point is now moving. From dynamics, the time rate of change of angular momentum is equal to the sum of the moments when the angular momentum and the moments are with respect to 1) a fixed point, 2) the center of mass, or 3) a moving point for which the velocity is parallel to the velocity of the center of mass. We shall take moments about the center of mass, as Lamb does in his textbook on Mechanics.



The moment of inertia about a horizontal axis through C is K_c. Consider the reactions at O. There is a vertical force - Mg and a horizontal force - H, where H is the force acting at the center of mass. We evaluate H using Newton's second law, and the principle that the center of mass moves as a particle with mass M would if the resultant of all the forces acting at points in the body acted on it. The absolute displacement of C is $x + r\Theta$, so $H = M\, d^2/dt^2\ (x + r\Theta)$

Then $$K_c\ddot{\Theta} = - Mgr\Theta - Hr$$
$$= - Mgr\Theta - Mr^2\ddot{\Theta} - Mr\ddot{x}$$

Using the parallel axis theorem

$$K\ddot{\Theta} + Mgr\Theta = - Mr\ddot{x}.$$

The relative motion of the pendulum inside its frame, given by Θ, is the same as if the force $- M\ddot{x}$ were applied at the center of mass, with the frame at rest.

Then $$\ddot{\Theta} + \frac{Mgr}{K}\Theta = - \frac{Mr}{K}\ddot{x}$$

or $$\ddot{\Theta} + \omega_n^2\Theta = - (1/\ell)\ddot{x}, \qquad (8\text{-}2)$$

where $\ell = K/Mr$ is the reduced pendulum length, and ω_n has the form derived in Section 5.22. A damping moment $- c\dot{\Theta}$ can be introduced, since the damping force depends only on relative motion. Then the equation becomes

$$\ddot{\Theta} + 2\zeta\omega_n\dot{\Theta} + \omega_n^2\Theta = -(1/\ell)\ddot{x} \qquad (8\text{-}3)$$

Then $$\ell\ddot{\Theta} + 2\zeta\omega_n\ell\dot{\Theta} + \omega_n^2\ell\Theta = - \ddot{x}$$

or $$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = - \ddot{x} \qquad (8\text{-}4)$$

where $z = \ell\Theta$ is the relative displacement of the center of oscillation. Equation (8-4) is identical to (7-10) for the mass-spring system. It is now clear that the point in the pendulum that behaves like the mass in the mass-spring system is the center of oscillation. Thus, the entire response characteristic of the physical pendulum is known from the discussion in

Sections 7.3 - 7.5, with z referring to the center of oscillation. For example, at high frequencies, it is the center of oscillation that has zero absolute motion, or is the "steady-point" of the seismometer. From another viewpoint, the center of oscillation acts like the point mass of a simple pendulum (Equation 8-1).

The case of the horizontal pendulum is most simply treated by remembering that the only effect of inclining the axis of rotation is to change the effective gravity force per unit mass from g to g sin i. For small values of i, the equation of motion becomes (no damping)

$$K\ddot{\theta} + Mgri\theta = - Mr\ddot{x}$$

$$\ddot{\theta} + \frac{gi}{\ell}\theta = -(1/\ell)\,\ddot{x}$$

$$\ddot{\theta} + \omega_n^2\theta = -(1/\ell)\,\ddot{x}, \text{ identical to (8-2),}$$

but with $\omega_n^2 = g^i/\ell$. The conclusions concerning the response are identical to the case of the ordinary physical pendulum, but now ω_n is much smaller, so that for the same ω_e, we are farther out on the flat part of the magnification curve. K/Mr is not changed by the inclination, and the center of oscillation is still the point in the pendulum that z refers to in all the results for relative motion.

8.3. Effects of general motion on pivoted systems (Byerly, 1952, Coulomb, 1956, Sect. 2, Sect. 9)

The analysis of the response of pivoted systems has been carried out on the assumption that the motion of the ground is a translation in the direction of the one degree of freedom of the system. In real ground motion, the direction of motion is quite arbitrary, and, as mentioned in the introduction, includes rotations as well as translations. Analyses of more general cases have been carried out by several authors, and Byerly presents a complete treatment, in which the conditions are revealed for which all motions except the one considered here, above, can be neglected.

If, for example, a horizontal pendulum is acted on by ground motion strictly parallel to the neutral plane, it will not respond, as there is no moment of the force $- M\ddot{x}$ about the axis of rotation. But suppose a motion perpendicular to the neutral plane, i.e., in the direction of intended response, acts at the same time. The boom is displaced, and "end-on accelerations" will have a moment. The moment arm is usually small and this effect is neglected. Byerly examines in detail the conditions in which this is justified, as well as the omission of tilts and rotations. He makes only reasonable assumptions about the construction of the pendulum, such as symmetry which is usually found, and constraint to rotation about an axis.

The end-on acceleration produces a moment, dependent on θ, as expected from the manner in which this quantity enters the problem. As shown by Byerly (his equation (4), the effect is that of a negative resting force. In principle, then, end-on accelerations can lead to instability of the system, as discussed by Coulomb, Section 9.

Byerly summarizes the conditions under which we may neglect rotations or tilts of the earth and accelerations other than that in the one degree of freedom of the pendulum when the frame is at rest, as follows. Our notation is introduced.

$\frac{gT_n^2}{2\pi\ell} \ll \frac{\Lambda}{x_m}$	(to neglect tilts)
$\frac{T_n^2}{\ell} \ll \frac{T_e^2}{x_m}$	(to neglect end-on acceleration) (can be troublesome for large T_n, and small T_e)
$2\pi T_n \ll \frac{T_e\Lambda}{x_n}$	(to neglect rotations)
$\frac{g}{2\pi} \ll \frac{\Lambda}{T_e^2} = \frac{V}{T_e}$	(to neglect tilts, horizontal pendulum)
$g \ll \frac{\Lambda^2}{x_m T_e^2} = \frac{V^2}{x_m}$	(to neglect tilts, vertical pendulum)
$2\pi\ell \ll \Lambda$	(to neglect rotations)
$4\pi^2 r \ll \frac{\Lambda^2}{x_m}$	(to neglect rotations).

where Λ = wave length along the surface of a simple harmonic ground motion
V = apparent surface velocity.

These conditions are easily satisfied in all but extreme cases.

Exercise: Test these conditions for a pendulum with T_n = 30 sec, ℓ = 20 cm, r = 18 cm, for waves with T_e = 1 sec, V = 10 km/sec, and T_e = 50 sec, V = 4 km/sec. Determine the largest x_m in each case for which the condition is satisfied.

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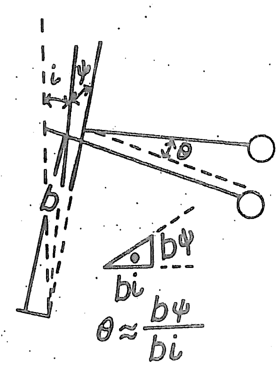
9. The effect of tilt on a horizontal pendulum

Although the dynamic tilts associated with the incidence of seismic waves have a negligible effect under the conditions outlined above, large and very slow tilts of the earth, associated with atmospheric pressure variations, temperature variations, and sometimes tectonic activity have pronounced effects on long period horizontal pendulum. A tilt of the instrument frame in the neutral plane will change the period (because the angle i changes), but the pendulum will not rotate about its axis. We shall call on angular displacement in the neutral plane an inclination, as before, and use "tilt" only for an angular displacement normal to the neutral plane about a horizontal axis.



The neutral plane is the plane of the vertical and the axis of rotation. If the axis of rotation of a horizontal pendulum at rest is tilted, the orientation of the neutral plane changes. The pendulum must then rotate about the axis of rotation until the center of mass is in the new neutral plane. We assume the lateral tilt is small enough that the change in i is negligible. Then the angular displacement of the boom about the axis is

$$\theta = \psi / i \tag{9-1}$$

Thus, for small values of i, i.e., long period pendulum, a small tilt produces a large displacement of the boom. Solving the period expression (5.2-3) for i and substituting

$$\theta = \frac{g T_n^2}{4\pi^2 \ell} \psi , \tag{9-2}$$

so that the tilt sensitivity increases as the square of the natural period.

If the seismograph is one that directly records $z = \ell\theta$, this response to tilt can seriously disturb the record, resulting in wild wandering of the trace. The McComb-Romberg seismograph (Macelwane, 1947, p. 161) is an example of a mechanical-optical system in which tilt compensation was provided. The insertion of an electromagnetic transducer, so that $\dot{z}$ is recorded rather than z, eliminates the wandering of the trace in response to slow tilt.

The sensitivity of the horizontal pendulum to tilt can also be put to use. By using a very long period and recording by direct coupling, the instrument can be used to record the tilts (one man's noise is another man's signal). Also, by applying a known tilt, ψ , to the frame, a known displacement can be given to the center of oscillation

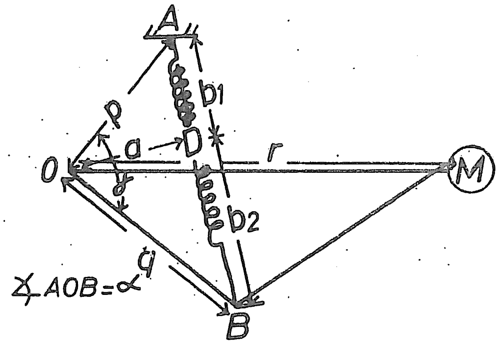
$$z = \ell\theta = \frac{g T_n^2}{4\pi^2} \psi$$

This provides a simple means of determining the static magnification of a direct recording seismograph. This technique will be discussed further below.

10. Non-linear effects in the long-period vertical seismometer

(Reference: J. Wilip, "Zur Theorie und Konstruktion von Vertical Seismographen", Gerl. Beit. z. Geophysik, 19:4, 1928; F.W. Sohon, Intro. Theoretical Seism., Part II - Seismometry, 1932, pp. 29-34)

In the analysis of Section 5.12, a system sensitive to vertical ground motion was considered. In this system, the period may be lengthened by decreasing a, the moment arm of the spring. The initial tension in the spring must then be increased in order to support the boom in the rest position. There is a practical limit to how long a period can be achieved this way, and greater flexibility is provided by two steps: connecting the spring to the boom below the horizontal plane through the center of mass, and letting the spring act on a line inclined to the vertical. We shall analyze such a system, following the discussion in Sohon.



$\overline{AB} = b_1 + b_2 = L$, the length of the spring in equilibrium

<u>Restoring moment of the spring</u> In this analysis we shall not neglect the change in the moment arm, a, of the spring, as we did in Section 5.12. In the triangle OAB, $L^2 = p^2 + q^2 - 2\,pq\cos\alpha$

Differentiating, and letting $dL/d\alpha = L'$, $2LL' = 2pq\sin\alpha$ (10-1)

The area of $\triangle OAB$ is $aL/2$ and $1/2\; pq\sin\alpha$, so that

$$aL = pq\sin\alpha \qquad (10\text{-}2)$$

From (1) and (2) $\quad L' = a$ (10-3)

Differentiating (2) $\quad a'L + aL' = pq\cos\alpha$

Using (3) $\quad a'L = pq\cos\alpha - a^2$ (10-4)

$$a''L + a'L' = -pq\sin\alpha - 2aa'$$

By (2) and (3) $\quad a''L = -aL - 3aa'$ (10-5)

Using $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$, and $\alpha = ∡AOD + ∡DOB$

$$\cot\alpha = \frac{1}{\tan\alpha} \quad \frac{1 - (b_1/a)(b_2/a)}{(b_1/a) + (b_2/a)} \quad \frac{a^2 - b_1b_2}{aL}$$

Using (2) $\quad aL\cot\alpha = pq\cos\alpha = a^2 - b_1b_2$ (10-6)

Then, from (4) $\quad a' = -b_1b_2/L$ (10-7)

and from (5) $\quad a'' = -a - \dfrac{3aa'}{L} = -a + \dfrac{3ab_1b_2}{L^2}$ (10-8)

(Note: From (7) $a' = 0$ for $b_2 = 0$, the case in Section 5.12, so the approximation made there is good.)

If we now displace the boom by $\Theta = d\alpha$, we can write the new value of the moment arm by means of Taylor's series, keeping only terms in Θ^2

$$a + \Delta a = a + a'\Theta + \tfrac{1}{2}a''\Theta^2 + 0(\Theta^3)$$

$$= a - \frac{b_1b_2}{L} + \tfrac{1}{2}\left(-a + \frac{3ab_1b_2}{L^2}\right)\Theta^2 \qquad (10\text{-}9)$$

Call the initial spring tension P (= $k\Delta$ in earlier sections). After displacement, it is

$$\begin{aligned} P + \Delta P &= P + k(\Delta L) \\ &= P + k(L'\Theta + \tfrac{1}{2}L''\Theta^2) \\ &= P + k(a\Theta + \tfrac{1}{2}a'\Theta^2) \\ &= P + k\left(a\Theta - (\tfrac{1}{2}b_1b_2/L)\Theta^2\right) \end{aligned} \qquad (10\text{-}10)$$

The spring moment after displacement is (c.w. is positive)

$$-(P+\Delta P)(a+\Delta a) = -Pa - (ka^2 - Pb_1b_2/L)\theta - \tfrac{1}{2}\left[-Pa - \frac{3ab_1b_2}{L}\left(k - \frac{P}{L}\right)\right]\theta^2 + 0(\theta^3) \qquad (10\text{-}11)$$

(Note: Again, with $b_2 = 0$, and keeping only terms in θ, we get the result of Section 5.12.)

The gravity moment is

$$Mgr\cos\theta = Mgr\,(1 - \theta^2/2) + 0(\theta^3) \qquad (10\text{-}12)$$

In equilibrium, $Mgr - Pa = 0$. It is important that the center of mass be on the horizontal plane through 0 to eliminate response to horizontal ground motion.

The total moment is, adding (11) and (12), and inserting the equilibrium condition

$$-(ka^2 - Pb_1b_2/L)\theta + \frac{3ab_1b_2}{2L}\left(k - \frac{P}{L}\right)\theta^2 \qquad (10\text{-}13)$$

The equation of motion is obtained by equating this to $K\ddot{\theta}$. Considering only first order terms, and using $P = Mgr/a$;

$$K\ddot{\theta} = -(ka^2 - Mgrb_1b_2/aL)\theta$$

so that $\omega_n^2 = ka^2/K - gb_1b_2/a\ell L$, using $K/Mr = \ell$ (10-14)

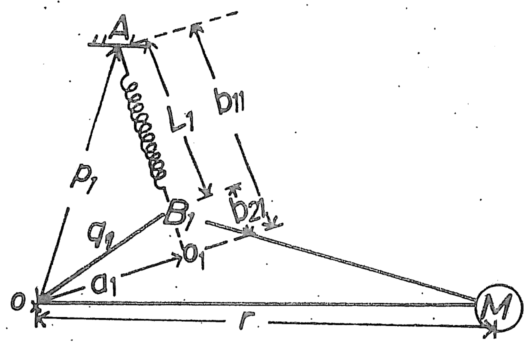
Thus a long period can be achieved by making 'a' small, thereby decreasing the first term and increasing the second. If we make $b_2 = 0$ as in Section 5.12, we lose the advantage of the second term. Physically, what has happened is that this arrangement we have managed to make the moment arm decrease as the spring lengthens, so that the restoring moment per unit angular deflection is smaller.

However, if we make the coefficient of θ small, in order to obtain a long period, the term in θ^2 is no longer negligible. The equation of motion is non-linear and the free oscillations are no longer simple harmonic motion. Physically, the term in θ in (10-13) changes sign with θ, but the term in θ^2 is always positive. Therefore, the restoring moment for a given positive θ is less than for the same angle in the negative(upward) direction. The motion is not symmetrical about the zero position, and the period becomes a function of θ. This is the fundamental difficulty with the long-period vertical originally built by Galitzin.

10.1. Wilip's solution

One solution to this problem was suggested by Wilip (1928), and led to the Wilip-Galitzin vertical seismograph that is still in use in a number of observatories. Wilip suggested fastening a second spring with its point of attachment above the horizontal plane through the pivot and center of mass. The result is that $L_1 = b_{11} - b_{21}$, as shown in the figure. An analysis similar to the one carried out above shows that the restoring moment of this spring for a positive (downward) θ, is

$$-\left(k_1a_1^2 + \frac{P_1b_{11}b_{21}}{L_1}\right)\theta - \frac{3a_1b_1b_2}{2L_1}\left(k_1 - \frac{P_1}{L_1}\right)\theta^2. \qquad (10\text{-}13a)$$



Because both terms in the coefficient of θ are positive, a long period can not be achieved by this arrangement alone. However, the important result is that the sign of the θ^2 term is opposite to that in (10-13). By combining the two configurations, it is possible to eliminate the term in θ^2 and still achieve a fairly long period. The condition to eliminate θ^2 is

$$\frac{3ab_1b_2}{2L}\left(k - \frac{P}{L}\right) = \frac{3a_1b_{11}b_{21}}{2L_1}\left(k_1 - \frac{P_1}{L_1}\right).$$

The natural angular frequency is

$$\omega_n^2 = \frac{\left(ka^2 + k_1a_1^2 - \frac{b_1b_2}{L} + \frac{b_{11}b_{12}}{L}\right)}{K}$$

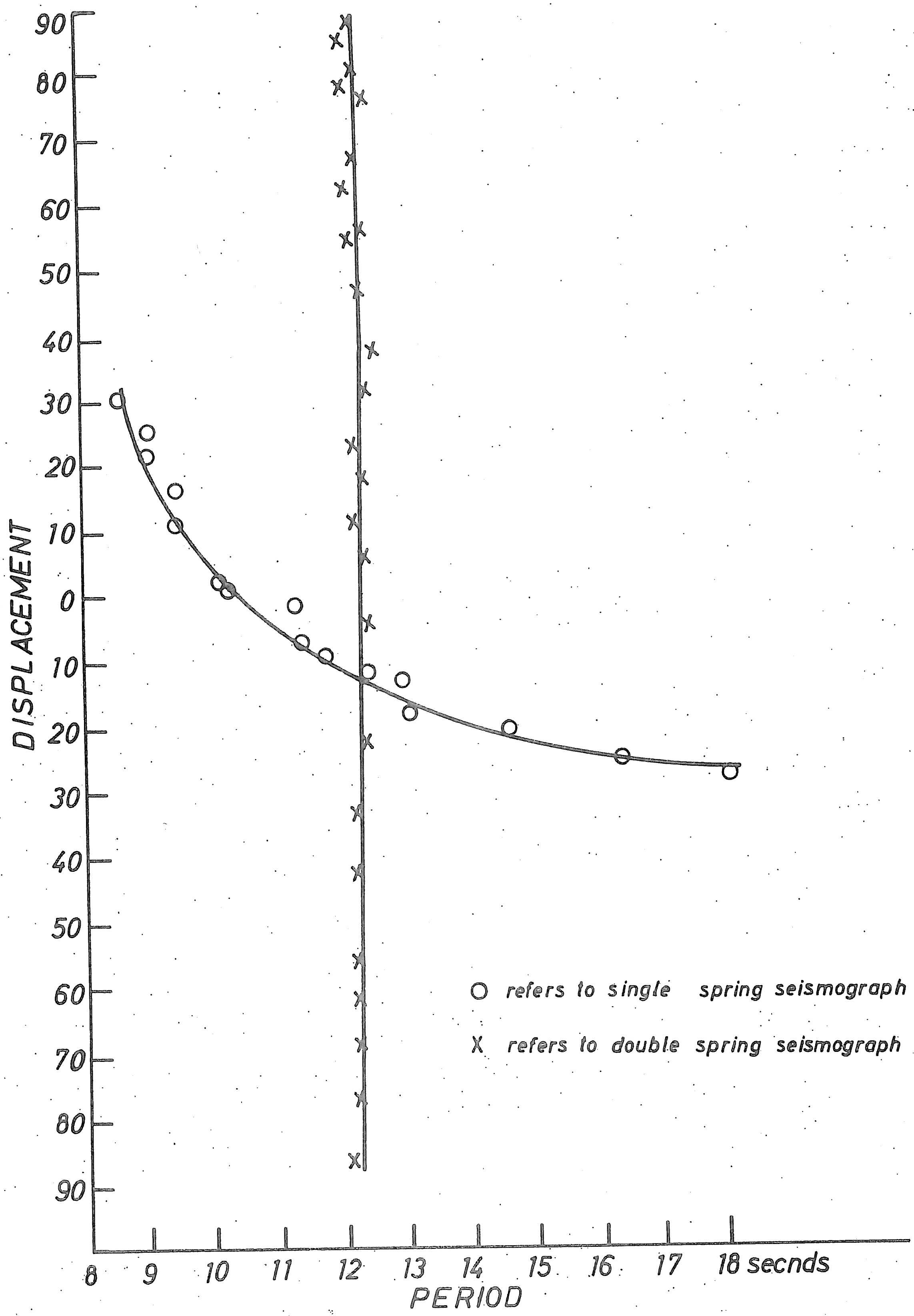
The Wilip-Galitzin is designed on this principle. The period is about 12 seconds, considered long at the time the instrument was developed, but only intermediate today. The improvement actually achieved over the single spring system is illustrated in the figure, taken from Wilip.

10.2. The zero-length spring

(Reference: L.J.B. La Coste, "A New Type, Long Period, Vertical Seismograph", Physics, 5:7, 1934, pp. 178-180.)

Inspection of (10-13) reveals another way of making the coefficient of θ^2 vanish: the use of a spring for which $(k - P/L)$ is zero. This implies that the initial spring tension be $P = kL$, that is we require a spring that exerts a force proportional to its length. Comparing this to the usual expression $P = k(L - L_o)$, we see this requires that the initial length, L_o, be effectively zero, and the elongation of the spring be equal to its length. Such a spring is called a zero-length spring.

A zero-length spring is made by bending the wire as the coils of the spring are turned in such a way that the coils are pressed together (La Coste, also Coulomb, p. 49). Then a certain amount of force is required to open the coils. Enough initial stress is put in to make the force to just open the coils equal to the spring constant times the actual physical initial length. Then for greater loads, the spring has the desired property.


90
80
70
60
50
40
30
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10
0
10
20
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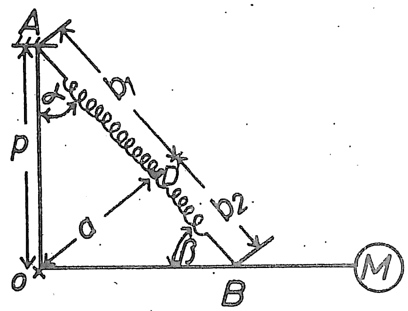
40
50
60
70
80
90
DISPLACEMENT
8
9
10
11
12
13
14
15
16
17
18 secnds
PERIOD
O refers to single spring seismograph
X refers to double spring seismograph

This type of spring is used in the Press-Ewing vertical seismograph and in the Sprengnether-Columbia vertical that is used in stations of the World-Wide Standard Seismograph Network.



A typical configuration is that in the figure. $\angle AOB$ is 90°. Then, putting $P/L = k$ in (10-13), with $b_1 = a/\tan\alpha$, $b_2 = a/\tan\beta$, the restoring moment for a positive θ is $-ka^2(1 - 1/\tan\alpha\,\tan\beta)$.

If dimensions are chosen so that $OA = OB$, $\angle\alpha = \angle\beta = 45^\circ$, then the moment is zero, and an infinite period results. A long, but finite, period can be achieved by making $\angle AOB$ slightly less than 90°, say $(90^\circ - \gamma)$, but keeping OM horizontal. Then, rewriting the expression for the moment in terms of $p = a/\sin\alpha$, with $\alpha = \beta$,

$$-kp^2(\sin^2\alpha - \cos^2\alpha)\theta = (kp^2\cos 2\alpha)\theta\ ..$$

Angle α is now $(45^\circ + \gamma/2)$.
The moment is $kp^2\cos(90 + \gamma)\theta = -(kp^2\sin\gamma)\theta$.

Then $$\omega_n^2 = kp^2\sin\gamma/K,$$

and the period is easily adjusted by changing γ. In the actual seismometer, provision is made for moving point A horizontally, and then changing the tension in the spring to keep the boom horizontal. The period can also be made finite by giving a small initial length to the spring, but this brings back the non-linearities that we sought to eliminate.

Exercise: Write the expression for the total moment on the zero-length spring system starting from the beginning, rather than from (10-13). Show that for $\overline{OA} = \overline{OB}$, and $\angle AOB = 90^\circ$, the moment is $(Mgr - kp^2)\cos\theta$. Thus if we make $kp^2 = Mgr$, the moment is zero for all θ, and the period is infinite. This shows that the result above is not only valid to terms in the second power of θ, but is exact.

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11. Some other problems of seismometry

It is straightforward to design on paper a seismometer that will have the desired characteristics. The realization of an actual instrument is not so easy, especially for long period systems. One of the first difficulties arises for vertical instruments because springs are not ideal instrument components. Three characteristics of springs are unfavorable: thermal expansion, thermal variation of the spring constant, and elastic creep. Some materials, such as quartz, have low temperature coefficients of expansion and elasticity. These have found extensive use in devices in which the suspended mass is small, such as gravity meters. Elinvar is the best material for springs able to carry the large masses employed in long-period vertical seismometers. It is still desirable to minimize temperature variations in the instrument vault.

Elastic creep is a problem primarily during the initial period of operation of an instrument. Many days may be required before a stable zero of the instrument is reached. If it is necessary to clamp the instrument, a new creep cycle begins when it is again unclamped. It is good practice for the manufacturer to stretch the spring to near its operating length for a long period before installation in the seismometer in order to minimize the time required for this adjustment.

Temperature variations in the vault can cause other troubles. If the temperature of the pier should be warmer than the air temperature, which can occur readily in the winter, convection of the air will result. This moving air can cause very bad disturbances of the pendulum, resulting in poor records. The problem is solved by either heating the entire vault, with the heat source at the ceiling, or by putting heaters into the instrument cases, near the top. In neither case should the heaters be thermostatically controlled, as the thermostat cycle will appear on the records. By trial and error the correct rate at which heat should be added to give stable operation is found.

Thermal stresses in the frame of the instrument can also result in disturbed records. The solution is again proper addition of heat, and thermal insulation of the frame itself will result in marked improvement. Heat treatment of the frame after assembly to relieve stresses introduced during manufacture has also proven useful.

A further problem is the effect of air pressure variations on long-period vertical seismometers (Ewing and Press, Trans. A.G.U., 34, 1953). As pressure systems associated with weather changes moves over the station, the density of the air in the vault will change. The suspended boom and mass of the vertical seismometer are acted on by a buoyant force proportional to air density, and so will respond to the density change. One method of compensating for this effect, suggested by Ewing and Press, is to fix a light-weight vessel with volume moment about the axis of rotation equal to the volume moment of the pendulum on the opposite side of the axis. Since the buoyant moment is proportional to the volume moment, any change acting on the pendulum is equalized by the change on the compensating vessel.

A more simple technique, and one now generally employed, is to enclose the seismometer in a sealed case of stiff metal. The seal only has to be good enough to keep out rapid pressure changes, as slow variations will not cause disturbances on the record because an electromagnetic transducer is employed.

The use of elinvar as the spring material has the disadvantage that this is a ferromagnetic material. The position and design of the permanent magnet used in the associated transducer must be proper or a non-linear force will act on the system. This accounts, for example, for the placement and configuration of the magnet in the Sprengnether-Columbia vertical.

A further trouble is encountered in seismometers that use copper vanes in the field of a permanent magnet to achieve viscous damping. Even the best electrolytic copper contains small amounts of ferromagnetic impurities which are acted on by the damping magnet, resulting in a slightly shortened period and departures from linearity. The damping vane-magnet technique has been dropped from modern designs.

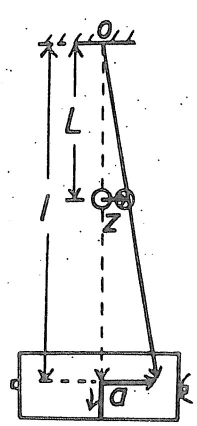
Instrument frames and the cases of portable seismographs should be stiff enough that the natural frequencies are very much higher than the frequencies to be recorded. Otherwise, strong impulsive inputs can excite parasitic vibrations that appear on the record. Lateral vibrations of the spring in a vertical seismometer excited by a high frequency input produce a typical transient on the record.

12. Mechanical-Optical (Direct recording) Seismographs

12.1. Static magnification

The direct way of using the systems discussed in the previous sections as the detector components of seismographs is to provide a means of recording a magnified version of the relative displacement, z. This displacement, as derived above, refers to the motion of the mass in a system involving translation, and to the motion of the center of oscillation in a pivoted system. In the oldest seismographs and in modern instruments in which strong earth motions are to be measured and only low magnifications are required, the magnification of the relative motion is achieved by a series of mechanical and optical levers. This magnification depends only on the geometry of the lever system and is independent of frequency. Of course the actual system magnification, from ground to record, is frequency dependent, as developed above.

If the final lever is mechanical, the arm is equipped with a device for marking the record, such as a stylus scratching on a smoked paper surface. If the final lever is optical, recording is done photographically. The form of the record may be a strip chart pulled past the recording point or a piece of paper wrapped on a drum. In the latter form of recording, either the drum or the recording lever is mounted on a helical screw, so that there is relative translation sideways of the recording point over the paper as the trace is drawn. By this scheme, a long time, say 24 hours, can be recorded on a piece of paper of convenient size and shape for handling. For normal earthquake recording chart speeds from 15 mm/min to 60 mm/min are used. Much slower speeds are used for recording earth tides, and much faster speeds for local earthquakes and explosions.

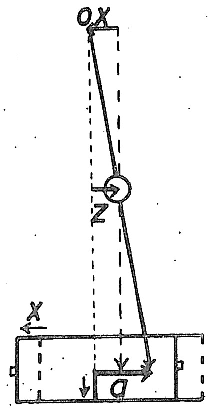


We have called the constant factor by which the lever system magnifies the relative motion z the static or geometric magnification. Suppose we consider a very simple (but rather impractical) horizontal seismograph, consisting of a simple pendulum with a long massless pointer extending from it, as shown in the figure. Then if the mass is given a static displacement z, a mark 'a' is drawn on the record, and the magnification is

$$a/z = I/L,$$

where I is the total length of the indicator.

Suppose now the frame of the same seismograph is shaken horizontally at a very high frequency, so that $|z| = x$. Taking into account the fact that the drum moves with the



earth, we see that the mark on the record is again such that

$$a/z = I/L$$

Thus, as derived earlier, we conclude that the magnification due to the geometry of the system is both the magnification of a static displacement of the mass with the frame at rest and the magnification of very high frequency ground motion. We have called this the static or geometric magnification, V_o, and it is equal to I/L in this system.

In an actual instrument, a set of levers is used that is designed to make the instrument compact. But for any instrument there is an equivalent indicator length I, and an equivalent pendulum length L, such that the static magnification is equal to I/L.

12.2. Calibration of a mechanical-optical seismograph

We have seen, Equation (7-12), that the actual magnification, V, depends on three constants of the instrument (neglecting Coulomb damping): V_o, T_n, and h. Calibration of a seismograph means determination of V as a function of fe, so that ground motion may be determined from the record. This can be done either by directly measuring the trace amplitude for a given ground motion at many frequencies, or by measuring the intrinsic constants, V_o, T_n and h and calculating the magnification from (7-12) (thereby demonstrating one's faith in the theory).

12.21. Shake-table calibration

The direct approach to calibration requires that a suitable shake-table be available. Most shake-tables in use for engineering studies are not suitable for seismograph calibration. The table must be capable of performing very small motions with displacements known to high accuracy, and must cover the frequency range of seismological interest. The low frequency, small amplitude requirement eliminates most commercially available systems. It is also necessary that the stray magnetic field at the instrument produced by the table be small. Electrodynamic systems fail in this regard, and hydraulic systems must be used. The table surface must not be ferromagnetic. Finally, the table must perform either vertical or horizontal motion with very little motion in directions other than that intended. These stringent specifications are met by only a few systems that are available for purchase.

Even if a seismometric shake-table is available, it may not be a satisfactory calibration tool in all cases. Portable and short-period observatory seismographs may be calibrated on a shake table. The method is much less satisfactory for long-period observatory instruments. It is almost impossible to clamp, move, reinstall, and unclamp a long period seismometer and end up with the same operating characteristics, especially period, as at the beginning. It is much better to calibrate long period instruments after they are in place, using another method.

The shake-table procedure offers one great advantage. Departures of the response from the idealized theoretical behavior can be detected. For example, parasitic resonances due to improper design of component parts of the seismometer will show up if measurements are made at sufficiently closely spaced frequencies. Because these resonances are usually associated with low damping, they are very sharp and a source of great trouble in recording. Because they are sharp, they may be missed if the frequencies at which calibration is done are widely spaced. If it is possible to drive the shake table with either a narrow pulse, or a step function, for which the spectra are continuous, these parasitic resonances will be found easily.

Most laboratories and observatories do not have a seismometric shake-table system, and other methods depending on the determination of the intrinsic constants, must be used. The following methods apply only to direct recording mechanical-optical instruments, and additional methods for treating electromagnetic instruments will be discussed later.

The determination of T_n and ζ is best done by observing free oscillations. If the damping can be removed, T_n is directly measured. The damping factor is calculated from the damping ratio, using (6-7a). Most direct recording instruments are damped about 0.5 to 0.7 critical, so that, large damping ratios are involved. It is possible, but not advisable, to find T_n by measuring T_d and ζ and using (6-5), because of the large damping ratio and the difficulty of measuring T_d accurately. In most mechanical instruments now in use, T_n changes very little with time, and needs to be checked only at long intervals, unless the instrument is dropped or otherwise abused.

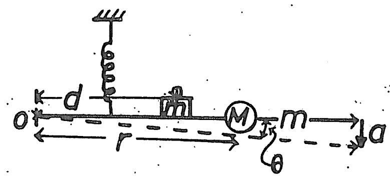
12.22. Test weight method for determining V_o

The previous discussion of seismometer theory suggests three methods of determining V_o, all static tests. The principle involved is to give a known displacement to the mass, in the case of a system with translation, or to the center of oscillation, in the case of a pivoted system.

A direct approach is to apply a known static displacement to the system by means of a micrometer screw. Then V_o is directly calculated from the resulting trace deflection. This is convenient during the final assembly of the instrument at the factory, and may be suitable for field application if the construction is such that the inertial member is easily accessible. If, for a pivoted system, the displacement is applied at any point other than the center of oscillation, a correction must be made by multiplying the applied displacement by the ratio of the reduced pendulum length to the distance from the pivot to the point displaced.

The reduced pendulum length must be known. This is best determined by removing the pendulum from the frame and supporting it so that it is free to swing as an ordinary physical pendulum. The axis of rotation, now horizontal, should be the same as when the pendulum is installed in the frame, and all elastic elements should be removed. In this configuration, the reduced pendulum length is the equivalent pendulum length, and is calculated from the natural period. While the pendulum is out of the frame, it is worthwhile to measure and record the total mass, M, and the location of the center of mass, given by r. It is advisable when ordering instruments to request these three mechanical constants from the manufacturer.

The static magnification of a pivoted system can also be determined by placing a small mass, m, on the inertial member at a known distance, d, from the axis of rotation so as to produce a static displacement. If the effective restoring moment per unit angular displacement is τ, the system will rotate through an angle θ, such that



$$mgd = \tau\theta .$$

The period of the system is

$$T_n = 2\pi\sqrt{K/\tau} ,$$

so that, eliminating τ

$$mgd = (4\pi^2 K/T_n^2)\theta$$

Dividing by Mr, and rearranging

$$T_n^2 mgd/4\pi^2 Mr = (K/Mr)\theta = \ell\theta = z$$

Thus

$$V_o = a/z = 4\pi^2 Mra/T_n^2 mgd , \qquad (12\text{-}1)$$

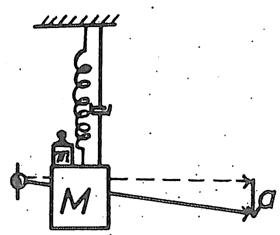
where 'a' is the trace displacement resulting from the addition of m.

This test requires knowledge of M, r, and T_n. It is especially important that T_n be carefully determined because it appears to the second power in the expression for V_o. The test can be applied to any instrument for which it is possible to produce a moment mgd by placing a test mass on the system.

Exercise: Show that the static magnification of a vertical seismograph based on a simple mass-spring system is given by

$$V_o = 4\pi^2 Ma/gT_n^2 m ,$$

where 'a' is the trace displacement produced by placing a mass m on the main mass M.



12.23. Tilt test method for determining V_o

It was shown in Section 9 that a displacement of the center of oscillation of a horizontal pendulum results from a small lateral tilt of the frame:

$$\ell\theta = (gT_n^2/4\pi^2)\psi \qquad (9\text{-}2)$$

Therefore, if the frame of the instrument is given a known, small lateral tilt ψ, and if the resulting trace displacement is 'a', the static magnification is

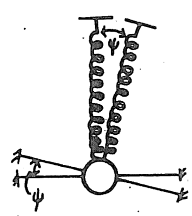
$$V_o = a/\ell\theta = (4\pi^2/gT_n^2)(a/\psi) \qquad (12\text{-}2)$$

The same result is obtained if the pendulum has elastic hinges, and for an inverted pendulum.

Exercise: Prove (12-2) is valid for an inverted pendulum.

Because, from (9-2), the displacement of the center of oscillation varies linearly with ψ, a small initial ψ (due to mislevelling) makes no difference. It is also clear that a much larger tilt is required to produce the same deflection of a short period instrument than one of long period.

The tilt test may also be applied to vertical seismographs, but the situation is more complicated. Indeed, the tilt test is also a standard method of calibrating gravity meters based on the hinged, spring-supported boom principle (Refer: Heiland, Geophysical Exploration, P. 134). Consider the simple spring-supported boom of Section 5.12, and assume that the system is exactly levelled initially, so that the pivot, the line of action of the spring, and the center of mass are in the same vertical plane. Let the restoring constant be τ, so that the natural period is $2\pi\sqrt{K/\tau}$.



Front view (greatly exaggerated)

Front view
(greatly exaggerated)

Let the frame of the instrument now be tilted laterally through an angle ψ, so that the axis of rotation makes an angle ψ with the horizontal, and the spring makes the same angle with the vertical. The gravity moment about the axis is reduced from Mgr to Mgr cos ψ. The boom must then rise, so that the spring tension is reduced by a corresponding amount. Note that the boom rises for tilting in either direction, assuming a perfectly vertical spring initially. If the initial elastic restoring moment is $\mathcal{M}$,

$$\mathcal{M} = Mgr \,.$$

After tilting

$$Mgr \cos\psi = \mathcal{M} - \tau\theta \,,$$

where θ is the angular displacement resulting from the tilt. For small tilts, $\cos\psi = 1 - \psi^2/2$, so that

$$Mgr - Mgr\,\psi^2/2 = \mathcal{M} - \tau\theta$$

Using the equilibrium condition, and replacing τ in terms of the period

$$\frac{K}{Mr}\,\theta = \frac{gT_n^2}{8\pi^2}\,\psi^2$$

$$z = \ell\theta = \frac{gT_n^2}{8\pi^2}\psi^2 \qquad (12\text{-}3)$$

Again, the known tilt results in a known displacement of the center of oscillation. Then

$$V_o = \frac{8\pi^2}{gT_n^2}\frac{a}{\psi^2} \qquad (12\text{-}4)$$

Because the displacement of the boom depends on ψ^2, the vertical seismometer is much less sensitive to tilt than the horizontal. This means a much larger tilt must be used to calibrate a vertical instrument. A further result of the second power dependence is that a small initial tilt due to mislevelling can lead to a large error in the result. This effect can be taken into account by tilting the instrument to both sides of the initial position. If an initial tilt is present, the resulting trace displacements will not be equal. Fortunately, the effect of an initial tilt can be removed by averaging the displacements produced by tilting both ways, so that

$$V_o = \frac{8\pi^2}{gT_n^2}\frac{\frac{1}{2}a(a_1 + a_2)}{\psi^2} \qquad (12\text{-}5)$$

where a_1 and a_2 are the displacements of the trace from zero for equal tilts to both sides. The test should always be done this way.

Exercise: Prove that averaging the two displacements removes the effect of initial tilt, correct to terms in the second power of the angles.

Find the initial tilt such that the displacement due to a tilt to one side is twice that due to equal tilt to the other side. How big a tilt is required to produce a displacement of 1 cm on the record of a horizontal seismograph with a period of 1 second and static magnification of 100? How big a tilt is required to produce the same displacement on the record of a vertical seismograph with the same constants?

12.3. Examples of mechanical-optical seismographs

12.31. The Wiechert horizontal seismograph (Macelw ..e, When the Earth Quakes, pp. 156-160)

One of the oldest seismographs that was widely used the early days, and which is still in use in some observatories was developed by Emil Wiechert at Gottingen. The horizontal component Wiechert was especially successful. The small Wiechert consists of an inverted pendulum with a mass of 80 kg. The center of mass is approximately 100 cm above a universal hinge. Two orthogonal components of horizontal motion are recorded from this single mass through a mechanical lever system. The natural period is adjusted to about 6 secs. Air damping, in which a piston attached to the lever system moves inside a cylinder, is employed. The damping is adjusted by means of a port at the bottom of the cylinder. Static magnification is about 100.

Recording is on smoked paper. The drum is weight-driven. Friction is quite large in this instrument due to the use of pivots on the stylus system and the contact between the stylus point and the surface of the record.

Adjustment of the Wiechert is difficult. It is especially hard to separate the two components of horizontal motion. The static magnification is measured by the test weight method.

Large Wiecherts, with masses of one to 17 tons, have also been in use since the early part of the century. The large mass is used to make effects of friction less important.

12.32. The Wood-Anderson horizontal seismograph (Anderson and Wood, Bull. Seis. Soc. Amer., 15, pp. 1-72, 1925).

At the other extreme so far as mass is concerned is the Wood-Anderson torsion seismograph. The system is a torsion pendulum (Section 5.3) with a small mass in the form of a cylinder or elongated rectangle of metal. The period is adjusted by changing the inclination angle, using the levelling screw. The original Wood-Andersons were operated with fairly long periods, but now torsion seismographs are adjusted to periods of 2 seconds or less (0.8 seconds is considered the "standard Wood-Anderson" period). The mass oscillates in the field of a permanent magnet, so that free oscillations are viscously damped by eddy currents in the mass. To eliminate violin-string vibrations of the torsion fiber, small oil-filled dampers enclose the suspension at two points.

Magnification is achieved optically. A small mirror is mounted on the mass, and a beam of light from a lamp on the recorder is reflected onto the drum. In some instruments a second mirror is mounted on the inside of the instrument case, so that the light beam is reflected from the moving mirror to this fixed mirror, back to the moving mirror and then to the drum. In this way twice the magnification is obtained with the same optical arm.

Exercise: Show that a beam of light reflected from a rotating mirror turns through twice the angle the mirror turns through.

Magnification of 1000 or more is easily achieved. The "standard Wood-Anderson" has a static magnification of 2800, and is damped 0.8 initial.

The static magnification is measured most easily by the tilt test (Section 12.23). The instrument is mounted on its base plate so that one of the three levelling screws controls lateral tilt. If the distance from this screw to the line joining the other two screws is d, and if the pitch of the screw (distance advanced per rotation) is p, a known tilt may be applied by turning this screw through a known angle. This angle may be observed by inserting a mirror in this screw and observing the deflection of a reflected scale in a telescope. If change in scale reading on a scale located a distance A_2 from this auxiliary mirror is 't', the tilt is

$$\psi = p^t / 4\pi A_2 d$$

The resulting deflection of the recording light spot can be read on a scale set up in front of the drum. Let the distance from the seismometer mirror to the drum surface be A_o and to the scale in front of the drum be A_1. If s is the deflection of the light spot across the scale corresponding to a reading 't' in the telescope, the static magnification is

$$V_o = \frac{16\pi^3}{gp} \frac{A_o A_2}{A_1 T_n^2} \frac{s}{t}$$

In modern versions of the Wood-Anderson, a lever is attached to this tilting screw that can be moved between two index marks, giving a standard tilt to the system.

12.33. Portable seismographs

Several short period portable mechanical-optical seismographs for measuring strong motions from local earthquakes or large explosions have been developed.

The Ishimoto seismograph is used as an accelerometer. Its natural period is 0.1 seconds. it employs air damping, and records through a stylus on smoked paper. The horizontal component is based on the inverted pendulum, and the vertical is a spring-supported hinged boom. Any of the methods of determining V_o, which is about 200, can be applied to this instrument because the suspended system is easily accessible. However, the tilt test is not very satisfactory for such a short period instrument. (Reference: Notes by N. Nakajima, I.I.S.E.E.).

The Sprengnether three-component portable seismograph is a mechanical-optical, photographic-recording, system with a natural period of 0.75 seconds, damped 0.55 critical. In this case also inverted pendulums are used for the horizontal components, and a spring-supported hinged boom for the vertical. Damping is by eddy-currents in a copper damping vane attached to each pendulum. The recording mirror is mounted on an axle that is turned through a bow-string and pulling arrangement by the relative motion of the pendulum. Magnification is fixed, and may be specified from 7.5 to several hundred, depending on the application. V_o is determined most easily by the tilt test, since the entire system is enclosed in a case.

The instrument is designed for recording for only a short time, about 90 seconds, so that the camera must be started just before the event to be recorded occurs. The chart speed is about 10 cm per second. The instrument acts as a displacement meter for frequencies above about 1 cps.

12.4. Non-sinusoidal ground motion: Integration of the Indicator Equation

(References: "The Determination of True Ground Motion by Integration of Strong Motion Records: A Symposium", Bull. Seis. Soc. Amer., 33: pp. 1-64, 1943, especially paper by F. Neumann; Coulomb, 1956, pp. 31-32; Kisslinger, Bull. Seis. Soc. Amer., 49: pp. 267-271, 1959)

The differential equation of the motion of the indicator of a direct recording seismograph is obtained by multiplying (7-10) or (8-4) by V_o:

$$\ddot{a} + 2\zeta\omega_n\dot{a} + \omega_n^2 a = -V_o\ddot{x} \qquad (12\text{-}6)$$

If x(t) is specified, this equation can be solved for a(t), as was done for the case of sinusoidal earth motion in deriving the steady-state simple harmonic response. Actual ground motion is much more complex, and x(t) is not known. If $\omega_e/\omega_n >> 1$ for the entire signal spectrum, as discussed in Section 7.4, the instrument can be treated as a displacement meter, and the record interpreted as a magnified copy of the ground motion. Under more general conditions, this interpretation is not valid.

A direct attack on this problem is to integrate equation (12-6) twice. Then

$$x(t) = -1/V_o \left[a(t) + 2\zeta\omega_n \int_0^t a\,dt + \omega_n^2 \int_0^t \int_0^t a\,dt\,dt - a(0) - V_o x(0) - 2\zeta\omega_n a(0)t - \dot{a}(0)t - V_o\dot{x}(0)t \right] \quad (12\text{-}7)$$

If we assume the ground and the seismometer system are at rest until t = 0, then a(0) = x(0) = 0. Further, $\dot{a}(0) = -V_o\dot{x}(0)$; i.e., at the first instant the relative velocity of the inertial system is equal to the ground velocity, but with the opposite sense. Then

$$x(t) = -1/V_o \left[a(t) + 2\zeta\omega_n \int_0^t a\,dt + \omega_n^2 \int_0^t \int_0^t a\,dt\,dt \right] \quad (12\text{-}8)$$

If a(t) is given by a seismogram, (12-8) can be mechanically or numerically integrated to give x(t). This process is simple in principle, but fraught with practical difficulties. For example, a small error in the choice of the zero position of the trace accumulates on integration, so that an initial small base line shift leads to a parabolic base line after two integrations (Neumann, 1943).

For the routine reduction of numerous seismograms, the double integration process is too time-consuming. Another approach is to assume an equation for the trace and perform the integration in (12-8) analytically. For example, Benndorf assumed the onset of the arriving wave train could be represented reasonably well by one-quarter cycle of a sine curve, i.e. $a(t) = a_m \sin \omega_e t$, for $0 \leq t \leq t_m$, where t_m is the time of the maximum on the record, which begins at t = 0. Note that this does not imply that the ground motion is sinusoidal. The period of the ground motion, $2\pi/\omega_e$, is taken as $4t_m$. The result of carrying out the integration is

$$x_m \approx x(t_m) = -a_m/V_o \left[1 + 2\zeta\,\omega_n/\omega_e + (\omega_n/\omega_e)^2 (\pi/2 - 1) \right. \quad (12\text{-}9)$$

so that Benndorf's formula for the magnification of the onset of a new phase is

$$V = \left| \frac{a_m}{x_m} \right| = \frac{V_o}{1 + 2\zeta\,\omega_n/\omega_e + (\omega_n/\omega_e)^2 (\pi/2 - 1)} \quad (12\text{-}10)$$

A discussion of the validity of Benndorf's formula is given in Kisslinger, 1959. Discussion of other representations of the seismogram trace is given by Sohon, 1932, Ch. IX.

Exercise: Derive (12-9) by putting $a(t) = a_m \sin \omega_e t$ in (12-8) and integrating to $t = \pi / 2\omega_e$.

Part II

Electromagnetic Seismographs

13. Introduction

The magnification that can be achieved through mechanical and optical levers is limited to a few thousand. To achieve higher magnification the mechanical energy represented by the motion of the inertial member relative to the frame is converted into electrical energy. With the signal in the form of a voltage, almost unlimited magnification can be achieved when needed by the use of high sensitivity recording devices, including electronic amplification. Further, the signal can be filtered selectively to reject unwanted noise components, thereby enhancing the ratio of useful information to noise. With the signal in electrical form, it can be recorded on magnetic tape for ease of later processing, and can also be converted easily on-line or subsequently to digital form for processing by a digital computer.

Any device which converts the mechanical energy of the seismometer into electrical energy is called a transducer. Several physical principles can be applied to the designs of a transducer, but the most commonly used devices are based on electromagnetic induction. Other common types of transducers are based on the change in capacitance that results when the separation of two capacitor plates changes and on the piezoelectric effect. Experimental instruments using lassers as transducers are now being tested, and other mechanical-electrical effects in materials have been proposed and tested for transducer used.

Although magnetic tape recording is now widely used for seismic prospecting and various seismological research purposes, most observatories record the transducer output by means of a galvanometer. Even when tape recording is the primary means, an auxiliary photographic record by means of a galvanometer is often made for monitoring purposes. Because of its inherent reliability and the need for costly play-back equipment for processing magnetic tape records, it is like that most observatories will continue to use galvanometric recording in the foreseeable future.

This discussion will deal primarily with electromagnetic transducers and galvanometric recording. The physical principles underlying these devices are simply stated by the laws of induction(MKS units used throughout):

1) Voltage developed in a circuit when the flux linking it changes

$$E = - d\Phi/dt \qquad (13\text{-}1)$$

where Φ is the flux in webers;

2) Electrical field across a conductor moving in a magnetic field:

$$\vec{e} = \vec{V} \times \vec{B} \qquad (13\text{-}2)$$

where $\vec{e}$ is the voltage per unit length (volts per meter);
$\vec{V}$ is the velocity of the conductor relative to the field (meters per second);
$\vec{B}$ is the flux density (webers per meter^2).

3) Force on a conductor in a magnetic field through which a current is passing:

$$\vec{F} = I\vec{\mathcal{E}} \times \vec{B} \qquad (13\text{-}3)$$

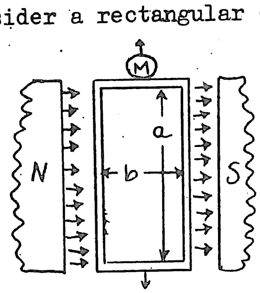
where $\vec{F}$ is the force per unit length acting on the conductor, relative to the structure that supports the source of the magnetic field (newtons per meter);
I is the current in the conductor (amperes);
$\vec{\mathcal{E}}$ is a unit vector parallel to the conductor, positive sense in direction of conventional current flow;
$\vec{B}$ is the flux density (webers per meter^2).

If a conductor of length L which is perpendicular to a uniform field $\vec{B}$ moves perpendicular to $\vec{B}$ and itself, the voltage across the ends of the conductor is $E = -BVL$. The sense of E is given directly by (13-2). This minus sign in the expression for E is usually included, as in (13-1), in recognition of Lenz's law.

If a current I is passed through the same conductor, the force acting is $F = BIL$. If the conductor is not perpendicular to the field, the length to be used is the projection perpendicular to the field, and if the velocity is not perpendicular to the plane of the field and conductor, its projection on this plane is used. These relations are readily handled as they arise by vector algebra.

14. Theory of the D'Arsonval Galvanometer (Refer: Sohon, 1932, Ch. VIII)

The D'Arsonval, or moving coil, galvanometer is the type used almost universally for seismic recording. The device consists of a coil of wire suspended in a magnetic field, with the suspension acting as a conductor through which the signal current flows. The magnet is designed so that the flux density, B, is uniform in the region in which the coil turns. Consider a rectangular coil, with average dimensions $a \times b$, with N turns.



If a steady current I passes through the coil, the force on one side is

$$F = BINa \text{ newtons,}$$

and the moment is $F \cdot b/2$.

By (13-3), the forces on the two sides are such that the moments they produce add, so the total moment is

$$2\,BINa \cdot b/2 = BINA \quad \text{newton-meters} \qquad (14\text{-}1)$$
$$= \gamma I$$

where A is the mean area of the coil, and $\gamma = BNA$ is a constant of the galvanometer, called the electrodynamic constant.

In response to this moment the coil will turn through an angle such that the torsional moment of the suspension is equal to the electromagnetic moment:

$$\tau_1 \theta = \gamma I \qquad (14\text{-}2)$$

The sensitivity of the galvanometer can be expressed by the ratio of the current I to the angle θ it produces. This ratio, which is constant over a large range of values of θ , is called the galvanometer constant, G. (See Appendix 14-A).

$$G = I/\theta = \tau_1/\gamma \qquad \text{(amperes per radian)} \qquad (14\text{-}3)$$

The sensitivity is more usually expressed by the current required to produce a deflection of one millimeter of a light spot reflected from the galvanometer mirror. If the galvanometer is installed in a box with a fixed scale, the deflection is on this scale. More generally, the deflection is taken to be on a scale one meter from the galvanometer mirror. Taking into account the fact that the reflected beam turns through twice the angle of the mirror, the current sensitivity is $g = G/2000$ amperes per millimeter on a scale at one meter.

The galvanometer is an oscillatory system, and must be analysed as such when varying curfents pass through it. It must be noted that in the construction of a galvanometer for seismographic application, it is essential that the line of the suspension pass through the center of mass of the coil, so that the galvanometer does not act as a torsion seismometer. This fequirement calls for careful control in the final assembly.

When the galvanometer coil is oscillating, the linear velocity of the side perpendicular to the flux is $(b/2)\dot{\theta}$. Each side produces an e.m.f. according to (13-2), and these add, so that the total veltage at the coil terminals is

$$\begin{aligned} E &= - BNa.\ 2(b/2)\dot{\theta} \\ &= - BNA\ \dot{\theta} \\ &= - \gamma\ \dot{\theta} \end{aligned}$$

Thus, γ has a second significance. It is both the moment per unit current and the voltage per unit angular velocity.

When the galvanometer is oscillating with the coil circuit open, the equation of motion is: $K_1 \ddot{\theta} + d\ \dot{\theta} + \tau_1\ \theta = 0$, where K_1 is the moment of inertia about the axis of rotation and d is the inherent viscous damping, usually very small.

If the coil circuit is closed through an external resistance R, the current in the circuit will be

$$i = E/(R + R_g) = - \gamma\dot{\theta}/(R + R_g) \qquad (14\text{-}5)$$

As a result of this current, a moment $\gamma i = - \gamma^2\dot{\theta}/(R + R_g)$ will act on the system, This terms has the form of a viscous damping term.

The equation of free oscillation with the circuit closed is

$$K_1\ddot{\theta} + \left\{ d + \gamma^2/(R + R_g) \right\}\dot{\theta} + \tau_1\theta = 0 \qquad (14\text{-}6)$$

or

$$\ddot{\theta} + 2h_1\omega_{n1}\dot{\theta} + \omega^2_{n1}\theta = 0$$

with

$$\omega^2_{n1} = \tau_1/K_1 \qquad (14\text{-}7)$$

$$2h_1\omega_{n1} = d/K_1 + \gamma^2/K_1(R + R_g) \qquad (14\text{-}8)$$

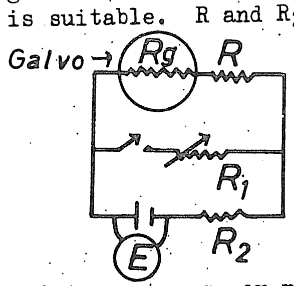
The damping is controlled by the external resistance R. The minimum damping corresponds to $R = \infty$, and is given by the open-circuit damping coefficient d. If R is small, the current i is large, and the resulting damping moment is large. For some value of R the system is critically damped. The critical external damping resistance is found as follows. The open-circuit damping is measured

$$h_o\omega_{n1} = d/2K_1$$

Appendix 14-A.

Calibration of a D'Arsonval galvanometer.

The galvanometer constant is determined by passing a small, known current through the coil. Care must be taken to protect the galvanometer against overloading, as it is easily damaged. The circuit in the figure is suitable. R and R_2 are very large compared to R_g, the internal resistance of the galvanometer, and R_1 is much smaller. The battery voltage is measured with a potentiometer, with the current flowing. Because $R_1 \ll R_2$, the current delivered by the battery is $I_2 = E/R_2$. The current in the galvanometer is $I = I_2R_1/(R_1 + R + R_g)$. G is determined by observing the deflection of the light spot. If the deflection is d on a scale at a distance L, $\theta = d/2L$, and $G = 2IL/d$. R_1, R_2, and R may be precision decade resistance boxes, or may be measured with a Wheatstone bridge.



R_g is still to be determined. This resistance should never be measured with an ohmmeter or Wheatstone bridge, as the large current way damage the galvanometer. A safe method is to use the same circuit, with the addition of a variable shunt, R_s, across the galvanometer. The shunt is adjusted until the current (deflection) is one-half the value for the same setting of R_1, without the shunt. Then

$$R_g = R_s(R_1 + R)/(R_1 + R - R_s) \approx R_s \text{ for } R \gg R_s \gg R_1.$$

(end of Appendix)

Then the damping h_1 is determined for an external resistance R_1 greater than the critical value, so that damping ratio can be measured.

$$h_1\omega_{n1} = h_o\omega_{n1} + \gamma^2/2K_1(R_1 + R_g)$$

$$\gamma^2/2k_1 = \omega_{n1}(h_1 - h_o)(R_1 + R_g)$$

If the critical external resistnace is R_x, for which $h_1 = 1$

$$\gamma^2/2K_1 = \omega_{n1}(1 - h_o)(R_x + R_g)$$

Equating and solving for R_x

$$R_x = R_1 - \frac{1 - h_1}{1 - h_o}(R_1 + R_g) \qquad (14\text{-}9)$$

Exercise: Knowing R_x, calculate h_1, for any other value of external resistance, R_1.

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exercise: Suppose you have measured T_{n1}, G, R_x, and h_o. Show how you can use these to derive the intrinsic constants of the galvanometer: γ, τ, K_1, and d.

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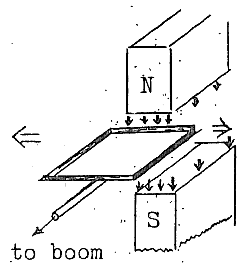
15. The moving-coil (Galitzin) transducer.

An Important advance in seismometry occurred about 1906 when B. Galitzin successfully mounted an electromagnetic transducer on a seismometer, and recorded the output galvanometrically. The form of this transducer, one still widely used, is a permanent magnet mounted to the frame, with a coil system mounted on the pendulum so that the coil moves in the gap of the magnet. The Galitzin and Wilip Galitzin instruments actually used four rectangular coils and two permanent magnets (see Figure 17, Coulomb, 1956). With a rectangular coil in this configuration, only one side of the coil must be in the field of the magnet; otherwise the e,m.f.'s developed in the two parallel sides would be equal and opposite.



The following analysis of the transducer action in a pivoted system follows M.T. Antune, "Les Sismographes Electromagnetiques et L'Enregistrement Conforme des Mouvements du Sol", Boleti , ia Sociedade Portuguesa de Cienias Naturais, Vol. III, a. A serie (Vol. XVIII), Fasc. 1, pp. 66-84, 1950.

Let $d\vec{s}$ be an element of length of the conductor in the coil. Its orientation is arbitrary, although in practice it is usually perpendicular to both the axis of rotation and the magnetic flux. Suppose the angular speed of the boom is $d\phi/dt = \dot{\phi}$ (Note: From this point on, ϕ will be used for the angular displacement of the seismometer, θ for the galvanometer). The velocity of the element is

$$\vec{V} = (\vec{k} \times \vec{R})\dot{\phi}$$

where $\vec{k}$ is a unit vector parallel to the axis of rotation, $\dot{\phi}\vec{k}$ is the angular velocity, and $\vec{R}$ is the position vector from the pivot to $d\vec{s}$.

By (13-2), the resulting electric field is

$$\vec{e} = \vec{V} \times \vec{B} = \dot{\phi}(\vec{k} \times \vec{R}) \times \vec{B}$$
$$= \dot{\phi}\,[\vec{R}\,\vec{B}\cdot\vec{k} - \vec{k}\,\vec{B}\cdot\vec{R}]$$

(Note: In usual arrangement, $\vec{R}\perp\vec{B}$, $\vec{B}\parallel\vec{k}$ so that $\vec{e} = Br\dot{\phi}\,\vec{e_r}$, where $\vec{R} = r\vec{e_r}$)

The voltage across $d\vec{s}$ is $\vec{e}\cdot d\vec{s}$, so that the total voltage across the terminals of the coil is

$$E = \oint_{coils} \vec{e}\cdot d\vec{s} = \dot{\phi}\oint_{coils}\left[\vec{B}\cdot\vec{k}\,\vec{R}\cdot d\vec{s} - \vec{B}\cdot\vec{R}\cdot\vec{k}\,ds\right]$$

The integral is a constant of the instrument, depending only on the geometry of the pendulum and the coil, and the flux density in the gap of the magnet. This constant is the electrodynamic constant, Γ, of the seismeter, and

$$E = \Gamma\dot{\phi} \qquad (15\text{-}1)$$

is the voltage developed by the transducer.

Exercise: In the Galitzin transducer, $\vec{B}\parallel\vec{k}$, and on the average over the coil, $\vec{R}\perp\vec{B}$. One side of each coil, approximately prallel to R, of length s, is in the gap of the magnet. If the flux in the gap is uniform and each of the four rectangular coils has N turns, show

$$\Gamma = 4BNsL_c ,$$

where L_c is the distance from the pivot to the center of the coil system. This is the result given on p. 85 of Sohon. The minus sign there is quite arbitrary.

If the coil terminals are closed through an external circuit (that will include the galvanometer eventually), a current I will flow. As soon as I flows, a froce is exerted on each coil element, according to (13-3):

$$I d\vec{s} \times \vec{B}$$

The moment of this force about the pivot is

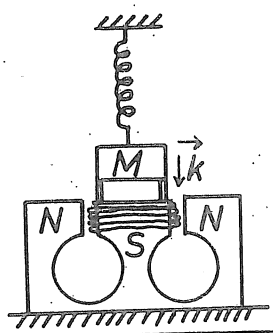
$$d\vec{M} = \vec{R}\times(I d\vec{s}\times\vec{B}) = I\left[\vec{B}\cdot\vec{R}\,d\vec{s} - \vec{R}\cdot d\vec{s}\,\vec{B}\right]$$

The moment about the axis of rotation is $d\vec{M}\cdot\vec{k}$, so that the total moment about the axis of rotation fesulting from the current in the coil is

$$M = \oint_{coils} d\vec{M}\cdot\vec{k} = I\oint\left[\vec{B}\cdot\vec{R}\,\vec{k}\cdot d\vec{s} - \vec{k}\cdot\vec{B}\,\vec{R}\cdot d\vec{s}\right]$$
$$= -\Gamma I \qquad (15\text{-}2)$$

The electrodynamic constant, therefore, is both the voltage per unit angular speed and the (negative) moment per unit current. The minus sign in (15-2) simply means that the moment is such as to oppose the displacement that gave rise to the current.

Exercise: Show that for a seismometer based on a system with translation, as in the figure, $\Gamma_T = \oint_{coil}\vec{k}\times\vec{B}\cdot d\vec{s}$, where the output voltage is $E = \Gamma_T\dot{z}$, $\vec{k}$ is a unit vector in the direction of translation, and $\dot{z}\vec{k}$ is the relative velocity of the coil with respect to the magnet. If the coil is circular



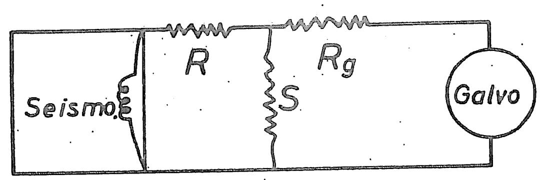
with radius r and the field is radial and uniform over the coil, show that

$$\Gamma_T = 2\pi rNB.$$

16. The electromagnetic seismograph: equation of motion

(References: Wenner, Research Paper No. 66, U.S. Bureau of Standards Jour. of Res., Vol. II 1929, pp. 963-999; Coulomb and Grenet, Annales de Pysique, 11[e] serie, Tome 3, 1935; Grenet, Annales de Geophysique, Tome 8, 1952; Eaton, J., "Theory of the Electromagnetic Seismograph", Bull. Seis. Soc. Amer., Vol. 47, 1957, pp. 37-75)

Suppose the transducer and galvanometer are now connected through a resistive network, as shown in the figure. R is the total resistance



in the seismometer side of the circuit, including the internal resistance of the coil, and Rg has the same meaning for the galvanometer branch of the circuit. S is a shunt resistor. This circuit is quite general, and allows for the insertion of damping resistors and attenuators between the seismometer and the galvanometer. Call the current in the seismometer coil I and in the galvanometer coil i. Then the equations of motion of the two components, for a pendulum-type seismometer with mechanical damping coefficient D and restoring constant τ, are

Seismometer: (using (15-2))

$$K_1\ddot{\phi} + D\dot{\phi} + \tau\phi = -Mr\ddot{x} - \Gamma I \qquad (16-1)$$

Galvanometer: (using (14-1) and assuming the galvanometer does not respond as a seismometer)

$$K_0\ddot{\theta} + d\dot{\theta} + \tau_1\theta = \gamma i \qquad (16-2)$$

Applying Kirchoff's laws to get the distribution of current in the elements and summing the voltages in each loop, using (15-1) and (14-4)

$$\Gamma\dot{\phi} - IR - (I - i)S = 0$$

$$\gamma\dot{\theta} + iRg + (i - I)S = 0$$

Solving for I and i

$$I = \frac{(R_g + S)\Gamma\dot{\phi} - \gamma S\dot{\theta}}{RR_g + RS + R_gS}$$

$$i = \frac{-(R + S)\gamma\dot{\theta} + \Gamma S\dot{\phi}}{RR_g + RS + R_gS}$$

Following Wenner, we let $RR_g + RS + R_gS = Q^2$. The moments acting on the two components as a result of these currents is

Seismometer: $$M = -\Gamma I = -\frac{R_g + S}{Q^2}\Gamma^2\dot{\phi} + \frac{S}{Q^2}\Gamma\gamma\dot{\theta}$$

Galvanometer: $$M_1 = \gamma i = -\frac{R + S}{Q^2}\gamma^2 + \frac{S}{Q^2}\Gamma\gamma\dot{\phi}$$

Inserting these in (16-1) and (16-2), we obtain the equations of motion.

Seismometer: $$K\ddot{\phi} + \left(D + \frac{R_g + S}{Q^2}\Gamma^2\right)\dot{\phi} + \tau\phi = -Mr\ddot{x} + \frac{S}{Q^2}\Gamma\gamma\dot{\theta}$$

or $$\ddot{\phi} + 2h\omega_n\dot{\phi} + \omega_n^2\phi = -\frac{\ddot{x}}{l} + \frac{S}{KQ^2}\Gamma\gamma\dot{\theta} \qquad (16\text{-}3)$$

where $$2h\omega_n = D/K + (R_g + S)/KQ^2\Gamma^2$$

$$\omega_n^2 = \tau/K$$

Galvanometer: $$K_1\ddot{\theta} + \left(d + \frac{R + S}{Q^2}\gamma^2\right)\dot{\theta} + \tau_1\theta = \frac{S}{Q^2}\Gamma\gamma\dot{\phi}$$

or $$\ddot{\theta} + 2h_1\omega_{n1}\dot{\theta} + \omega_{n1}^2\theta = \frac{S\Gamma\gamma}{Q^2K_1}\dot{\phi} = k\dot{\phi} \qquad (16\text{-}4)$$

as in (14-6) - (14-8).

These are the general operating equations of the Galitzin-type seismograph. Taken together, they enable us to go from the ground motion, x, to the galvanometer deflection, Θ.

Exercise: Let S go to infinity (no shunt) in (16-3) and (16-4) and write the equations for a directly connected seismometer-galvanometer system.

The right-hand side of (16-4) shows how the galvanometer is driven by the seismometer. The coefficient of ϕ was named the transfer factor, k , by Galitzin (originally for $S = \infty$). We note, however, that an analogous term appears on the right-hand side of (16-3), a term that shows how the oscillating galvanometer drives the seismometer. This effect has been called the galvanometer "reaction", and will appear subsequently in <u>the coupling factor</u> of the system.

If K is made large and K_1 small, as in many systems, the transfer factor can be large and the reaction small. Galitzin assumed the reaction was negligible, but we shall investigate this question more carefully.

We see also that the damping of both components can be controlled by the resistance in the circuit. In older designs, D was large, due to a damping vane, and external damping predominated in the seismometer. In modern designs, the damping vane is eliminated, and damping both components is due primarily to the currents in the circuits.

Adopting the notation of Coulomb and Grenet, we write the total damping coefficient of the pendulum as

$$\Delta = D + \frac{S + Rg}{Q^2}\Gamma^2; \quad 2h\omega_n = \Delta/K;$$

and of the galvanometer as $\delta = d + \frac{R + S}{Q^2}\gamma^2, \quad 2h_1\omega_{n1} = \delta/K_1$

Now define $\sigma_s = \Gamma\gamma S/\Delta Q^2$; so that $\Gamma\gamma S/KQ^2 = 2h\omega_n\sigma_s$; (16-6)

and $\sigma_g = \Gamma\gamma S/\delta Q^2$, so that $\Gamma\gamma S/K_1Q^2 = 2h_1\omega_{n1}\sigma_g = k$, the " transfer factor" (16-7)

The equations of motion become

Seismometer: $\ddot{\phi} + 2h\omega_n\dot{\phi} + \omega_n^2\phi = -\ddot{x}/\ell + 2h\omega_n\sigma_s\theta$ (16-8)

Galvanometer: $\ddot{\theta} + 2h_1\omega_{n1}\dot{\theta} + \omega_{n1}^2\theta = 2h_1\omega_{n1}\sigma_g\dot{\phi}$ (16-9)

We eliminate ϕ between these equations by differentiating (16-8) one time, solving (16-9) for $\dot{\phi}$, and differentiating to get $\ddot{\phi}$ and $\dddot{\phi}$,and substituting. Then multiplying through by $2h_1\omega_{n1}\sigma_g$ and collecting terms, we get the equation for the galvanometer in terms of ground motion.

$$\theta^{(IV)} + (2h_1\omega_{n1} + 2h\omega_n)\dddot{\theta} + (\omega_{n1}^2 + 4hh_1\omega_n\omega_{n1} + \omega_n^2 - 4hh_1\omega_n\omega_{n1}\sigma_s\sigma_g)\ddot{\theta} + \ldots$$

$$\ldots\ldots + (2h\omega_n\omega_{n1}^2 + 2h_1\omega_n^2\omega_{n1})\dot{\theta} + \omega_n^2\omega_{n1}^2\theta = -\frac{2h_1\omega_{n1}\sigma_g}{\ell}\dddot{x} \quad (16\text{-}10)$$

Since the trace is given by $2A_o\theta$, where A_o is the distance from the galvanometer mirror to the record surface, (16-10) when multiplied by

$2A_o$ is the differential equation for the indicator. Letting multiplication of differentiating operators mean successive differentiation, (16-10) can be factored

$$\left\{\left[\frac{d^2}{dt^2} + 2h\omega_n \frac{d}{dt} + \omega_n^2\right]\left[\frac{d^2}{dt^2} + 2h_1\omega_{n1}\frac{d}{dt} + \omega_{n1}^2\right] - 4hh_1\omega_n\omega_{n1}\sigma^2 \frac{d^2}{dt^2}\right\}\theta$$

$$= - \frac{k}{\ell}\ddot{X} \qquad (16\text{-}11)$$

where $\sigma^2 = \sigma_s\sigma_g$. The only term in which the coupling between the seismometer and galvanometer, or the "reaction term" appears in the indicator equation is the term involving σ^2. σ^2 is called <u>the coupling factor</u>.

The steady-state response to simple harmonic ground motion is found by putting $x = x_o e^{i\omega_e t}$, letting $\theta = \theta_o e^{i\omega_e t}$ and finding θ_o so that this is a solution. θ_o will be a complex number, the modulus of which gives the amplitude response, the argument of which gives the phase response. The result, after some tedious algebra is

$$\frac{\ell\theta_o}{X_o} = \frac{i\, 2h_1\sigma_g \frac{\omega_o}{\omega_n}}{\left[\frac{\omega_e^2}{\omega_n\omega_{n1}} - \frac{\omega_n}{\omega_{n1}} - \frac{\omega_{n1}}{\omega_n} + \frac{\omega_n\omega_{n1}}{\omega_e^2} - 4hh_1(1-\sigma^2)\right] + i\left[2h_1\left(\frac{\omega_n}{\omega_e} - \frac{\omega_e}{\omega_n}\right) + 2h\left(\frac{\omega_{n1}}{\omega_e} - \frac{\omega_e}{\omega_{n1}}\right)\right]} \qquad (16\text{-}12)$$

$$= \frac{i\, 2h_1\omega_{n1}\sigma_g\omega_e^3}{\left[(\omega_e^2 - \omega_n^2)(\omega_e^2 - \omega_{n1}^2) - 4hh_1\omega_n\omega_{n1}(1-\sigma^2)\omega_e^2\right] + i2\omega_e\left[h_1\omega_{n1}(\omega_n^2 - \omega_e^2) + h\omega_n(\omega_{n1}^2 - \omega_o^2)\right]} \qquad (16\text{-}13)$$

Multiplying by the optical magnification, $2A_o$, the system response, from ground motion to record, is

$$\frac{a_o}{x_o} = \frac{2A_o\left(\frac{k}{\ell}\right)\omega_e^3\, e^{i\left(\frac{\pi}{2} - 4\right)}}{\left\{ [R]^2 + [I]^2 \right\}^4} \qquad (16\text{-}14)$$

where R and I are the real and imaginary parts of the denominator of (16-13), and $\varphi = \tan^{-1} I/R$. The contribution of $+\pi/2$ to the phase shift comes from the differentiation of the relative motion by the transducer.

Hagiwara (Bull. Earthquake Research Institute, vol. 36, Part 2, 1958, 139-164) has shown that (16-14) can be written in a form convenient for calibration of the system as the product of two factors, one involving only the system constants, the other the frequency dependence.

$$\frac{a_m}{x_m} = w \cdot f \qquad (16\text{-}14a)$$

where $W = Mr \left(\frac{2\pi}{T_n}\right) \left(\frac{2\pi}{T_{n1}}\right)^2 \cdot S_1 S_2 \mu_1 \frac{1}{Z_{11}}$, with

S_1, the sensitivity of the seismometer as a galvanometer

$$S_1 = \frac{\phi}{I} = \frac{\Gamma}{\tau} \doteq \frac{\Gamma T_n^2}{4\pi^2 K};$$

S_2, the sensitivity of the galvanometer, including the optical magnification

$$S_2 = 2A_o \frac{\theta}{i} = 2A_o \frac{\gamma}{\tau_1} = \frac{2A_o \gamma T_{n1}^2}{4\pi^2 K_1}$$

Z_{11}, the resistance into which the transducer works, R in series with S and R_g in parallel,

$$Z_{11} = \frac{Q^2}{S + R_g}$$

μ_1, the attenuation factor, $\frac{I_g}{I}$, with the galvanometer clamped

$$\mu_1 = \frac{S}{R_g + S}$$

Thus, $W = \dfrac{k}{\ell\omega_n} = \dfrac{2h_1 \omega_{n1} \sigma_g}{\ell\omega_n}$

The factor f is $\dfrac{\omega_e / \omega_{n1}}{F(\omega_e)}$, where $F(\omega_e)$ is the modulus of the denominator of (16-12).

Exercise: Write out the modulus and argument of the denominator of (16-13). Write the amplitude and phase responses of the system.

Exercise: From (16-12), write the velocity sensitivity, $\theta/\dot{x}$, of the system. Compare with equation (18) in Willmore.

Exercise: Find the amplitude response for $\sigma^2 = 1$, the case of tight coupling.

We note that the reaction of the galvanometer on the seismometer appears only in the coupling factor σ^2. The importance of this reaction depends on the magnitude of σ^2 relative to 1. σ^2 can only be zero if σ_s is zero. We evaluate this factor now. Let the part of the damping factor that depends on the current in the circuit be h', so that $h = h_o + h'$. Then, for the seismometer

$$2h\omega_n = 2\,(h_o + h')\,\omega_n = D/K + (S + R_g)/KQ^2\Gamma^2$$

Equating the expressions for the damping due to current, and solving for $2K\omega_n$

$$2K\omega_n = \Gamma^2(S + Rg)/h'Q^2$$

Then, from (16-6)
$$\sigma_s = \frac{\Gamma\gamma S}{Q^2}\cdot\frac{1}{2h\omega_n K} = \frac{\Gamma\gamma S}{Q^2}\cdot\frac{1}{h}\cdot\frac{h' Q^2}{\Gamma^2(S+Rg)}$$

$$= \frac{h'}{h}\cdot\frac{\gamma}{\Gamma}\cdot\left(\frac{S}{S + Rg}\right) \qquad (16\text{-}14a)$$

A parallel derivation for the galvanometer gives

$$\sigma_g = \frac{h_1'}{h_1}\cdot\frac{\Gamma}{\gamma}\cdot\left(\frac{S}{S + R}\right) \qquad (16\text{-}14b)$$

then
$$\sigma^2 = \sigma_s\sigma_g = \frac{h' h_1'}{h h_1}\;\frac{S^2}{(S + Rg)\,(S + R)}$$

The coupling factor has the maximum value 1 if S is infinite (no shunt) and all the damping of both components is due to current in the circuit. σ^2 can be made small by making h' or h_1' small, or by making the shunt resistnance small. If the seismometer damping is achieved by a damping vane, so that $h \approx h_o$, with K very large, h' will be small. On the other hand, there is an advantage in eliminating the damping vane, and controlling the damping of the seismometer by the resistance it works into. In that case, the coupling is made small by making S small compared to R and Rg. This measure also reduces σ_g, so that the sensitivity of the system is lower than for a directly coupled system. The electrodynamic constant of the seismometer transducer must then be big enough to yield the necessary sensitivity. A system in which the coupling factor is made small by the use of a shunt is said to be <u>decoupled</u>, or loosely coupled.

The system response, Equation (16-14), is symmetric in the constants of the seismometer and galvanometer. A seismograph based on a deismometer with a given damping and period, and a galvanometer with a given damping and period can be replaced by one in which these constants are interchanged.

17. Seismographs with zero coupling.

If $\sigma^2 = 0$ in (16-14), the magnification is given by

$$\frac{\ell a_o}{2A_o k x_o} = \frac{\omega_e^3}{\left\{\left[(\omega_e^2-\omega_n^2)(\omega_e^2-\omega_{n1}^2) - 4hh_1\omega_n\omega_{n1}\omega_e^2\right]^2 + 4\omega_e^2\left[h_1\omega_{n1}(\omega_n^2-\omega_e^2)+h\omega_n(\omega_{n1}^2-\omega_e^2)\right]^2\right\}^{1/2}} \qquad (17\text{-}1)$$

Factoring ω_e^4 out of the denominator, expanding, regrouping, and factoring

$$\frac{\ell a_o}{2A_o k x_o} = \frac{1}{\omega_e\left\{\left[\left(1-\frac{\omega_n}{\omega_e}\right)^2\right)^2 + \left(2h\frac{\omega_n}{\omega_e}\right)^2\right]\left[\left(1-\frac{\omega_{n1}}{\omega_e}\right)^2\right)^2 + \left(2h_1\frac{\omega_{n1}}{\omega_e}\right)^2\right]\right\}^{1/2}} \qquad (17\text{-}2)$$

$$\frac{\pi\ell a_o}{A_o k x_o} = \frac{T_e}{\left\{\left[\left(1-\left(\frac{T_e}{T_n}\right)^2\right)^2 + \left(2h\frac{T_e}{T_n}\right)^2\right]\left[\left(1-\left(\frac{T_e}{T_{n1}}\right)^2\right)^2 + \left(2h_1\frac{T_e}{T_{n1}}\right)^2\right]\right\}^{1/2}} \qquad (17\text{-}3)$$

$$= \frac{T_e}{U\,U_1} \qquad (17\text{-}4)$$

Exercise: Show that the phase lag for $\sigma^2 = 0$ is

$$\alpha = (\delta + \delta_1 - \pi/2) \qquad (17\text{-}5)$$

where $\theta_o = |\theta_o|e^{-i\alpha}$, $\theta = \theta_o e^{i\omega_e t} = |\theta_o| e^{i(\omega_e t - \alpha)}$

$$\delta = \tan^{-1}\frac{2h\frac{T_e}{T_n}}{\left(\frac{T_e}{T_n}\right)^2 - 1}$$

$$\delta_1 = \tan^{-1}\frac{2h\frac{T_e}{T_n}}{\left(\frac{T_e}{T_{n1}}\right)^2 - 1}$$

As with the direct-recording seismograph, there is an arbitrary factor of ± 1, corresponding to an arbitrary shift by π, in the specification of the phase, because the leads to the galvanometer can be connected in one of two ways. The result above agrees with the convention used in the World Wide Standard Seismograph network, for which the phase lag is written as $3\pi/2$ for $T_e = 0$ and $-\pi/2$ (a phase lead) for $T_e \to \infty$.

The phase relations in the electromagnetic seismograph with $\sigma^2 = 0$ can be derived directly from the equations of forced motion. If the ground motion is $x_0 \sin \omega_e t$, the relative motion of the pendulum from (7-11) is $\phi_0(\omega_e) \sin(\omega_e t - \delta)$, where

$$\tan \delta = \frac{2\zeta \omega_e/\omega_n}{1 - (\omega_e/\omega_n)^2}$$

The voltage driving the galvanometer is $\Gamma\dot{\phi} = \Gamma\omega_e \phi_0(\omega_e) \sin(\omega_e t + \frac{\pi}{2} - \delta)$.

The driving moment is proportional to this voltage. The response of the galvanometer is then

$$\theta = \theta_0(\omega_e) \sin(\omega_e t + \pi/2 - \delta - \delta_1),$$
$$= \theta_0(\omega_e) \sin\left[\omega_e t - (\delta + \delta_1 - \pi/2)\right],$$

where $\theta_0(\omega_e)$ is given by (16-14), and δ_1 has the form analogous to δ. See the figure, page 71.

Exercise: For $T_n = 30$ secs, $\zeta = 1.5$, $T_{n1} = 100$ secs, $\zeta_1 = 1$, $\sigma^2 = 0$, find the phase lag for $T_e = 1$, 30, 100, and 200 seconds. Sketch the phase response curve for this system.

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17. 1 The Galitzin adjustment.

The original Galitzin electromagnetic seismograph was designed with $\zeta = \zeta_1 = 1$, $T_n = T_{n1}$, $\sigma^2 = 0$. In this case

$$U = U_1 = 1 + \left(\frac{T_e}{T_n}\right)^2$$

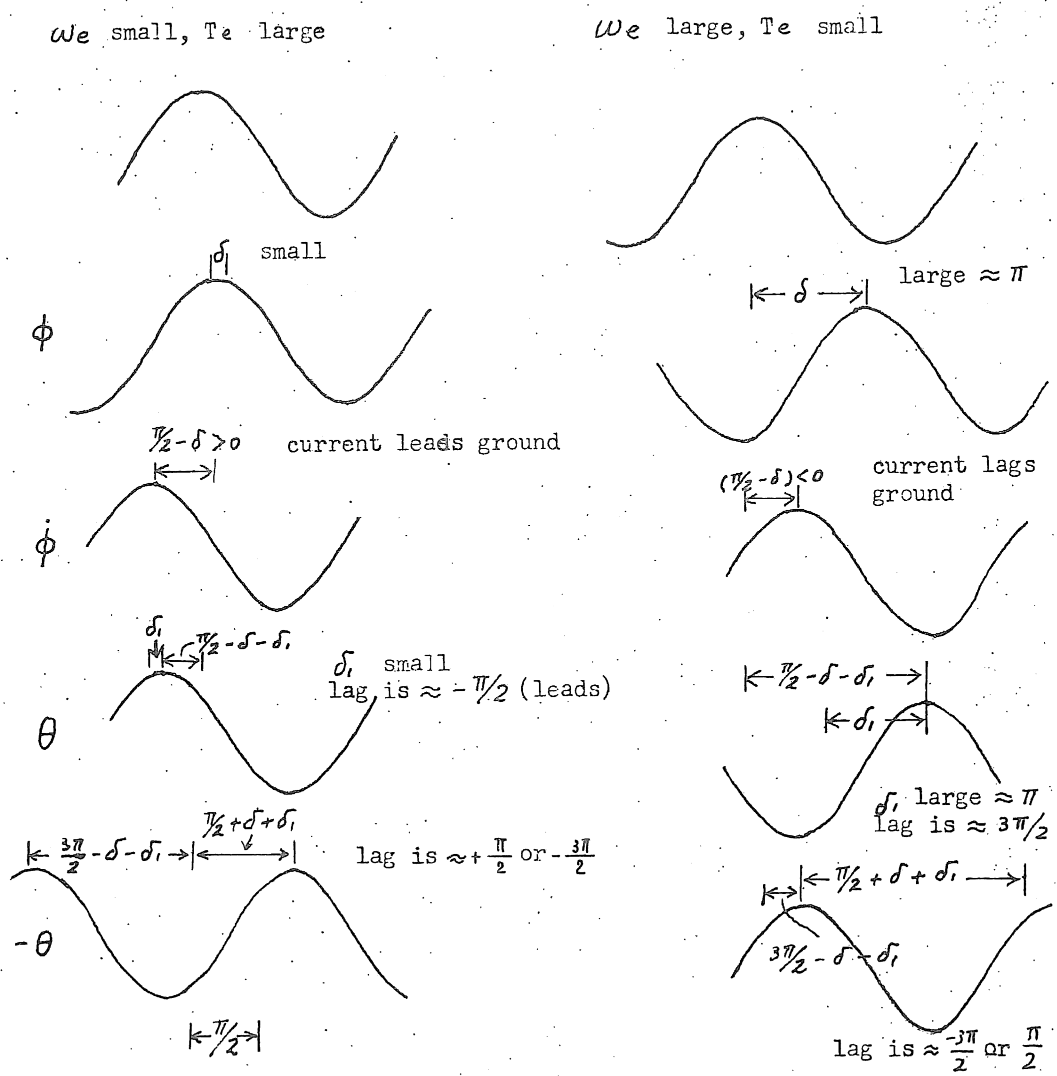
The magnification becomes

$$V = \frac{a_0}{x_0} = \left(\frac{A_0 K}{\pi \ell}\right)\left(\frac{T_e}{\left[1 + \left(\frac{T_e}{T_n}\right)^2\right]^2}\right) = \frac{A_0 K T_n}{\pi \ell} \frac{T_e/T_n}{\left[1 + (T_e/T_n)^2\right]^2} \qquad (17\text{-}6)$$

Exercise: For the ideal Galitzin adjustment

1) Show that $V \to 0$ for both $T_e \to 0$ and $T_e \to \infty$, so that noth high frequency noise and the effects of slow disturbances, such as tilts, are eliminated.

Phase lag relations in electromagnetic seismograph with $\sigma^2 = 0$



$$x = x_0 \exp(i\,\omega_e t)$$

$$\theta = \theta_0(Te)\exp\{i(\omega_e t - \alpha)\}$$

$$\tan\delta = \frac{2h\,Te/Tn}{(Te/Tn)^2 - 1} \quad ; \quad \tan\delta_1 = \frac{2h_1\,Te/Tn_1}{(Te/Tn_1)^2 - 1}$$

If use θ , d = ($\delta + \delta_1 = -\pi/2$), α varies from $3\pi/2$ at Te = 0 to $-\pi/2$ as Te $\rightarrow \infty$ (WWSSN Manual) $0 \leq \delta, \; \delta_1 \leq \pi$

If use $-\theta$, $\alpha = (\delta + \delta_1 + \pi/2)$, but $0 \geq \delta, \delta_1 \geq -\pi$, α varies from $+\pi/2$ at Te = 0 to $-3\pi/2$ as Te $\rightarrow \infty$. (Coulomb any many others).

2) Find the period for which the magnification is a maximum, and evaluate this peak magnification.

3) Find the magnification for $T_e = T_n$, and compare with the peak magnification.

4) Find the phase angle of the Galitzin at $T_e/T_n = 1$, and find the value of T_e for which the phase shift is zero. The result is the same as Fig. 18 in Coulomb, if π is substracted from the values for the phase lag found here (He chose to multiply by -1, p. 58).

17.2 Approximate response curves for heavily damped systems, $\sigma^2 = 0$. (Reference: P.L. Willmore, "Some Properties of Heavily Damped Electromagnetic Seismographs" Geophys. Jour. of Roy. Astro. Soc., Vol. 4, 1961, pp. 389-404.)

We shall take advantage of the merits of working with velocity sensitivity rather than magnification, and the log-log presentation discussed on page 35. From (16-12), since $(x)_{max} = i\omega_e x_o$, the velocity sensitivity with $\sigma^2 = 0$, can be written

$$\theta_o/(\dot{x})_m = -(2h_1 \sigma_g/\ell\omega_n)\left[2h_1 + i(\omega_e/\omega_{n1} - \omega_{n1}/\omega_e)\right]^{-1} \cdot \left[2h + i(\omega_e/\omega_n - \omega_n/\omega_e)\right]^{-1}$$

$$= -\frac{\gamma\Gamma S}{\ell\omega_n\omega_{n_1}Q^2K_1} \cdot \frac{1}{\left[2h + i\left(\frac{\omega_e}{\omega_n} - \frac{\omega_n}{\omega_e}\right)\right]\left[2h_1 + i\left(\frac{\omega_e}{\omega_{n1}} - \frac{\omega_{n1}}{\omega_e}\right)\right]} \quad (17\text{-}7)$$

Using $\ell = k/Mr$,

$$\frac{\theta_o}{(\dot{x})_m} = \frac{-\gamma\Gamma SMr}{Q^2} \; \frac{1}{K\omega_n\left[2h + i\left(\frac{\omega_e}{\omega_n} - \frac{\omega_n}{\omega_e}\right)\right]} \cdot \frac{1}{K_1\omega_{n1}\left[2h_1 + i\left(\frac{\omega_e}{\omega_{n1}} - \frac{\omega_{n1}}{\omega_e}\right)\right]}$$

(17-8)

Now consider the motion of a rotational system (not necessarily a seismometer or galvanometer driven by a moment $M_o e^{i\omega_e t}$.

$$K\ddot{\theta} + \Delta\dot{\theta} + \tau\theta = M_o e^{i\omega_e t}$$

$$\theta = \theta_o e^{i\omega_e t}, \text{ where}$$

$$\theta_o = M_o/\left[(\tau - \omega_e^2 K) + i\omega_e\Delta\right]$$

$$(\dot{\theta})_m = i\omega_e\theta_o = M_o/\left[\Delta + i\left[\omega_e K - \tau/\omega_e\right]\right]$$

$$M_o/(\dot{\theta})_m = \Delta + i\left[\omega_e K - \tau/\omega_e\right] \quad (17\text{-}9)$$

$$= 2h\omega_n K + i\left[\omega_e K - K\omega_n^2/\omega_e\right]$$

$$= K\omega_n\left[2h + i(\omega_e/\omega_n - \omega_n/\omega_e)\right] \quad (17\text{-}10)$$

The ratio of the driving moment to the angular velocity it produces, (17-9) is called <u>the mechanical impedance</u> of the system:

$$Z_{mech} = M_o/(\dot{\theta})_m$$

The response of any simple, viscously damped oscillator can be expressed in the form

$$\dot{\theta}_m = M_o/Z_{mech}$$

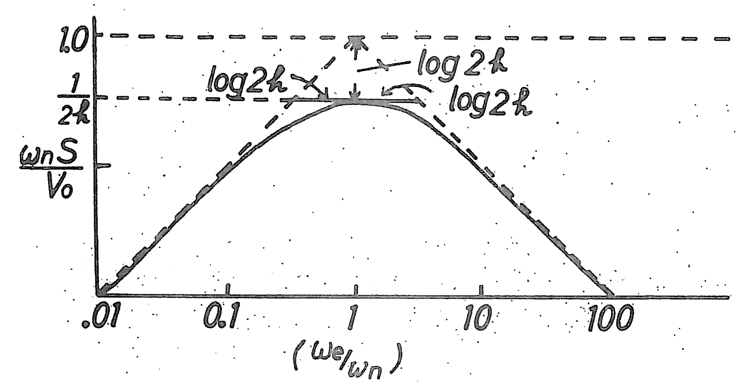
$\dot{\theta}_m$ is a complex number that includes the phase information.

Exercise: Derive the mechanical impedance of a mass-spring-dashpot system driven by $f(t) = F_o e^{i\omega_e t}$.

Thus, the velocity response of the seismograph, (17-8), can be written as the product of the velocity responses of two simple oscillators:

$$\frac{\theta_o}{(\dot{x})_m} = \frac{-Mr}{Z_{seis}} \cdot \frac{\gamma \Gamma S}{Q^2 Z_{galv}} \qquad (17\text{-}11)$$

From page 35, we know that the velocity sensitivity always peaks at $\omega_e/\omega_n = 1$, is symmetric about this value of ω_e/ω_n, the peak value is $\omega_n S_p/V_o = 1/2h$, and the curve is asymptotic to a line with slope + 1 at low frequencies, -1 at high frequencies.



From the figure, for $h > 0.5$, a horizontal line through the peak response cuts the asymptotes at two values of the frequency ration, $1/2h$ and $2h$. Compare Fig. 2 in Willmore, 1960. If the damping is high, say critical or greater, the response can be reasonably well approximated by three straight line segments. The velocity sensitivity along the segment with slope +1, and the magnification is constant along the segment with slope -1. Because of the symmetry of the velocity response the shape of the curve is unchanged if T_e/T_n is used instead of the frequency ratio. However, in this case, the curves of constant magnification and acceleration sensitivity are reversed, i.e., magnification is constant along the curve with slope + 1. (The abscissae on Fig. 3, Willmore, 1961, are mislabelled, should be relative period.)

We can now apply this approach to (17-11), and use the fact that algebraic multiplication is equivalent to adding logarithms of the factors.

Given the period and damping (assumed fairly heavy) of both the seismometer and galvanometer, we plot approximate response curves for each, and then simply add them, graphically, to get the system response. The factors involving the constants of the system (M, r, Γ, etc.) only affect the level of the response, not the shape. They must be adjusted, of course, to get the desired peak magnification.

If it is desired to estimate the numerical value of the response from given system constants, (17-11) must first be multiplied by $2A_o$, and, using $a_m = 2A_o\theta_m$, the velocity response, as recorded, is

$$\frac{a_m}{\dot{x}_m} = - \frac{Mr}{Z_{seis}} \quad \frac{2A_o\gamma\Gamma S}{Q^2 \cdot Z_{galv}} \qquad (17\text{-}11a)$$

Rewriting (17-10)

$$\dot{\theta}_m = \frac{M_o}{Z_{mech}} = \frac{M_o}{K\omega_n} \quad \frac{1}{2h + i\left(\frac{\omega_e}{\omega_n} - \frac{\omega_n}{\omega_e}\right)}$$

$$= \frac{M_o}{K\omega_n} \quad \frac{\frac{\omega_e}{\omega_n}\exp\left\{- i2h\frac{\omega_e}{\omega_n} / \left[1 - \left(\frac{\omega_e}{\omega_n}\right)^2\right]\right\}}{\left\{\left(2h\frac{\omega_e}{\omega_n}\right)^2 + \left[\left(\frac{\omega_e}{\omega_n}\right)^2 - 1\right]^2\right\}^{1/2}}$$

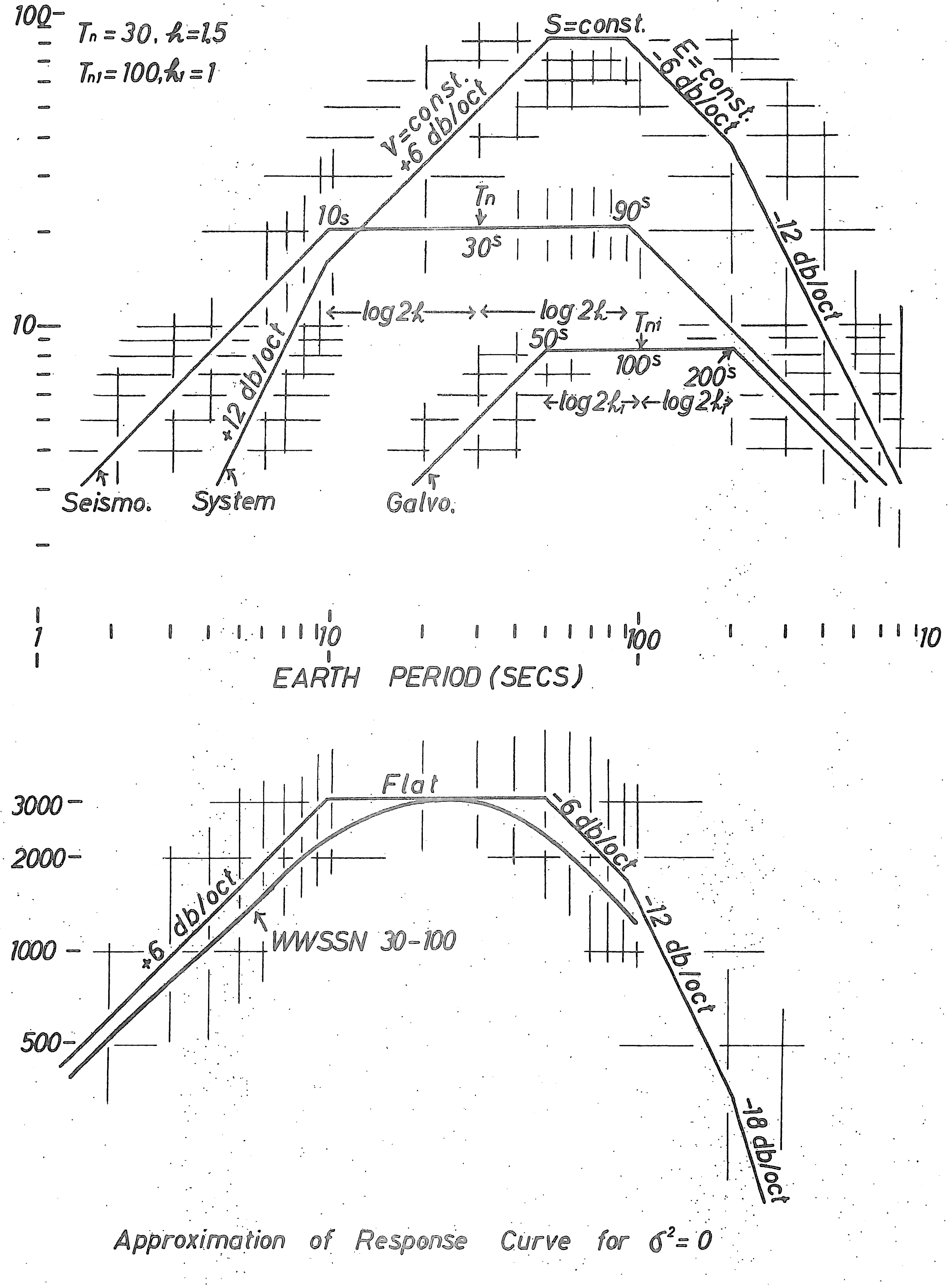
$$\text{or} \quad \left|\frac{K\omega_n}{M_o}\dot{\theta}_m\right| = \frac{\omega_e/\omega_n}{\left\{\left[1 - \left(\frac{\omega_e}{\omega_n}\right)^2\right]^2 + \left(2h\frac{\omega_e}{\omega_n}\right)^2\right\}^{1/2}}$$

which is the same expression as $\frac{\omega_n S}{V_o}$ in (7-14a) with peak value $\frac{1}{2h}$ at $\frac{\omega_e}{\omega_n} = 1$. Therefore a log-log plot of $\dot{\theta}_m$ has the same properties as a similar plot of S, with peak value of $\frac{M_o}{K\omega_n} \cdot \frac{1}{2h}$ at $\frac{\omega_e}{\omega_n} = 1$.

Therefore the first factor in (17-11a) peaks at $\frac{Mr}{2h\omega_n K}$ at $\omega_e = \omega_n$, the second at $2A_o\gamma\Gamma S/2h\omega_{n1}K_1Q^2 = 2A_o k / 2h_1\omega_{n1} = 2A_o\sigma_g$, at $\omega_e = \omega_{n1}$.

Given the instrinsic constants of the system, one can calculate these factors, and their product is the peak velocity sensitivity. Knowing this value and the shape of the curve from the procedure discussed above, the approximate velocity sensitivity curve can be plotted, and the magnification curve derived from it if desired.

For example, let $T_n = 30$ secs, $h = 1.5$, $T_{n1} = 1{,}00$ secs, $h_1 = 1$. The approximate velocity response curve is shown in the firure. In the lower part of the figure, the curve has been replotted as a magnification curve with the peak set at 3000. The magnification curve for the long



Approximation of Response Curve for $\sigma^2 = 0$

period seismograph used in the WWSSN, which has approximately these constants, is also shown. The method is certainly not exact, especially for damping as low as that in this example, but it does give a good indication of the shape of the curve. The errors in the method are discussed in detail by willmore, 1961.

Exercise: Show that the same response curve could be achieved with $T_n = 67$ secs, $h = 0.671$, $T_{n1} = 44.7$ secs, $h_1 = 2.236$. Is this a practical design for achieving this response curve?

Exercise: Find the approximate response curve, and designate the frequency range in which the magnification is constant, and the one in which the velocity sensitivity is constant for the following system (Antune, 1950): $T_n = 20$, $h = 0.5$, $T_{n1} = 1$, $h_1 = 10$. Repeat for $h = 0.707$.

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18. Seismographs with arbitrary coupling.

18. 1. Introduction of dimensionless time variable.

We return to equations (16-3), (16-4) and (16-10) to investigate the response with arbitrary coupling. Following Coulomb and Grenet (also see Eaton, 1957, and Willmore, 1961) we first simplify the equations by changing to a dimensionless time variable. Let $\omega' = \sqrt{\omega_n \omega_{n1}}$ and let $\tau = \omega' t$. Further, let $df/d\tau = f'$. Then, $d^nf/dt^n = (\omega')^n \, d^nf/d\tau^n$. Define $\rho = \sqrt{\omega_n/\omega_{n1}} = \omega_n/\omega' = \omega'/\omega_{n1}$, a dimensionless angular frequency. The seismometer equation, (16-3), becomes

$$\omega'^2 \phi'' + 2h\omega_n \omega' \phi' + \omega_n^2 \phi = -\frac{\omega'^2}{\ell} x'' + 2h\omega_n \sigma_s \omega' \theta'$$

or

$$\phi'' + 2h\rho\phi' + \rho^2 \phi = -\frac{x''}{\ell} + 2h\rho\sigma_s\theta' \qquad (18\text{-}1)$$

similarly, for the galvanometer

$$\theta'' + \frac{2h_1}{\rho}\theta' + \frac{\theta}{\rho^2} = \frac{2h_1}{\rho}\sigma_g\phi' \qquad (18\text{-}2)$$

The indicator equation becomes:

$$\theta^{(iv)} + \left(\frac{2h_1}{\rho} + 2h\rho\right)\theta''' + \left(\frac{1}{\rho^2} + \rho^2 + 4h_1h(1-\sigma^2)\right)\theta'' + 2\left(\frac{h}{\rho} + h_1\rho\right)\theta' + \theta = \frac{-2h_1\sigma_g}{\rho\ell}x''' \qquad (18\text{-}3)$$

or

$$\theta^{(iv)} + A\theta''' + B\theta'' + C\theta' + \theta = Fx''' \qquad (18\text{-}4)$$

Coulomb and Grenet demenstrate that by adjusting the coupling, σ^2, an infinite number of seismographs, all with the same response curve, can be designed, at least theoretically. The shape of the response curve is determined by A, B, and C. F fixes the absolute level of the magnification. Expressing h and h_1 in terms of A and C, and substituting in B:

$$B = (\rho^2 + \rho^{-2}) + (1 - \sigma^2)\left[(\rho A - C/\rho)(\rho C - A/\rho)\right]/(\rho^2 - \rho^{-2})^2$$

Let $(\rho^2 + \rho^{-2}) = (\omega_n/\omega_{n1} + \omega_{n1}/\omega_n) = a$:

$$B = a + (1 - \sigma^2)\left[\rho^2 AC - A^2 - C^2 + AC/\rho^2\right]/(a^2 - 4)$$

or

$$(a^2 - 4)(a - B) + (1 - \sigma^2)\,AC(a - A/C - C/A) = 0$$

Suppose A, B, C are fixed, thereby fixing the response curve. Then, for any value of σ^2, which has been shown to be between 0 and 1, this cubic equation in "a" has one or three real roots. Each root fixes two values of ω_n/ω_{n1} that are reciprocals of each other, corresponding to the fact the constants of the seismometer and galvanometer can be interchanged without changing the response. Thus

$$(\omega_n/\omega_{n1}) = a/2 \pm \sqrt{(a/2)^2 - 1}$$

The case for $\sigma^2 = 0$, the decoupled system, is only one of an infinity of equivalent seismographs. Decoupling offers some practical advantages, especially the possibility of designing a resistive network that can be inserted between the two components to act as a damping control and at the same time a step attenuator that will not change the damping of either component.

18. 2 Laplace transformation of the equations (Refer: Eaton, 1957).

Let $\Phi(s) = \mathcal{L}[\phi(\tau)] = \int_0^\infty e^{-s\tau}\phi(\tau)\,d\tau$, with similar definitions for $\Theta(s)$ and $X(s)$. Let the angular displacements and velocities of the seismometer and galvanometer at $\tau = 0$ be $\phi(0)$, $\phi'(0)$, $\theta(0)$, $\theta'(0)$. From (18-1),

$$(s^2 + 2h\rho s + \rho^2)\,\Phi(s) = \phi'(0) + (s + 2h\rho)\,\phi(0) - 2h\rho\sigma_s\,\theta(0)$$
$$+ 2h\rho\sigma_s s\,\Theta(s) - (1/l)\left[s^2X(s) - sx(0) - x'(0)\right] \quad (18\text{-}5)$$

From (18-2)

$$(s^2 + 2(h_1/\rho)s + 1/\rho^2)\,\Theta(s) = \theta'(0) + (s + 2h_1/\rho)\,\theta(0)$$
$$- 2(h_1/\rho)\sigma_g\,\phi(0) + 2(h_1/\rho)\sigma_g s\,\Phi(s) \quad (18\text{-}6)$$

Solving for $\Phi(s)$ and $\Theta(s)$, transform of seismometer response to x(t):

$$\Phi(s) = \frac{\left[(s^2 + 2\frac{h_1}{\rho}s + \frac{1}{\rho^2})(s + 2h\rho) - 4hh_1\sigma^2 s\right]\phi(0) + (s^2 + 2\frac{h_1}{\rho}s + \frac{1}{\rho^2})\phi(0) -}{\Delta(s)}$$

$$\frac{-\frac{2h}{\rho}\sigma_s\theta(0) + 2h\rho\sigma_s s\theta'(0) - \frac{1}{l}(s^2 + 2\frac{h_1}{\rho}s + \frac{1}{\rho^2})\left[s^2X(s) - sx(0) - x'(0)\right]}{\Delta(s)}$$

(18-7)

transform of galvanometer response:

$$\Theta(s) = \frac{\left[(s^2 + 2h\rho s + \rho^2)(s + 2\frac{h_1}{\rho}) - 4hh_1\sigma^2 s\right]\theta(0) + (s^2 + 2h\rho s + \rho^2)\,\theta'(0)}{\mathcal{J}(s)}$$

$$\frac{2h_1\rho\sigma_g\phi(0) + 2\frac{h_1}{\rho}\sigma_g s\phi(0) - \frac{1}{\ell}2\frac{h_1}{\rho}\sigma_g s\left[s^2X(s) - sx(0) - x'(0)\right]}{\mathcal{J}(s)} \qquad (18\text{-}8)$$

where $\mathcal{J}(s) = (s^2 + 2h\rho s + \rho^2)(s^2 + 2\frac{h_1}{\rho}s + \frac{1}{\rho^2}) - 4hh_1\sigma^2 s^2$ (18-9)

Equation (18-6) is the solution of the problem for arbitrary coupling, arbitrary initial conditions, and arbitrary ground excitation (as long as $X(s)$ exists). The major task of finding $\theta(\tau) = \mathcal{L}^{-1}[\Theta(s)]$ remains. Before examining that question, we can write the transform of the solution for some inputs of interest. For example, the solution for a sinusoidal ground motion that begins suddenly from a condition of rest is:

$$x(\tau) = x_0 \sin\omega\tau,\ \tau \geqslant 0 \qquad X(s) = x_0\omega/(s^2 + \omega^2)$$
$$= 0 \qquad ,\ \tau < 0$$

(Note: $\omega\tau = \omega_e t$, so that $\omega_e = \omega\omega_n$, or $T_e = \sqrt{T_n T_{n1}}/\omega$)
Initial conditions are: $x(0) = \phi(0) = \theta(0) = \theta'(0) = 0$; $x'(0) = \omega x_0$, $\phi'(0) = -x'(0)/\ell = -\omega x_0/\ell$. Substituting in (18-6), the harmonic response for arbitrary coupling is

$$\Theta(s) = \frac{(2h_1/\rho)\sigma_g s(-\omega x_0/\ell) - 1/\ell(2h_1/\rho)\sigma_g s)\left[x_0\omega s^2/(s^2 + \omega^2) - \omega x_0\right]}{\mathcal{J}(s)}$$

$$= -\frac{2h_1\sigma_g\omega x_0}{\ell\rho}\left(\frac{s^3}{(s^2 + \omega^2)\mathcal{J}(s)}\right) \qquad (18\text{-}10)$$

A test formerly used for calibration of electromagnetic seismographs in the "tapping test". A tap of very short duration relative to T_n is given to the pendulum, giving it an initial velocity. The resulting amplitude of the pendulum motion and the response of the galvanometer are observed. From the data, the transfer factor of the transducer, and thus the magnification can be claculated. The conditions are:

$$x(\tau) = x(0) = x'(0) = 0 \quad \text{(ground at rest)}$$
$$\phi(0) = \theta(0) = \theta'(0) = 0$$
$$\phi'(0) \neq 0$$

The motion of the seismometer is the inverse transform of

$$\Phi(s) = \frac{(s^2 + 2\zeta_1/\rho s + 1/\rho^2)\,\phi'(0)}{\mathcal{Z}(s)} \qquad (18\text{-}11)$$

and the galvanometer

$$\Theta(s) = \frac{(2\zeta_1/\rho)\sigma_g s\,\phi'(0)}{\mathcal{Z}(s)} \qquad (18\text{-}12)$$

Other standard tests may be treated in a similar way.

Exercise: Let $X(s) = 1$ in (18-6), with $x(0) = x'(0) = \phi(0) = \phi'(0) = \theta(0) = \theta'(0) = 0$. The corresponding $x(\tau) = \delta(0)$, the dirac delta function. The resulting $\theta(t)$ is the impulse response of the system.

Find $\Theta(s)$, and show that $\Theta(i\omega_e)$ with ω, ρ replaced in terms of ω_e, ω_n, and ω_{n1}, is exactly the steady-state harmonic response given by (16-12). Since $\Theta(i\omega_e)$ is the Fourier transform of $\theta(t)$, this result demonstrates that the Fourier transform of the impulse response is the steady-state harmonic response, including both amplitude and phase responses. This point will be taken up again when calibration methods are discussed.

The evaluation of the inverse transform to find $\theta(\tau)$ by the H aviside partial fractions technique (see Appendix 18-A) is straightforward if the linear factors of $\mathcal{Z}(s)$ are known. But $\mathcal{Z}(s)$ is a fourth degree equation, and its general solution is difficult to find. The problem is easy if $\sigma^2 = 0$, and we have essentially done this case for simple harmonic ground motion in deriving (17-1). We did not, however, write out the transient part of the solution, which we would find if we put $\sigma^2 = 0$ in (18-8) and solved for the inverse transform. The transient terms come from the zeros of $\mathcal{Z}(s)$, the steady-state terms from the zeros of the denominator of $X(s)$.

If we expand $\mathcal{Z}(s)$ in (18-7), we find that it is exactly the left hand side of (18-4), the indicator equation, if we replace $(d/d\tau)^n$ by s^n

$$\mathcal{Z}(s) = s^4 + As^3 + Bs^2 + Cs + 1 \qquad (18\text{-}13)$$

Coulomb and Grenet pointed out that this polynomial can be simplified, and the zeros found easily for the special case $A = C$. Physically this means that

$$2(\zeta_1/\rho + \zeta\rho) = 2(\zeta/\rho + \zeta_1\rho)$$

$$\text{or} \qquad (\zeta - \zeta_1)(\rho - 1/\rho) = 0,$$

so that the simplification is achieved if either $\zeta = \zeta_1$ or $\omega_n = \omega_{n1}$. Thus, if we are willing to adjust the seismograph so that either the two components are equally damped, or the two periods are equal, we can simplify the expression for $\mathcal{Z}(s)$. (Note: In the ideal Galitzin, for example, both of these conditions are satisfied, as well as $\sigma^2 = 0$.)

with $\mathcal{Z}(s) = s^4 + As^3 + Bs^2 + As + 1$, we wish to solve

$$(s^2 + 1/s^2) + A(s + 1/s) + B = 0$$

Following Eaton, we change variables, letting $s + 1/s = Z$, so that $s^2 + 1/s^2 = Z^2 - 2$. The equation to be solved is

$$Z^2 + AZ + B - 2 = 0$$

which can be handled by the quadratic formula.

Eaton presents the roots of this equation, and the corresponding value of s for the two cases, $\zeta = \zeta_1$ and $\omega_n = \omega_{n1}$. With the linear factors of $\mathcal{Z}(s)$ known, he then proceeds to find the inverse transforms of expressions like (18-8), (18-9), and (18-10) for a variety of inputs, and for components that are overdamped, critically damped, and underdamped.

Thus, in principle, (18-6) gives the system response for the most general case, but the actual time history of the response for arbitrary inputs is not so easy to obtain for the general case. However, the chief reason for wanting to be able to find $\theta(t)$ for a variety of inputs has been to make the interpretation of standardized calibration tests possible. We see from Eaton's results that it can only be done at present if special choices of the instrument constants are made. In recent years, calibration, techniques have been developed which are quite general and which test the system response as a whole rather than depending on the determination of intrinsic constants of the seismometer and galvanometer and the subsequent calculation of the response. Thus the need for special adjustments to make calibration possible no longer exists, and completely arbitrary combinations of natural periods, damping factors, and coupling can be used if a particular response curve is desired. It is desirable in the design stage of a new instrument to be able to predict the response, and this can be done by assigning numerical values to A, B, C in (18-11), and then finding to zero of $\mathcal{Z}(s)$ by algebraic or numerical techniques. It is much more likely that a designer will depend on his experience to get him close to the desired response, and then adjust the parmeters in a prototype until he is satisfied.

Exercise: Given the following constants of a seismograph system (all units are c.g.s.).

Seismometer:	Galvanometer:
$M = 500$ gm	$K_1 = 1.62$
$K = 2.5 \times 10^5$	$d = .05$
$Mr = 10^4$	$\tau_1 = .318$
$T_n = 12.5^5$	$\gamma = 6.23 \times 10^5$
$\mathcal{T} = 6.25 \times 10^4$	$R_g = 10^{10}$
$D = 2.5 \times 10^3$	$A_o = 100$ cm
$\Gamma = 1.87 \times 10^8$	
$R = 4.1 \times 10^{10}$ (abohms)	
$S = 8.2 \times 10^{10}$	

Find: T_{n1}, ζ_o, ζ_{o1}, $2\zeta\omega_n$, $2\zeta_1\omega_{n1}$, σ_s, σ_g, σ^2, ℓ.
Write the expression for the steady-state harmonic response. and find the magnification at $T_e = 6.28^s$, 12.56^s.

Appendix 18-A

The Heaviside Partial Fractions Expansion to obtain the Inverse Laplace Transform.

If $F(s) = \mathcal{L}[f(t)] = p(s)/q(s)$, were p, q are polynomials, q of higher degree than P:

1. If $(s - a)$ is a distinct linear factor of $q(s)$, and if $\phi(s) = (s - a)F(s)$, the factor contributes a term to the inverse transform $= \phi(a)e^{at}$.
2. If $(s - a)$ is repeated r times, and $\phi(s) = (s - a)^r F(s)$, the contribution to the inverse transform is

$$e^{at} \sum_{n=1}^{r} \frac{\phi^{(r-n)}(a)}{(r-n)!} \frac{t^{(n-1)}}{(n-1)!}, \text{ where } \phi^{(m)} = d^m\phi/dt^m.$$

For r = 2, the contribution is $e^{at}[\phi(a)t + \dot{\phi}(a)]$

3. If $[(s + b)^2]$ is a distinct quadratic factor of $q(s)$, $\phi(s) = (s + b)^2 + a^2 \; F(s)$, and $\phi(-b + ia) = \phi_1 + i\phi_2$, the factor contributes the following to the inverse transform

   $1/a \; e^{-bt} \; (\phi_2 \cos at + \phi_1 \sin at)$
4. If $F(s) = \phi(s)/[(s + b)^2 + a^2]^2$, and $\phi(-b + ia) = \phi_1 + i\phi_2$; $\dot{\phi}(-b + ia) = \phi_3 + i\phi_4$, the repeated factor contributes

$$(1/2a^3)e^{-bt}[(\phi_2 - a\phi_3)\cos at + (\phi_1 + a\phi_4)\sin at + at(\phi_2 \sin at - \phi_1 \cos at)]$$

(end of Appendix)

19. Auxiliary galvanometers as band-rejection filters (Reference: Pomeroy and Sutton, Bull. Seis. Soc. Amer., Vol. 50, 1960, pp. 135-154)

Following upon previous work by Grenet and by Desvaux, Pomeroy and Sutton have investigated the use of a second galvanometer in the circuit as a mechanical filter for rejecting a selected frequency band. Their purpose was to develop a means of eliminating the energy in the 4-10 second microseism band from high-sensitivity, long period records. They show that if a lightly-damped galvanometer with natural frequency at the center of the band to be rejected is inserted in the circuit, a sharp notch is formed in the response curve at this frequency, but the curve is not too badly distorted at other frequencies. The width of this notch can be varied by changing the damping of the filter galvanometer; the higher its damping, the wider the notch.

The method works because the filter galvanometer itself executes very large motion at its resonant frequency. The energy in the signal in the neighborhood of this frequency is effectively absorbed by the filter galvanometer, so that the recording galvanometer doesn't see it.

Pomeroy and Sutton present results for systems in which the seismometer and two galvanometers are tightly coupled in a single series circuit, and well as systems in which the seismometer and filter galvanometer are tightly coupled, but the recording galvanometer decoupled, and the filter and recording galvanometers tightly coupled to each other, but decoupled from the seismometer. In all cases, a notch is formed at

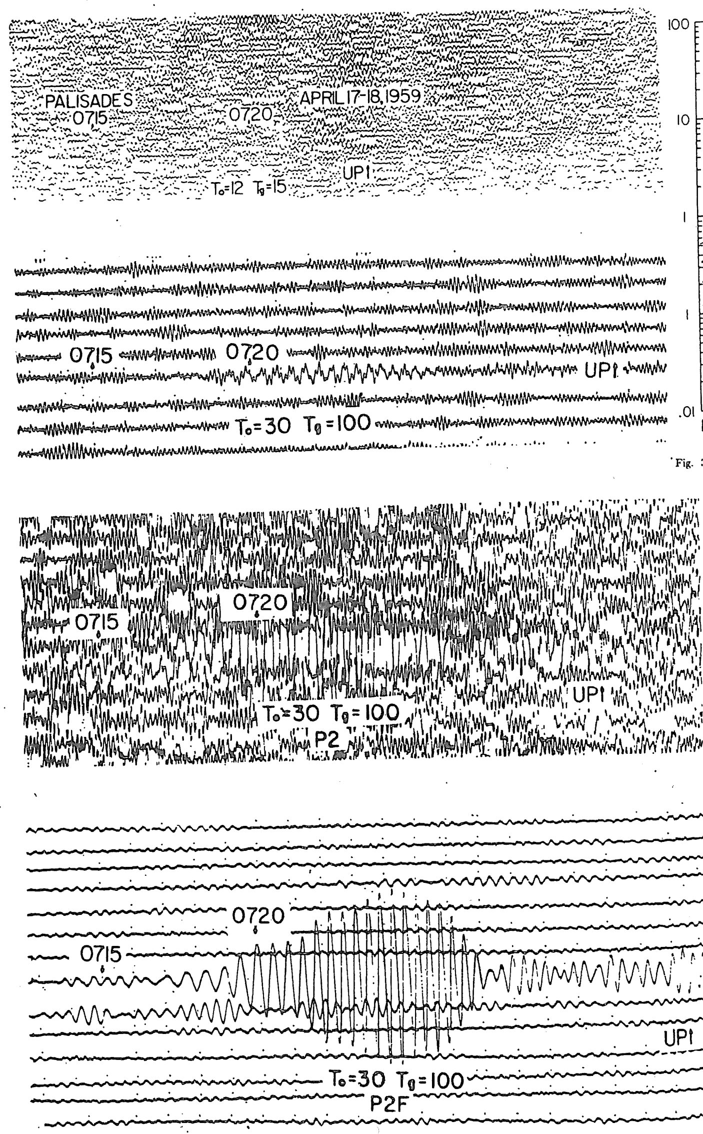


Fig. 2. Palisades vertical seismograms, April 17 and 18, 1959.

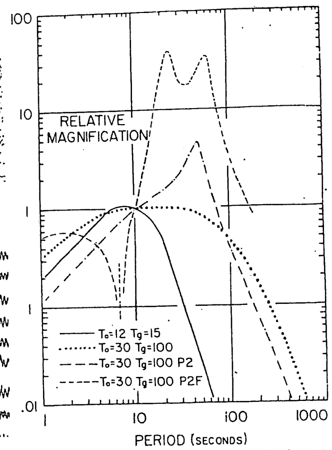


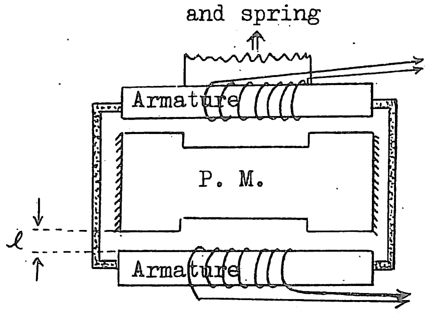
Fig. 3. Seismograph magnification curves, normalized at 10-seconds period, for records shown in figure 2.

the filter galvanometer frequency. The response curve is smoother for the recording galvanometer cecoupled from the seismometer for either position of the filter galvanometer, or for no filter galvanometer. They present records for the tightly coupled system which demontrate beyond question the efficiency of their technique. A copy of their records is shown in the figure above.

The presence of the filter galvanometer badly distorts the phase response curve in the neighborhood of the rejected frequency band, but leaves it relatively undisturbed otherwise. The instruments are used mainly to record long-period surface waves, for which accurate phase shift information is essential. At these periods a reasonably accurate phase shift correction can be made.

20. The Variable Reluctance (Benioff) Transducer (References: Benioff, H., Bull. Seis. Soc. Amer., Vol. 22, 1932, pp. 155-169; Bullard, E.C., Mon. Not., Roy. Astro. Soc., Geophys. Suppl., Vol. 4, 1938, pp. 336-340; Devlin, J.J., Bull. Seis. Soc. Amer., Vol. 28, 1938, pp. 255-258; Byerly, P., Seismology, 1942, pp. 140-143; Willmore, 1960. pp. 248-252)

Another way using the principle of electromagnetic induction to build a transducer in employed in the variable reluctance transducer. Rather than use the relative motion between a conductor and a fixed magnetic field, this transducer works by the change of the total flux in a magnetic circuit. Two air gaps that change length with the relative motion are part of the circuit, and changes in the length of these gaps change the reluctance of the circuit. This type of transducer is usually called a Benioff transducer because, although he did not invent the device, H. Benioff first designed a very successful seismograph incorporating it, an instrument now used world-wide.



In the vertical component Benioff, a translating mass on a spring, the armature and coil assembly is fastened to the suspended mass. A permanent magnet is fixed to the instrument frame. Only one armature was used in the original design, but this led to non-linearities which are eliminated by using two, one each side of the permanent magnet. In the horizontal component Benioff, the assembly is in the center of the moving mass, which also moves in translation on an elastic suspension.

The flux in a magnetic circuit is equal to the magnetomotive force of the magnet divided by the relucta ce of the circuit;

$$\phi = \frac{\text{m.m.f}}{\text{reluctance}}$$

analogous to Ohm's law for an electrical circuit. The reluctance of a magnetic path is

$$\text{reluctance} = \mathcal{R} = \frac{\ell}{\mu A}$$

when ℓ is the length of the path; μ, the permeability of the material; and A, the cross-sectional area.

The permeability of the permanent magnet and the armature cores is very high compared to the permeability of air (ca .10^4 ; 1), so the reluctance of the magnetic circuit is dominated by the air gaps. Considering only one branch of the circuit, and neglecting the reluctance of the magnet and armature core

$$\text{m.m.f.} = 2\phi\mathcal{R} ,$$

where ϕ is the flux, $\mathcal{R}$ is the reluctance of each air gap. Since m.m.f. is constant ,

$$\phi \, d\mathcal{R} + \mathcal{R} \, d\phi = 0$$

or

$$d\phi = - \phi \, d\mathcal{R} / \mathcal{R}$$

Since $\mathcal{R} = \ell/A$ ($\mu \approx 1$ for air), $d\phi = - \frac{d\ell}{\ell} \phi$, is length of gap, A its cross-section. The change in gap length, $d\ell$, is dz for the lower circuit, and -dz for the upper circuit for dz positive downward.

The forces acting on the mass, in addition to the usual spring force and force of mechanical damping, are a magnetic force due to induced magnetism in the cross and electromagnetic force due to the current in the circuit. The equation of relative motion for the double-armature system (see Byerly, 1942, or Willmore, 1960) is

$$\frac{d^2z}{dt^2} + \frac{c}{M} \frac{dz}{dt} + \left(\frac{k}{M} - \frac{k'}{M}\right)z + \frac{2\phi_m}{nAM} Li = - \frac{d^2x}{dt^2}$$

where z, x, c, M, k are as before.

ϕ_m = total flux in the magnet at rest position.

$$k = \frac{\phi_m^2}{A\ell}$$

L = inductance of coils, assuming no leakage flux.
n = number of turns on coils.
i = current in coils.

The term in k' is the acceleration of the suspended mass because of magnetic attraction between the two sides of the gaps. It appear as a negative spring constant. The term involving i is the acceleration resulting from the flux generated by current flow in the coil.

The output of the transducer with the coil circuit open is found by evaluating $E = - n \, d\phi/dt = - n \, (d\phi/dz)(dz/dt)$. The coefficient $-n \, d\phi/dz$ is a constant of the transducer, which, from an analysis of division of flux in the circuit, turns out to be

$$- n \frac{\phi_m}{2\ell_o}$$

for each armature, or, if two coils are connected in series adding;

$$E = - n \frac{\phi_m}{\ell_0} \frac{dz}{dt}$$

$$= \Gamma_T \, \dot{z}$$

where Γ_T is the electrodynamic constant of the variable reluctance transducer, and ℓ_o is the length of the gap at rest.

As soon as the circuit is closed and a time-varying current begins to flow, an additional voltage L di/dt is generated, so that the total voltage at the coil terminals, is $E \Gamma_T \, dz/dt + L \, di/dt$. The voltage resulting from the inductance modifies the behavior considerably, especially at high frequencies, and the analysis becomes more complicated. Sparks and

Hawley (Geophysics, Vol. 4, 1939) have analyzed the motion including the inductance term, but not the galvanometer reaction. For a discussion of the effect of the inductance term see Willmore, 1960, pp. 250-252.

Savill, Carpenter, and Wright (Geophysical Jour. Vol. 6, No. 4, 1962, pp. 409-425) have derived equations parallel to those in Section 16, but including the effect of the inductance of the transducer coil. They have solved the resulting fifth order differential equation for θ in terms of x by means of an analogue computer for a Benioff (high inductance) seismograph, and a Willmore (low inductance) seismograph. They conclude that the inductance of the Benioff produces a sharper high frequency cutoff and a longer rise time of the transient response. This leads to a reduction of the amplitude of the first motion on the record.

21. Calibration of electromagnetic seismographs.

(References: Byerly, P., Seismology, 1942, pp. 136-140: Willmore, P.L., Bull. Seis. Soc. Amer., Vol. 49, 1959, pp. 99-114: Espinosa, A.F., Sutton, G.H., Miller, H.J., Bull. Seis. Soc. Amer., Vol. 52, 1962, pp. 767-ff; The Geotechnical Corporation, Operation and Maintenance Manual, World-Wide Seismograph System, Model 10700, April 1962).

One of the most important advances in seismology during the last decade has been the establishement of a network of calibrated seismographs over much of the world. Prior to this time seismologists worked almost exclusively with the times of arrival of events and could do little with amplitude data. Yet information about some of the most challenging problems in seismology, such as focal mechanisms of earthquakes, inelastic properties of earth materials, and details of earth structure (Low-velocity layers, etc.) is contained in the amplitude of seismic waves incident on the surface. Precise work on surface-wave dispersion requires knowledge of the phase response of the instruments. Methods of calibration have been known for a long time: the problem has been to devise techniques of applying the known methods that could be applied routinely in the field by personnel with minimum training.

Calibration means the determination of the amplitude and phase response of the complete seismograph system. If the steady-state harmonic response is known over the whole frequency band, the ground motion giving rise to any waveform on the record can be determined. Conversely, as will be seen below, if the response to any known ground motion with a continuous spectrum is observed, the steady-state response can be derived.

21. 1 The transfer function.

Given a linear differential equation with constant coefficients

$$(a_o D^n + \cdots\cdots + a_{n-1} D + a_n)u = f(t) \qquad (21\text{-}1)$$

then $a_o s^n + \cdots\cdots + a_{n-1} s + a_n = 0$ is the characteristic equation.

$$Y(s) = \frac{1}{a_o s^n + \cdots\cdots + a_{n-1} s + a_n}$$ is called

the transfer function of the differential equation. An important property of the transfer function is that if $f(t) = Ae^{i\omega t}$, A constant, ω real,

$$u(t) = Y(i\omega)Ae^{i\omega t} \qquad (21\text{-}2)$$

is a particular solution of the differential equation, provided $(i\omega)$ is not a root of the characteristic equation. This can be verified by substitution.

If $A = Re^{i\alpha}$ is the input amplitude, so that $f(t) = Re^{i(\omega t + a)}$, R, a, real, the output is $|Y(i\omega)|\ Re^{(i\omega t + \alpha + \psi)}$. $|Y(i\omega)|$ is a frequency-dependent amplification factor that we have called the amplitude response of the system described by the differential equation, and $\psi = \arg Y(i\omega)$ gives the phase response.

Let $U(\omega)$ be the Fourier transform of $u(t)$,

$$U(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t)e^{-i\omega t}\, dt \qquad (21\text{-}3a)$$

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(\omega)e^{i\omega t}\, d\omega \qquad (21\text{-}3b)$$

It can be shown that the Fourier transform of $d^k u/dt^k$ is equal to $(i\omega)^k$ times the Fourier transform of u. Applying this to the original differential equation (21-1), with $F(\omega)$ the Fourier transform of $f(t)$

$$(a_o(i\omega)^n + \ldots\ldots\ldots\ + a_{n-1}(i\omega) + a_n)U(\omega) = F(\omega)$$

or

$$\frac{U(\omega)}{Y(i\omega)} = F(\omega)$$

so that

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Y(i\omega)F(\omega)e^{i\omega t}\, d\omega$$

This says that each frequency in the input, with amplitude and phase given by $F(\omega)d\omega$ comes through the system multipleid by the value of the transfer function at that frequency, $Y(i\omega)$, and the total output is the synthesis of all these frequency components.

Suppose we observe the output $u(t)$ for a known input $f(t)$. Then the harmonic response function can be calculated by dividing the Fourier transform of the output by the Fourier transform of the input

$$Y(i\omega) = \frac{U(\omega)}{F(\omega)} \qquad (21\text{-}4)$$

The nature of $f(t)$ is not particularly important, except that $F(\omega)$ must exist and should contain all frequencies in the pass band of the system described by (21-1), and it is sensible to choose $f(t)$ so that $F(\omega)$ is easy to calculate.

In this approach to determining the response of a system, sometimes called the "black box" approach, no attention is paid to the details of what goes on inside the system. It is only necessary that the system be linear, so that it can be described by (21-1) and the superposition called for in (21-3b) is valid.

21. 2 Pulse calibration of a seismograph.

In the case of a seismograph, the straightforward way to apply (21-4) would be to give a known displacement to the <u>frame of the seismometer.</u> This can be done if a suitable seismographic shake-table system is available, as discussed in Section 12.21, and if the seismograph is of a type that can be moved around without changing its properties.

The most simple input is a unit impulse in ground displacement, so that $x(t) = \delta\ (0)$, $X(i\omega) = 1$. The Fourier transform of the response to this impulse is then directly the harmonic-displacement response of the system. This was arrived at in a different way in the exercise on pages 107-108, where the equality between the Laplace transform with $(i\omega)$ as the transform variable and the Fourier transform for functions that are equal to zero for negative time was used. Unfortunately, this test is not usually practical, because there is no convenient way of providing this excitation.

An alternative is to provide an excitation of the inertial member of the seismometer system, whether a translating mass or a pendulum. It has been shown earlier that a ground acceleration $\ddot{x}$ applied to the frame of a system with translation produces the same relative motion as the motion produced by a force $-M\ddot{x}$ applied to the mass with the frame at rest. Similarly, for a rotating system, the acceleration $\ddot{x}$ of the frame is equivalent, for relative motion, to a force $-M\ddot{x}$ applied at the center of mass. Such tests as the tapping, release, and displacement tests (see Eaton, pp. 49-52) drive the inertial member by mechanical means. However, one can use the transducer in the seismometer as a means of driving it. Suppose the electrodynamic constant of the coil is known. If a current flows in the transducer coil, <u>a moment</u> $-\Gamma_R I$ acts in the case of a hinged system, or a force $-\Gamma_T I$ in the case of the translating mass. In either case the waveform of the current corresponds to the waveform of an equivalent ground acceleration, $\Gamma_R I/Mr$ for a hinged system, as in (16-1), or $\Gamma_T I/M$ for a translating system, as in the Benioff equation, Section 20. The corresponding output can be treated as in the previous section to get the acceleration response of the system. This, in turn can be converted to the magnification curve, or displacement response, by multiplying by ω_e^2, and adding π to the phase angle. Even if Γ is not known, this procedure will yield the relative magnification curve.

It is necessary to apply the current to the coil in such a way that the galvanometer is not driven directly by the test signal. This can be done by inserting the seismometer and galvanometer in an impedance bridge, as will be discussed in detail below. The bridge is balanced so that the application of the test current with the seismometer clamped does not result in any deflection of the galvanometer. Of course, it is possible to provide the seismometer with a separate calibrating coil, independent of the galvanometer circuit, with known electrodynamic constant. Either means of driving the seismometer electrically provides a method for remote calibration when needed, as in bore-hole seismometers and other applications in which the detector is not easily accessible.

The method is not actually as good as tests which drive the frame. The assumption is made that the frame and supporting members act in a perfectly rigid manner. Pulse calibration on a shake-table will bring out departures from ideal behavior, whereas tests in which the inertial member is driven may not.

In this method we again choose an input with a known Fourier transform. A current pulse is not generated as easily as a current step, although it can be obtained if desired. A step in current is easily applied by connecting a dry-cell to the coil (assuming the inductance is not too high), a matter of closing a switch. The transform of a unit step function is $F(\omega) = \frac{1}{i\omega}$, so that the transform of the output of the galvanometer for this input must be multiplied by ω, and $\pi/2$ added to the phase, to get the acceleration response. Comparison of results obtained for a current pulse, a current step, and a sine-wave input is presented by Pomeroy and Sutton.

It is necessary to know Γ_T or Γ_R, the electrodynamic oonstant of the transducer or calibrating coil, to get an absolute calibration. One way of determining this is by observing the damping factor of the seismometer with open circuit and with a closed circuit with resistance R, including the transducer coil resistance. For a hinged system, the contribution of the current to the total damping moment is $\Gamma_R{}^2\dot{\phi}/R$, and

$$2h\omega_n = 2h_o\omega_n + \frac{\Gamma_R{}^2}{KR},$$

so that

$$\Gamma_R{}^2 = 2\omega_n(h - h_o)K\,R.$$

For a translating system, M replaces K, and Γ_T replaces Γ_R.

An alternate way of determining Γ is to compare the effects of a step in current with the effect of a step in force applied to the inertial member. A step in force is generated most easily by the "weight-snatch" technique, in which a test weight is applied in such a way as to produce a static displacement, and then is removed quickly. Consider a vertical component seismometer based on a translating mass. The removal of the test weight, mg, is equivalent to a step in force in the direction of sensitivity. But a calibration current step also is equivalent to a step in force. If the two forces are equal, $\Gamma_T I = mg$. In general, they will not be equal, but the responses of the system will have the same shape, with only the levels different. Suppose the first peak of the response transient is read from the seismogram for the two kinds of experiments, a_m corresponding to the weight snatch, a_I corresponding to the curfent step. Then, for a translating system

$$\frac{a_m}{a_I} = \frac{mg}{\Gamma_T I}\ ;\ \text{or}\quad \Gamma_T = \frac{mg}{I}\ \frac{a_I}{a_m}$$

and the electrodynamic constant is determined from easily observable quantities.

In the case of a pendulum system, the test weight must be arranged to apply a moment mgd about the axis of rotation. Then, $\Gamma_R I = mgd$ for equal inputs, or if the inputs are different

$$\frac{a_m}{a_I} = \frac{mgd}{\Gamma_R I} \; ; \qquad \Gamma_R = \frac{mgd}{I} \quad \frac{a_I}{a_m} ,$$

where again a_m and a_I are corresponding amplitudes, most conveniently the first peak of the transient response in the two tests. With reference to the WWSSN manual, equation 10, page 10 of Appendix A, $\Gamma_R = g_c d_c$, and $G_c{}^* = \Gamma_R/r$ for the rotational system.

Question: Why is a weight-snatch preferable to the sudden addition of a test mass?

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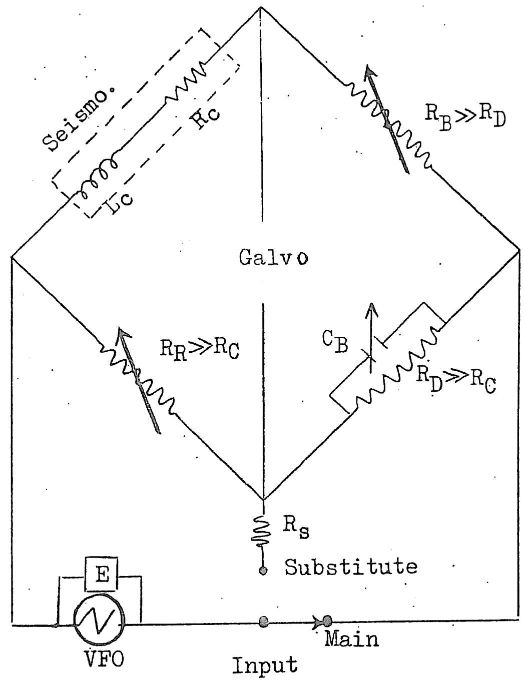
21. 3 Sine-wave calibration.

The main difficulty with the pulse calibration technique is the necessity of determining the Fourier transform of the output in order to get the whole response curve. For this reason, the chief application has been to the determination of the response at only one frequency in order to set the level of the response curve, with the assumption that the shape of the curve is known from the design constnants, T_n, T_{n1}, h, $h1$, and ^{2}. The height of the output peak alone is enough to set the magnification at one frequency. The only use usually made of the calibration pulses put on records twice a day is to provide a quick check on the operation of the instrument. A marked change in either the height or shape of the pulse is an easily detectable signal that something has happened to the instrument.

Thus pulse calibration is easy to do, but requires time and computing facilities to interpret. On the other hand, a sine-wave calibration of the system over the whole pass band takes much more time, but requires no further interpretation. If the input is a continuous sine wave, $F(\omega)$ is a line at the selected frequency, and the output is the response at that frequency. If the seismometer is driven with a sinusoidal current at many different frequencies, so that current is the same at all frequencies, the input corresponds to a constant level of sinusoidal acceleration of the frame at all frequencies, and the acceleration response curve is given by the output. It is only necessary to determine the equivalent ground acceleration to make the calibration absolute. Velocity sensitivity of magnification can then be calculated by multiplying the acceleration sensitivity by ω_e or $\omega_e{}^2$, respectively.

This is the technique developed by P.L. Willmore. His main contribution was the use of an impedance bridge to eliminate the direct driving effect of the test signal on the galvanometer. Independent information required to make the calibration absolute is the mass in the case of a translating system or the moment of intertia and distance of the transducer coil from the hinge for a rotating system. The moment of inertia may be calculated from $K = Mr\,\ell$.



From Willmore, 1959

In the bridge circuit, the seismometer-galvanometer circuit has been changed by the insertion of R_R in series, and $(R_B + R_D)$ shunting the galvanometer. Obviously the effect is minimized by making R_R small and B_B and R_D large. Pomeroy and Sutton suggest building the bridge into the permanent circuit. All elements outside the seismometer, such as damping resistors and attenuators are put into the galvanometer branch. The inductance of the seismometer coil is balanced by adjusting the capacitor C_B. Even if L_C is small, so that the electrical impedance of the seismometer coil is almost purely resistive, the use of C_B is necessary to get a very good balance of the bridge.

The steps in the calibration are:

1) With the seismometer clamped, balance the bridge so that the galvanometer deflection is negligibly small for strong excitation. The balance conditions are R_c/R_R, $L_C/R_R = R_BC_B$. R_C and L_C can be calculated from these conditions.

2) The seismometer is unclamped, and the galvanometer deflection is observed as a function of frequency for a constant level of input. This gives the relative acceleration response directly. The phase response can also be observed if needed.

3) The input is switched to the "Substitute" position, putting it across R_R. Again the galvanometer deflection is read as a function of frequency. As will be seen, this provides enough information to make the calibration absolute if the mechanical properties of the inertial member mentioned above are known.

We shall derive the equations for a translating system, and then modify them as needed for the rotating system. The output of the transducer coil is $e_c = \Gamma_T \dot{z}$, as before, and the force due to a current i_c flowing in the coil is $- \Gamma_T i_c$. But this force acting directly on the mass is equivalent to an acceleration of the frame, Section 7.3,

$$\ddot{x} = \frac{\Gamma_T i_c}{M}$$

When the seismometer is clamped, this force is balanced by the clamping mechanism. With the bridge balanced, the motion of the galvanometer is entirely a result of the response to this force when the seismometer is unclamped. If i_c has a constant amplitude, the equivalent peak acceleration is constant, and the acceleration response is obtained directly.

Call the current in the seismometer coil in the first step (system clamped) i_{c1}, and in the second step, (system unclamped), i_{c2}. The force driving the moving mass is $- \Gamma_T i_{c2}$, so that the velocity of the mass is

$$\dot{z}_2 = - \frac{\Gamma_T i_{c2}}{Z_M},$$

where Z_M is the mechanical impedance of the seismometer system (see equation (17-9)

$$Z_M = c + i\left[\omega_e M - \frac{k}{\omega_e}\right]$$

$$= c + iM\left(\omega_e - \frac{\omega_n^2}{\omega_e}\right).$$

The motion of the mass is caused by the force $\Gamma_T i_{c1}$, the force acting in the clamped position, but the current actually in the coil changes to i_{c2} because current flows in the galvanometer branch. Define Z_g as an additional mechanical impedance, representing the galvanometer response, thag changes i_{c1} into i_{c2}:

$$i_{c2} = i_{c1} \frac{\dot{Z}_M}{Z_M + Z_g}$$

so that

$$\dot{z}_2 = - \frac{\Gamma i_{c1}}{Z_M + Z_g}$$

In the third step, using the "Substitution" input (this is Stage 4 in Willmore, and we shall use subscript '4' to refere to it for ease of comparison), the current in the coil is i_{c4} and

$$\dot{z}_4 = - \frac{\Gamma_T i_{c4}}{Z_M}$$

As a result of this motion, a voltage is generated by the transducer,

$$e_{c4} = \Gamma_T \dot{z}_4 = -\frac{\Gamma_T^2 i_{c4}}{Z_M}$$

Call the input voltage across R_R in this step e_{s4}. Then the total voltage driving the galvanometer to produce the deflection θ_4 is $e_{c4} + e_{s4}$. in the second step, the galvanometer was driven by the voltage e_{c2}.

$$e_{c2} = \Gamma \dot{z}_2$$

$$= -\frac{\Gamma_T^2 i_{c2}}{Z_M}$$

Assuming that the galvanometer responds linearly to the applied voltage,

$$\frac{e_{c4} + e_{s4}}{\theta_4} = \frac{e_{c2}}{\theta_2} .$$

Solving for e_{s4}, and substituting for e_{c2} and e_{c4},

$$e_{s4} = -\frac{\Gamma_T 2}{Z_M} \frac{\theta_4}{\theta_2} \left[i_{c2} - \frac{\theta_2}{\theta_4} i_{c4} \right] .$$

The current that produces θ_2 is the difference between i_{c2} and i_{cl}. In other words, the voltage e_{c2} drives a current $i_{c2} - i_{cl}$ through the galvanometer, producing this deflection, so that

$$\frac{i_{c2} - i_{cl}}{\theta_2} = \frac{i_{c4}}{\theta_4}$$

or

$$\frac{\theta_2}{\theta_4} i_{c4} = i_{c2} - i_{cl}.$$

Substituting

$$e_{s4} = -\frac{\Gamma_T 2}{Z_M} \frac{\theta_4}{\theta_2} i_{cl}.$$

If E_s is the voltage out of the source in step 3,

$$e_s = \frac{E_s R_R}{R_s + R_R}$$

$$i_s = \frac{E_s}{R_s + R_R}$$

If E_M is the voltage applied across the bridge during steps 1 and 2,

$$i_{c1} = \frac{E_{M1}}{Z_C + R_B}$$

where Z_C is the electrical impedance of the coil. This is known from the balance conditions. For low frequencies and small coils, $Z_C \approx R_C$. Thus, Γ_T is given in terms of the ratio of the amplitudes in the two stages, measurable electrical quantities, and Z_M. Z_g is eliminated. Z_M could be determined by measuring M, T_n, and , and calculating c. An easier way, if M is known, is to use the data already collected, as follows;

$$|Z_M|^2 = c^2 + M^2 \left(\omega_e - \frac{\omega_n^2}{\omega_e} \right)^2$$

$|Z_M| = c$ at $\omega_e = \omega_n$, the minimum value. Therefore,

$$\left| \frac{\theta_4}{\theta_2} \right| = \frac{e_{s4} \, |Z_M|}{\Gamma_T^2 i_{c1}}$$

will have a minimum at $\omega_e = \omega_n$.

$\left| \frac{\theta_4}{\theta_2} \right|$ is plotted against frequency on semi-logarithmic coordinate paper (logarithmic frequency axis), and the frequency at which the minimum occurs is the natural undamped frequency of the seismometer. The sharpness of this minimum depends on the magnitude of c, the mechanical damping coefficient. If no damping vane is used in the system, the minimum is quite sharp and ω_n can be determined accurately.

Now,

$$\left(\frac{\theta_4}{\theta_2}\right)^2 = \frac{e_{s4}^2 \; Z_M^2}{(\Gamma_T^2 i_{c2})^2} = \left(\frac{e_{s4}}{\Gamma_T^2 i_{c1}} \right)^2 \left[c^2 + 4\pi^2 M^2 \left(\frac{f_e^2 - f_n^2}{f_e} \right)^2 \right],$$

so that the graph of $\left(\frac{\theta_4}{\theta_2} \right)^2$ vs. $\left[2\pi M \left(\frac{f_e^2 - f_n^2}{f_e} \right) \right]^2$ is a straight line with slope $\left(\frac{e_{s4}}{\Gamma_T^2 i_{c1}} \right)^2$. If this graph is drawn, and the measured slope is m, then

$$\Gamma_T^2 = \frac{e_{s4}}{m^{\frac{1}{2}} i_{c1}}$$

and Γ_T is determined. In practice, this graph will show two straight lines with slightly different slopes for values of f_e above and below f_n. This is a result of a small error in picking f_n from the first graph. It is sufficiently accurate to use the average of the two slopes to determine Γ_T, or one may adjust f_n until the points fall on a single line.

With Γ_T known, the driving acceleration $\Gamma_T i_{cl}/M$ can be calculated and the acceleration sensitivity determined as a function of frequency from the values θ_2.

For a hinged system the equations must be modified. First, we have defined Γ for a hinged system so that $-\Gamma_R i_c$ is the moment father than the force that drives the system. Γ_R in our notation is equal to $K \ell_c$ in Willmore's paper. The equivalent ground acceleration for a hinged system is

$$\frac{\Gamma_R i_{cl}}{Mr}.$$

Everyshere that Γ_T appears in the equations above, Γ_R is substituted, and M is replaced by K, the moment of inertia. Angular velocities, $\dot{\phi}$, replace linear velocities $\dot{z}$ throughout. Thus, after making the same observation, the slope of $\left(\frac{\theta_4}{\theta_2}\right)^2$ vs. $\left[2\pi K\left(\frac{f_e^2 - f_n^2}{f_e}\right)\right]^2$ is

$$m^{\frac{1}{2}} = \frac{e_{s4}}{\Gamma_R^2 i_{cl}} ,$$

and Γ_R can be calculated.

If MKS units are used, the voltages will be in volts, the currents in amperes, and resistances in ohms. With c.g.s. units, abvolts, abamps, and abohms must be used.

Part III

Other Topics

22. The linear strain seismograph.

(References: H. Benioff, Bull. Seis. Soc. Amer., Vol. 25, 1935, pp. 283 ff.; Bull. Geol. Soc. Amer., Vol. 70, 1959, pp. 1019 - 1032; H. Benioff and B. Gutenberg, Bull. Seis. Soc. Amer., Vol. 42, 1952, pp. 229 - 238.)

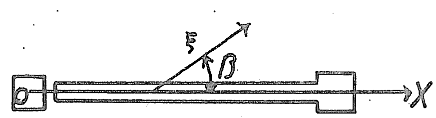
All of the instruments considered so far have been based on the concept of an inertial member loosely coupled to the moving earth. As has been shown, such instruments provide a signal containing information about the ground displacement. One important kind of seismic instrument does not respond to the displacement of the earth at a point, but to the difference in displacements at two separated points, in other words, to the integrated strain between the two points. This kind of instrument is called a strain meter or extensometer.

Piers are constructed at the two points, and then a rigid rod fastened to one of these piers comes very near the second, so that changes in the distance between the piers can be measured conveniently. The idea is an old one. Milne built a strain meter in 1888 with a distance of 3 feet between the piers, but it was very insensitive, providing a magnification at 1 second of only 1/30. Oddone used a three meter distance, with an hydraulic indicator giving a magnification of 3600. However, Oddone's instrument did not write a record.

By using a much longer distance between piers and his variable reluctance transducer, Benioff developed a very sensitive strain seismograph. His original instrument, 1935, was based on two piers 60 feet apart. The piers were 12 inch iron pipes, sunk 1.5 meters into granite. The rod was a two inch pipe insulated by a layer of asbestos and supported at twelve places along its length by three radial wires.

Modern strain meter tubes are made of fused quartz to reduce their temperature sensitivity. Lengths as great as 100 meters, as at Matsushiro Observatory, have been used. M. Major at the Colorado School of Mines has developed a temperature-compensated strain tube for use in a shallow trench. Most installation are in tunnels or maines in order to provide as constant a temperature environment as possible.

It is assumed that the rod acts as an incompressible, perfectly rigid body, so that motion of the pier to which it is attached appears instantly at its free end, without change. This requires that the period of the fundamental mode of free longitudinal vibrations in the rod be very short compared to the ground periods. In the original Benioff this period was 0.016 second.



Plan View

Consider a coordinate system with origin in the free pier and x-axis extending toward the other pier. Let ξ be the horizontal displacement at x, β the angle between ξ and x-axis. The component of ξ along the rod is $\xi \cos \beta$. The linear strain is $(\partial \xi / \partial x) \cos \beta$, so that the total displacement of one pier with respect to the other is

$$y = \int_0^L \cos \beta \frac{\partial \xi}{\partial x} dx , \qquad (22\text{-}1)$$

where L is the distance between the piers.

If an electromagnetic transducer is used, the output voltage is

$$\Gamma_T \dot{y} = \Gamma_T \frac{d}{dt} \int_0^L \cos \beta \frac{\partial \xi}{\partial x} dx \qquad (22\text{-}2)$$

If the ground motion is a plane wave, with wave length great compared to L, β and $\partial \xi / \partial x$ are constant, so that

$$y = L \cos \beta \frac{\partial \xi}{\partial x} \qquad (22\text{-}3)$$

$$\Gamma_T \dot{y} = L \cos \beta \frac{\partial^2 \xi}{\partial t \, \partial x} \qquad (22\text{-}4)$$

We represent the disturbance by $\xi(t - r/V)$, where V is the apparent surface velocity. V is the same as the true velocity of propagation only for waves tranvelling horizontally. Otherwise, $V = v/\sin i$, where v is the true velocity and i the angle of incidence at the surface.

For compressional waves, or motion in the plane of incidence, ξ is along r, so that $r = x \cos \beta$, and

$$\xi = \xi\left(t - \frac{x \cos \beta}{V}\right)$$

$$\frac{\partial \xi}{\partial x} = - \frac{\cos \beta}{V} \xi'$$

$$\frac{\partial \xi}{\partial t} = \xi'$$

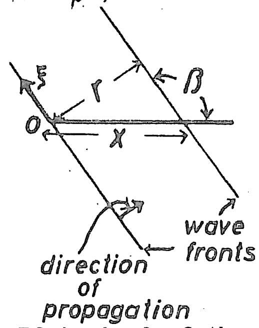
so

$$\frac{\partial \xi}{\partial x} = - \frac{\cos \beta}{V} \frac{\partial \xi}{\partial t}$$

The mechanical response is, therefore

$$y = - \frac{L \cos^2 \beta}{V} \frac{\partial \xi}{\partial t}$$

For shear waves, ξ is perpendicular to r, so that $r = x \cos(\beta - \frac{\pi}{2}) = x \sin \beta$, and



$$\xi = \xi(t - \frac{x \sin\beta}{V}) ,$$

$$\frac{\partial \xi}{\partial x} = - \frac{\sin\beta}{V} \frac{\partial \xi}{\partial t}$$

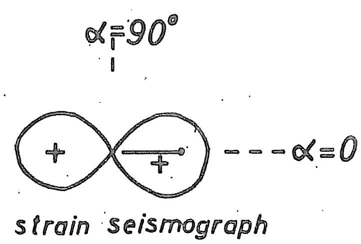
$$y = - \frac{L}{V} \sin\beta \cos\beta \frac{\partial \xi}{\partial t}$$

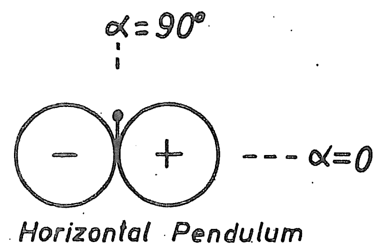
If instead of the angle β, between ξ and x, we use the angle between r and x, α

For compressional waves, $\alpha = \beta$, $\quad y = - \frac{L}{V} \cos^2\alpha \frac{\partial \xi}{\partial t}$ (22-5)

For shear waves, $\alpha = \beta - \pi/2$, $\quad y = \frac{L}{V} \sin\alpha \cos\alpha \frac{\partial \xi}{\partial t}$ (22-6)

We note that the dependence of the response on the direction to the source is quite different for a strain seismograph and a pendulum. Imagine a source of compression at various positions around the two types of instruments, with $\alpha = 0$ corresponding to either of the two directions of maximum sensitivity to p-waves.

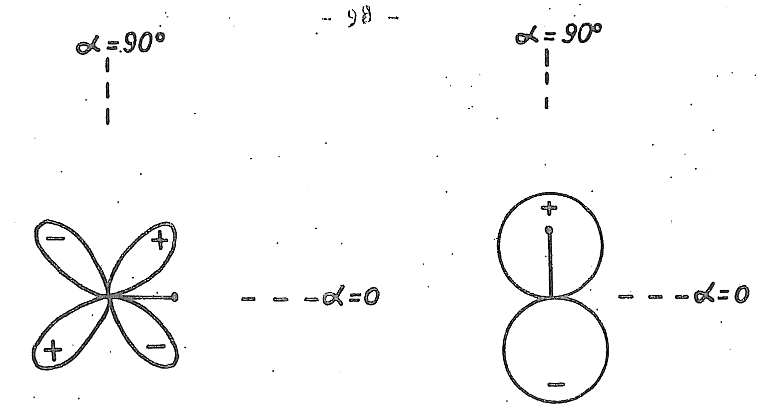




P - wave response

Because the pendulum response varies as cos α, the response will be maximum at 0°, and be of opposite sign for the source in the second and third quadrants. On the other hand, the strain meter sensitivity varies as $\cos^2\alpha$, so the directivity is sharper, and the sign of the signal does not change. It doesn't make any difference whether you push from right or left, the strain meter sees only whether the two piers become closer together or farther apart.

The response of a pendulum to S waves (assume the same polarity of the source at all azimuths) varies as sin α, so that the response is zero at $\alpha = 0$ and 180°, and is maximum at 90° and 270°, but with opposite signs. The strain meter response to S-waves, on the other hand, has four zero, corresponding to $\sin\alpha = 0$ and $\cos\alpha = 0$.



strain seismograph *Horizontal Pendulum*

S - wave response

Suppose two strain seismographs are oriented at right angles, so that a wave approaching one at the angle α approaches the other at $(\alpha - 90°)$. Let the outputs, y, be added. The response to motion in the place of incidence is

$$-\frac{L}{V}\frac{\partial \xi}{\partial t}\left[\cos^2\alpha + \sin^2\alpha\right] = -\frac{L}{V}\frac{\partial \xi}{\partial t},$$

independent of α. At the same time, the response to transverse motion is

$$+\frac{L}{V}\frac{\partial \xi}{\partial t}\left[\sin\alpha \cos\alpha - \sin\alpha \cos\alpha\right] = 0 .$$

Thus, the combination eliminates transvers motion, such as Love waves, but records horizontal motion in the plane of incidence with sensitivity that is independent of azimuth. Thus Rayleigh waves are separated from Love waves.

If we omit the directional factors from (22-5) and (22-6)(put the minus here with the direction) and substitute in (22-4), the output of an electromagnetic transducer mounted between the rod and the free pier

$$E = \frac{\Gamma_T L}{V}\frac{\partial^2 \xi}{\partial t^2}$$

If this transducer is connected to a galvanometer, so that the total resistance in the circuit is R, the current is $\frac{\Gamma_T L}{VR}\ddot{\xi}$, and the moment driving the galvanometer is

$$\frac{\gamma \Gamma_T L}{VR}\ddot{\xi}.$$

The equation of motion becomes

$$\ddot{\theta} + 2h_1\omega_{n1}\dot{\theta} + \omega_{n1}^2\theta = \frac{\gamma \Gamma_T L}{K_1 VR}\ddot{\xi}$$

This is exactly the equation for the response of a direct-recording seismograph (8-3) with the constant $(-1/\mathcal{L})$ replaced by the coefficient of $\ddot{\xi}$. Thus, a strain seismograph with an electromagnetic transducer has the frequency response of whatever galvanometer is used. With long period galvanometers available, the frequency response equivalent to long period pendulums can be achieved. Since the trace amplitude is $2A_1\theta$, where A_1 is the distance from the galvanometer mirror to the scale, the equivalent static magnification, V_o, is

$$\frac{2A_1\gamma\Gamma_T L}{K_1 RV},$$

The sensitivity is therefore dependent on the apparent surface velocity.

Exercise: What is the sensitivity of a strain seismograph to waves travelling vertically upward? Explain in terms of the relative motion of the two piers.

If a strain metef is to be used for recording very slow strains, such as those associated with earth tides and secular strain increments, a displacement transducer rather than a velocity transducer is required. Benioff (1959) used a bow-string and mirror mechanism to achieve mechanical-optical magnification of relative displacement of 10^4, so that with a length of 24.08 meters, a trace displacement of 1 mm was produced by a strain of 4.15×10^{-9}.

In the same paper, Benioff described a capacitance displacement transducer that is now widely used. Two parallel plates are anchored to the pier and electrically insulated. A third plate mounted on the quartz tube is located between the other two, and is grounded. The two capacitors thus formed are part of two L-C circuits with the same resonant frequency. A stable 5 MCPS oscillator is coupled to the two resonant circuits, whose frequencies are slightly different from the oscillator frequency. Displacement of the center plate relative to the other two changes the two capacitances by the same amount, but in opposite senses. Thus, one of the resonant circuits is brought closer to the oscillator frequency, the other is taken farther from it. The currents in the two circuits are now different, and the difference is proportional to the displacement of the center plate.

With this transducer, magnification of more than 10^5 is available. Practically the magnification is limited by the maximum tidal strain. In the instrument described by Benioff, magnification was 80,000, and with a 24 meter tube, the sensitivity is such that a trace amplitude of 1 mm corresponds to a strain of 5×10^{-10} or 1 mm in 2000 km.

23. Special applications of seismic instruments.

By means of the theory in the preceding sections, the various special designs of instruments for particular applications may be analyzed. The underlying principles are the same -- only the frequency range, sensitivity level, and recording medium must be selected for the task at hand.

Conventional earthquake observatory instruments apply these principles in a straightforward and unadorned fashion. The goal is reliablility and stability of performance, 24 hours a day, year after year. Modern electronic devices, which can give great flexibility and versatility to an instrument system, are excellent for use in research installations and for instruments that record for short periods of time, Under the watchful eye of a skilled technician they are of great value. Such devices have been generally shunned, however, for routine observatory use.

23. 1 Strong-motion instruments.

The purpose of strong-motion instruments is to record the complex and violent motions within the area of destruction of earthquakes, in order to provide structural engineers input data for earthquake-resistant construction. Motion associated with industrial blasting operations is also monitored by such instruments. Low magnification and sensitivity to high frequencies are required.

The instruments may be designed as accelerometer ($T_n \ll T_e$), as the Ishimoto or Wenner accelerometers, or as displacement meters, $T_n \gg T_e$. Because of the need for a fast chart speed to make the resolution of high frequency motion possible, economy requires that strong-motion instruments operate only while an event is in progress. Instruments for strong earthquake motion observations are equipped with a pendulum starter that is activated when the ground motion exceeds a predetermined level. This starter engages a holding relay which cause the recorder to run for a time considered long enough to include the whole motion, say one minute.

This technique means that the onset of the motion is not recorded, a loss that could be disastrous to a seismologist, but not so serious to the earthquake engineer. In a few cases a small foreshock has started the recorder, so that it was running when the main shock occurred. The records in these cases are obviously very valuable.

A technique introduced by Gane (<u>Bull</u>. <u>Seis</u>. <u>Soc</u>. <u>Amer</u>., 1949, p. 117) in South Africa combines the economy of the self-starter with the advantages of continuous recording. A closed magnetic tape loop is used as a delay line. This loop runs continuously, and the seismometer output is recorded. Along the loop, at position representing a delay of some six seconds in Gane's instrument, a playback head is located, followed by erasure, and back to the record head. If no event of sufficient size to trigger the recorder occurs, the output of the seismometer is discarded. If a big enough event occurs, the delay between record and playback is long enough that the recorder is running before the onset signal reaches the playback head, and the entire event is recorded.

Other strong motion instruments such as the Leet and Sprengnether instruments are designed for blast recording. The origin time of the event is known ahead of time, so that the recorder can be turned on manually just before the event.

23. 2 Seismic exploration instruments.

In the seismic reflection technique, chief interest is in the 20-100 cps pass-band, whereas refraction methods employ somewhat lower frequencies. The seismometer used in reflection word are high-frequency vertical component units with small masses, suitable for deployment in large numbers to form arrays.

The output of the seismometers in subjected to extensive processing, either on-line or subsequently. The signals are amplified, filtered, and signals from different channels are composited to give the optimum record of the reflections. Filtering is done in time, using networks, or in space, using arrays of seismometers.

It is common for the electronic amplification of the system to be programmed so that the gain increases in a predetermined manner as time passes. In this way deep, weak reflections and shallow, strong reflections can be accommodated on the same record. Automatic gain control is also comployed, a non-linear prosess that makes the recovery of actual ground motion impossible. This is usually of no concern to the exploration seismologist, as he is interested in time of arrival primarily. However, as exploration targets become more difficult, amplitude data are being incorporated in the analysis.

Photographic records may use displays in the form of a conventional "wiggly line", variable area recording, variable density recording, or combination of these. Dramatic improvement in the percdptibility of reflections results from the use of these presentations.

Initial recording broad-band on magnetic tape is now almost universal, with filtering done later. Corrections for elevation, normal move-out, weathered-layer thickness, etc. can be applied in the playback unit. In the last few years systems have come into use in which the seismometer magnetic tape. With digital recording in the field, processing by digital computers is practical and makes the application of modern concepts of time-series analysis to routine seismic interpretation practical.

23.3 Ocean-bottom, bore-hole, and lunar seismographs.

Progress in geophysics has always followed the extended ability of research workers to put instruments where they want them, and this applies in seismology as well as the other geophysical sciences.

With almost three-fourths of the earth's surface covered by deep water, the value of an ocean-bottom seismograph is obvious. Instruments abld to operate in the difficult environment of the sea floor have been developed in the United States, and an instrument is under development in Japan. In one concept, the instrument is completely self-contained. A 30-day magnetic tape recorder is incorporated, and the assembly is dropped to the ocean bottom with no connections to land or even to the surface. At the end of a predetermined time interval, the instrument package is broken loose from the base plate, and pops up to the surface. A radio beacon and a flashing light are activated to make recoverly possible. Recovery of the units is perhaps the most difficult part of the operation.

In another approach, the instrument is connected by calbe to shore. Emplacements as far as 100 kilometers off-shore have been used. The original installation is much more costly, but onee succdssfully in place, offers a more permanent recording site. Because of limitations imposed by the need to lay cables, this techniwue can only extend existing land-based observing a small amoutn out from the margins of continents. However, a glance at a map of world seismicity reveals that an extension of even 100 kilometers from continental margins or important islands brings some of the most seismic areas on earth within the reach of direct seismic observations.

Another rather hostile environment in which seismometers are now routinely operated is in boreholes. These may be from less than 100 meters to several kilometers deep. The goal is to place the instruments deep enough that they are beneath high-frequency noise propagating as Rayleigh waves. Because the amplitude of an elastic wave is smaller in the interior than at the surface by a factor as big as 2, depending on the angle of incidence, the instrument must be deep enough that the noise attenuation is greater than 2 before any improvement in signal-to-noise ratio is achieved.

Bore-hole instruments must be waterproof, and equipped with cables and connectors able to operate for a long time in water. The cable must be arranged so that surface noise is not transmitted down the cable to the instrument.

The successful emplacement of a seismograph on the moon will provide important data for understanding that body, as well as the planetary system in general (F. Press, et al, Jour. Geophys. Res., Vol. 65, pp. 3079-3105, 1960). Instruments have been designed and built for use on the moon (F.F. Lehner, et al., Jour. Geophys. Res., Vol. 67, pp. 4779-4794, 1962), and even sent on theif way in the early Ranger flights. Unfortunately, none of these instruments was delivered as intended, and so the seismological community is still awaiting its first lunar seismic data.

23.4 Seismic arrays.

If a number of seismometers are emplaced in a pattefn and the outputs combined, the resulting seismic array is an instrument itself, with capabilities beyond the individual elements that go to make it up. The analysis of seismic arrays is beyond the scope of this course, but considerable experience in their use has been gained in both seismic exploration and in super-sensitive observatories. A discussion of underlying princip,es is found in N.F. Barker, "Fourier methods in Geophysics", in vol. 2 of Methods and Techniques in Geophysics, 1966, Section 6. Directional Arrays.

Compare the response of a group of seismometers scattered over the earth's surface to two signals, one of which has a very long apparent wave length along the surface compared to the dimensions of the array, the other rather short. Examples would be a P wave coming up almost vertically, so that the wave front is almost parallel to the surface, and a train of high-frequehcy Rayleigh waves, travelling, of course, horizontally. The motion associated with the long apparent wave length is practically in phase at all detectors, and if the outputs are added, t is signal is reinforced. However, the signal with shorter wave length will be out of phase at some detectors compared to others, so that in the summation, this signal is attenuated.

A further step is to take advantage of the fact that seismic noise propagates with different speeds than the seismic signals that are usually of interest. By time-shifting the output of the variaus detectors by an amout corresponding to the apparent surface velocity of the desired signal across the array, this signal is further enhancedover the straight summation process, and the noise not so enhanced, giving further improvement in signal-to-noise ratio.

By comparing the time of arrival of an event at the various lelments of the array, the direction of approach of the signal may be determined. From the apparent surface velocity and knowledge of the true velocity in the surface layer of the kind of body wave (P or S) under observation. the angle of incidence can be derived, providing an added tool for identification of phases, especially crustal phases from regional and local events.