Notes on stress, strain and rotation

June 8, 2021

Robert B. Herrmann

Contents

1 Green's functions *page* 1
- 1.1 Introduction 1
- 1.2 Introduction 2
- 1.3 Wave equation solutions in cylindrical coordinates 2
- 1.4 Green's functions 3
- 1.5 Computation of stress, strain and rotation 6
  - 1.5.1 Wavenumber integration 7
  - 1.5.2 Modal superposition 7
- 1.6 Conversion of cylindrical strain to cartesian 8

***Appendix A* Stress, strain and rotetion** 10
- A.1 Introduction 10
- A.2 Strain 10
  - A.2.1 Cartesian coordinate system rotation 11
- A.3 Cylindrical coordinate systems 12
- A.4 Computation of Cartesian strains, stresses and rotations 15
- *Bibliography* 17

1

Green’s functions

1.1 Introduction

Many texts give expressions for displacements in plane-layered media due to point forces and moment tensors (Levshin and Yanson, 1971; Takeuchi and Saito, 1972; Keilis-Borok, 1989; Aki and Richards, 2002). These solutions are useful in regional moment tensor studies. However there are occasions when one in interested in the stresses and strains generated by a seismic source. Ground motions of a large earthquake may be such to change the stresses acting on neighboring faults in a way to facilitate local faulting. The newly introduced Distributed Acoustic Sensors (DAS) systems measure the strain in a fiber optic cable with great spatial detail. Interpreting earthquake data requires codes for predicting the observed strain.

In a cylindrical coordinate system for isotropic or transverse isotropic media, 15 Green’s functions must be computed to represent the displacement wave field due to point force and moment tensor sources. If one is interested in strain, then the partial derivatives of the displacement with respect to the z and r coordinates at the observation point will require an additional 30 functions to be computed. The partials with respect to azimuth will not require any significant computational effort since .

Computationally one performs a wavenumber integration to obtain the complete solution, but when the epicentral distance is large compared to the wavelength, a superposition of surface-wave models can provide a reasonable approximation to the exact solution by modeling the larger signals following S. This discussion review s the Green’s functions and and shows the modifications required to generate strain, stress and rotation time series.

1.2 Introduction

Progress in seismology has always involved the interaction of advances in instrumentation and computation. Recently there has been an emphasis on measuring strain using DAS (distributed acoustic sensors) and rotation. Thus the question of quantitative monitoring of these observations arises. Strain measurements have long been a topic in earthquake seismology (Benioff) using strain meters and dilatometers (Sacks) as well as in experimental rock mechanic procedures. Stress, which is derivable from strain one the material properties are know, is of interest in the remote triggering of earthquakes (Landers - Spudich).

The focus on this chapter is the generation of stress, strain and rotation time series for point force and moment tensor sources in plane-layered isotropic media. In line with the development through out the text, the synthetics will be computed using a cylindrical coordinate system. Strains were defined for a Cartesian system in §**??** and for a cylindrical coordinate system in §1.3. Before adapting the previous development to give stress, strain and rotation, we should review some continuum mechanics and derive expression for stress and strain in cylindrical coordinates (**??**-**??**).

1.3 Wave equation solutions in cylindrical coordinates

For a cylindrical coordinate system with coordinates (r, ϕ, z), the equations of motion for the displacement $\mathbf{u} = (u_r, u_\phi, u_z)$ are (Love, 1944; Aki and Richards, 2002)

$$
\begin{aligned}
\rho \frac{\partial^2 u_z}{\partial t^2} &= \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\phi z}}{\partial \phi} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + F_z \\
\rho \frac{\partial^2 u_r}{\partial t^2} &= \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} + F_r \\
\rho \frac{\partial^2 u_\phi}{\partial t^2} &= \frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\partial \sigma_{\phi z}}{\partial z} + \frac{2\sigma_{r\phi}}{r} + F_\phi
\end{aligned}
$$

The local coordinate system is assumed to be such that z is positive downward.

In an isotropic medium, the stresses are related to displacements in a cylindrical coordinate system through the relations (Hughes and Gaylord, 1964)

$$
\begin{aligned}
\sigma_{rz} &= 2\mu e_{rz} & \sigma_{zz} &= \lambda\Delta + 2\mu e_{zz} \\
\sigma_{\phi z} &= 2\mu e_{\phi z} & \sigma_{rr} &= \lambda\Delta + 2\mu e_{rr} \\
\sigma_{rz} &= 2\mu e_{rz} & \sigma_{\phi\phi} &= \lambda\Delta + 2\mu e_{\phi\phi}
\end{aligned}
\qquad (1.3.1)
$$

where the strains are defined as

$$
\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r} & e_{r\phi} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) \\
e_{\phi\phi} &= \frac{1}{r}\left(\frac{\partial u_\phi}{\partial \phi} + u_r\right) & e_{rz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \\
e_{zz} &= \frac{\partial u_z}{\partial z} & e_{\phi z} &= \frac{1}{2}\left(\frac{\partial u_\phi}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \phi}\right)
\end{aligned}
\tag{1.3.2}
$$

and the dilatation Δ is given by

$$
\Delta \;=\; \nabla \cdot \mathbf{u} \;=\; \frac{1}{r}\frac{\partial (r u_r)}{\partial r} \;+\; \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} \;+\; \frac{\partial u_z}{\partial z}. \tag{1.3.3}
$$

The rotations are defined as

$$
\begin{aligned}
\omega_{r\phi} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi} - \frac{\partial u_\phi}{\partial r} - \frac{1}{r}u_\phi\right) \\
\omega_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) \\
\omega_{\phi z} &= \frac{1}{2}\left(\frac{\partial u_\phi}{\partial z} - \frac{1}{r}\frac{\partial u_z}{\partial \phi}\right)
\end{aligned}
\tag{1.3.4}
$$

The derivation of these expressions for strain and rotation are given in the Appendix.

1.4 Green's functions

For problems in which the material properties only vary in the z-direction, define the displacements as

$$
\begin{aligned}
u_z(r,z,\omega) &= \sum_n \Big(A_n \cos n\phi + B_n \sin n\phi\Big) \\
&\quad \cdot \int_0^\infty U_z(k,z,\omega) J_n(kr) k dk \\
u_r(r,z,\omega) &= \sum_n \Big(A_n \cos n\phi + B_n \sin n\phi\Big) \\
&\quad \cdot \int_0^\infty \left[U_r(k,z,\omega)\frac{\partial J_n(kr)}{\partial r} - \frac{n}{r} U_\phi(k,z,\omega) J_n(kr)\right] dk \qquad (1.4.1) \\
u_\phi(r,z,\omega) &= \sum_n \Big(A_n \sin n\phi - B_n \cos n\phi\Big)
\end{aligned}
$$

$$\cdot \int_0^\infty \left[U_\phi(k,z,\omega)\frac{\partial J_n(kr)}{\partial r} - \frac{n}{r} U_r(k,z,\omega) J_n(kr) \right] dk$$

and the force per unit volume as

$$\begin{aligned}
F_z(r,z,\omega) &= \sum_n \left(A_n \cos n\phi + B_n \sin n\phi \right) \\
&\quad \cdot \int_0^\infty f_z(k,z,\omega) J_n(kr) k dk \\
F_r(r,z,\omega) &= \sum_n \left(A_n \cos n\phi + B_n \sin n\phi \right) \\
&\quad \cdot \int_0^\infty \left[f_r(k,z,\omega)\frac{\partial J_n(kr)}{\partial r} - \frac{n}{r} f_\phi(k,z,\omega) J_n(kr) \right] dk \qquad (1.4.2) \\
F_\phi(r,z,\omega) &= \sum_n \left(A_n \sin n\phi - B_n \cos n\phi \right) \\
&\quad \cdot \int_0^\infty \left[f_\phi(k,z,\omega)\frac{\partial J_n(kr)}{\partial r} - \frac{n}{r} f_r(k,z,\omega) J_n(kr) \right] dk
\end{aligned}$$

If one defines the transformed stresses as

$$\begin{aligned}
T_r &= \mu \left(\frac{dU_r^{(n)}}{dz} + kU_z^{(n)} \right) \\
T_z &= (\lambda + 2\mu)\frac{dU_z^{(n)}}{dz} - k\lambda U_r^{(n)} \qquad (1.4.3) \\
T_\phi &= \mu \frac{dU_\phi^{(n)}}{dz}
\end{aligned}$$

then the following ordinary differential equations must be solved for P-SV

$$\frac{d}{dz}\begin{bmatrix} U_r \\ U_z \\ T_z \\ T_r \end{bmatrix} = \begin{bmatrix} 0 & -k & 0 & \frac{1}{\mu} \\ \frac{k\lambda}{\lambda+2\mu} & 0 & \frac{1}{\lambda+2\mu} & 0 \\ 0 & -\rho\omega^2 & 0 & k \\ -\rho\omega^2 + \frac{4k^2\mu(\lambda+\mu)}{\lambda+2\mu} & 0 & \frac{-k\lambda}{\lambda+2\mu} & 0 \end{bmatrix} \cdot \begin{bmatrix} U_r \\ U_z \\ T_z \\ T_r \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ f_z \\ f_r \end{bmatrix} \qquad (1.4.4)$$

and

$$\frac{d}{dz}\begin{bmatrix} U_\phi \\ T_\phi \end{bmatrix} = \begin{bmatrix} 0 & 1/\mu \\ \mu k^2 - \rho\omega^2 & 0 \end{bmatrix} \begin{bmatrix} U_\phi \\ T_\phi \end{bmatrix} - \begin{bmatrix} 0 \\ f_\phi \end{bmatrix} \qquad (1.4.5)$$

for SH. In these equations, the transformed displacements are functions of wavenumber, angular frequency and vertical position. Because none of the terms within the square matrices involve derivatives with respect to z, and since the medium parameters vary continuously or piecewise continuously with the z-coordinate, we immediately see that the parameters U_r, U_z, U_ϕ, T_r, T_z and T_ϕ must be continuous at depths where the force terms are zero. Discontinuities in these parameters will occur when crossing the source layer. These discontinuities are used with the propagator and reflection matrix techniques for solving these differential equations. Modal superposition techniques work directly with the forces.

Solutions of the wave equation in cylindrical coordinates for a point force and/or moment tensor source can be written as follows:

$$\begin{aligned}
u_z(r,z,h,\omega) &= (F_1\cos\phi \ + \ F_2\sin\phi)\,ZHF \ + \ F_3 ZVF \\
&+ \ M_{11}\left[\frac{ZSS}{2}\cos(2\phi) - \frac{ZDD}{6} + \frac{ZEX}{3}\right] \\
&+ \ M_{22}\left[\frac{-ZSS}{2}\cos(2\phi) - \frac{ZDD}{6} + \frac{ZEX}{3}\right] \\
&+ \ M_{33}\left[\frac{ZDD}{3} + \frac{ZEX}{3}\right] \\
&+ \ M_{12}\left[ZSS\ \sin(2\phi)\right] \\
&+ \ M_{13}\left[ZDS\ \cos(\phi)\right] \\
&+ \ M_{23}\left[ZDS\ \sin(\phi)\right] \\
u_r(r,z,h,\omega) &= (F_1\cos\phi \ + \ F_2\sin\phi)\,RHF \ + \ F_3 RVF \\
&+ \ M_{11}\left[\frac{RSS}{2}\cos(2\phi) - \frac{RDD}{6} + \frac{REX}{3}\right] \\
&+ \ M_{22}\left[\frac{-RSS}{2}\cos(2\phi) - \frac{RDD}{6} + \frac{REX}{3}\right] \\
&+ \ M_{33}\left[\frac{RDD}{3} + \frac{REX}{3}\right] \\
&+ \ M_{12}\left[RSS\ \sin(2\phi)\right] \\
&+ \ M_{13}\left[RDS\ \cos(\phi)\right] \\
&+ \ M_{23}\left[RDS\ \sin(\phi)\right] \\
u_\phi(r,z,h,\omega) &= (-F_1\sin\phi + F_2\cos\phi)\,THF \\
&+ \ M_{11}\left[\frac{TSS}{2}\sin(2\phi)\right] \\
&+ \ M_{22}\left[\frac{-TSS}{2}\sin(2\phi)\right] \\
&+ \ M_{12}\left[-TSS\ \cos(2\phi)\right] \\
&+ \ M_{13}\left[TDS\ \sin(\phi)\right] \\
&+ \ M_{23}\left[-TDS\ \cos(\phi)\right]\ .
\end{aligned} \tag{1.4.6}$$

This expression assumes that the moment tensor, M_{ij}, is symmetric. The terms associated F_1, F_2 and F_3 are the medium response to a point force, while the other functions are the response to specific moment tensor expressions. The functions, e.g., ZSS, within the square brackets are the Green's functions for one particular representation of forces. The terminology used for these basic force and moment tensor solutions is simple. The leading Z, R or T, indicates the component of motion. The SS indicates that the solution is due to a strike-slip source, with only $M_{12} \neq 0$ or with $M_{11} = -M_{22}$ with other elements zero. The DS solution is associated with a vertical dip-slip source with only $M_{13} \neq 0$ or $M_{23} \neq 0$. The EX solution is for an isotropic center of expansion source with $M_{11} = M_{22} = M_{33}$. The DD solution does not correspond to a fault source, but can be understood as that part of a vertical or radial displacements for a 45° dip-sip source (e.g., $M_{22} = -M_{33}$) observed at an azimuth of 45°. The DD component is multiplied by the terms $2M_{33} - M_{11} - M_{22}$ which is known as a compensated linear vector dipole.

1.5 Computation of stress, strain and rotation

Rather than using explicit expressions for strain, stress and rotation, the decision was made to first generate the time series for the partial derivatives of the displacement with respect to r, ϕ and z, and then later combine them to make the desired time series for strain or rotation. To illustrate the required steps, we consider the vertical displacement for a strike-slip source defined by M_{12} with all other terms equal to zero. Starting with

$$u_z(r,z,h,\omega) \ = \ + M_{12} \ [ZSS \ \sin(2\phi)] \tag{1.5.1}$$

The partial derivatives are

$$\frac{\partial u_z}{\partial r}(r,z,h,\omega) \ = M_{12} \left[\frac{\partial ZSS}{\partial r} \ \sin(2\phi)\right] \tag{1.5.2}$$

$$\frac{\partial u_z}{\partial \phi}(r,z,h,\omega) \ = M_{12} \ [2ZSS \ \cos(2\phi)] \tag{1.5.3}$$

$$\frac{\partial u_z}{\partial z}(r,z,h,\omega) \ = M_{12} \left[\frac{\partial ZSS}{\partial z} \sin(2\phi)\right] \tag{1.5.4}$$

1.5.1 Wavenumber integration

Thus it is necessary to compute time series for ZSS, $\partial ZSS/\partial r$ and $\partial ZSS/\partial z$. From (1.4.1) we see that

$$\frac{\partial ZSS}{\partial r} = \int_0^\infty U_z \frac{\partial J_n(kr)}{\partial r} kdk \tag{1.5.5}$$

$$\frac{\partial ZSS}{\partial z} = \int_0^\infty \frac{\partial U_z^{SS}}{\partial z} J_n(kr)kdk \tag{1.5.6}$$

$$= \int_0^\infty \frac{1}{\lambda + 2\mu}\left(T_z^{SS} + kU_r^{SS}\right)J_n(kr)kdk \tag{1.5.7}$$

The latter expression arise from the definition of T_z in (1.4.3).

For a arbitrary source, We note that 15 functions, e.g., ZDD, ..., THF must be computed to obtain the displacements and $\partial/\partial\phi$ and another 30 for obtain the $\partial/\partial r$ and $\partial/\partial z$.

1.5.2 Modal superposition

For media that have locked mode solutions, modal superposition techniques can be used to form the time series corresponding to the pole contributions (Levshin and Yanson, 1971; Takeuchi and Saito, 1972; Keilis-Borok, 1989; Aki and Richards, 2002). Following Levshin and Yanson (1971), the far-field pole contribution of integrals of the form

$$u_z(r, z, h, \omega) = \int_0^\infty U_z(k, z, h, \omega)J_n(kr)kdk \tag{1.5.8}$$

is of the form

$$u_z(r, z, h, \omega) = -\pi i A_{L,R} D(k_m, h, \omega)U_z(k_m, z, \omega)H_n^{(2)}(k_m r) \tag{1.5.9}$$

$$= -\pi i A_{L,R} D(k_m, h, \omega)U_z(k_m, z, \omega)\sqrt{\frac{2}{\pi_m r}} e^{-i(k_m r - \frac{\pi}{4} - \frac{n\pi}{2})} \tag{1.5.10}$$

$$\tag{1.5.11}$$

In these equations h is the source depth, z is the receiver depth, $k_m = \omega/c_m$ is the wavenumber corresponding to the phase velocity c_m. The U_z evaluated at the receiver depth is the eigenfunction corresponding to the homogeneous solution of (1.4.4, 1.4.5). The $D()$ is a function of wavenumber and the eigenfunctions at the source depth.

To obtain the required partial derivatives of the displacement, the $\partial/\partial\phi$ is com-

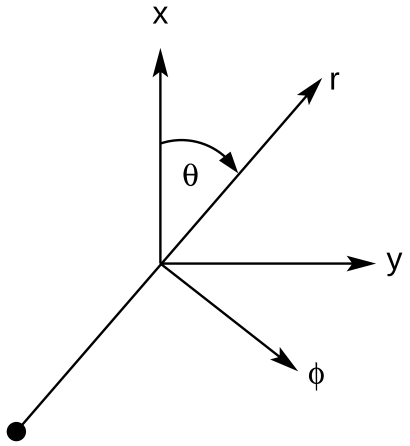

Figure 1.1 Transformation between (r, ϕ, z) coordinate system to an (x, y, z) coordinate system. The z-coordinate is down into the figure. The ϕ component of motion is the transverse component. Often the (x, y) axes are aligned north and east, respectively. In the case of DAS systems, one might align the x-axis with the direction of the fiber.

puted as before. The other partials are

$$\begin{aligned}
\frac{\partial u_z}{\partial r}(r, z, h, \omega) &= -\pi i A_{L,R} D(k_m, h, \omega) U_z(k_m, z, \omega)(-ik_m) \sqrt{\frac{2}{\pi_m r}} e^{-i(k_m r - \frac{\pi}{4} - \frac{n\pi}{2})} \\
\frac{\partial u_z}{\partial z}(r, z, h, \omega) &= -\pi i A_{L,R} D(k_m, h, \omega) \frac{\partial U_z}{\partial z}(k_m, z, \omega) \sqrt{\frac{2}{\pi_m r}} e^{-i(k_m r - \frac{\pi}{4} - \frac{n\pi}{2})} \qquad (1.5.12) \\
&= -\pi i A_{L,R} D(k_m, h, \omega) \frac{1}{\lambda + 2\mu} \Big(T_z + k U_r \Big) \sqrt{\frac{2}{\pi_m r}} e^{-i(k_m r - \frac{\pi}{4} - \frac{n\pi}{2})}
\end{aligned}$$

1.6 Conversion of cylindrical strain to cartesian

The choice of using a cylindrical coordinate system to describe wave propagation was made for computational efficiency, since the cartesian displacement at any point can be obtained from the cylindrical through a simple coordinate system rotation. Consider the coordinate system shown in Figure 1.1. The displacements in the (x, y, z) coordinate system are related to those in the (r, ϕ, z) system through the transformation

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_r \\ u_\phi \\ u_z \end{bmatrix}$$

Bower (2010) showed how to relate stresses in a cylindrical coordinate system to those in a Cartesian system. Rearranging those gives

$$\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\sigma_{rr} & \sigma_{r\phi} & \sigma_{rz}\\ \sigma_{r\phi} & \sigma_{\phi\phi} & \sigma_{\phi z}\\ \sigma_{rz} & \sigma_{\phi z} & \sigma_{zz}\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

A similar transformation is used to relate cartesian strains to cylindrical strains.

$$\begin{bmatrix}e_{xx} & e_{xy} & e_{xz}\\ e_{xy} & e_{yy} & e_{yz}\\ e_{xz} & e_{yz} & e_{zz}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}e_{rr} & e_{r\phi} & e_{rz}\\ e_{r\phi} & e_{\phi\phi} & e_{\phi z}\\ e_{rz} & e_{\phi z} & e_{zz}\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

Finally the expressions for rotation are

$$\begin{bmatrix}0 & \omega_{xy} & \omega_{xz}\\ -\omega_{xy} & 0 & \omega_{yz}\\ -\omega_{xz} & -\omega_{yz} & 0\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & \omega_{r\phi} & \omega_{rz}\\ -\omega_{r\phi} & 0 & \omega_{\phi z}\\ -\omega_{rz} & -\omega_{\phi z} & 0\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

Appendix A

Stress, strain and rotetion

A.1 Introduction

The focus on this appendix is the expression of stress, strain and rotation in cylindrical coordinates. Strains for a Cartesian system in are defined as $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ and for a cylindrical coordinate system in §1.3. This appendix derives the expressions for strain oin cylindrical coordinates.

A.2 Strain

Following (Sollberger et al., 2020), consider a point $\mathbf{x}$ where there is a displacement $\mathbf{u}$, e.g., $\mathbf{u} = \mathbf{u}(\mathbf{x})$. At a position $\mathbf{x} + \delta\mathbf{x}$, the displacement would be $\mathbf{u}(\mathbf{x} + \delta\mathbf{x}) \approx \mathbf{u}(\mathbf{x}) + \delta\mathbf{u}$ where

$$\delta\mathbf{u} = \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{bmatrix} = \mathbf{G} \begin{bmatrix} \delta x_1 \\ \delta u_2 \\ \delta x_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta u_2 \\ \delta x_3 \end{bmatrix} \tag{A.2.1}$$

The matrix $\mathbf{G}$ can be written as

$$\mathbf{G} = \frac{1}{2}\left(\mathbf{G} + \mathbf{G}^T\right) + \frac{1}{2}\left(\mathbf{G} - \mathbf{G}^T\right) \tag{A.2.2}$$

The differential displacement can also be written as

$$\delta\mathbf{u} = \epsilon\,\delta\mathbf{x} + \mathbf{\Omega}\,\delta\mathbf{x} \tag{A.2.3}$$

The first term on the right is the infinitesimal strain tensor and the second is the rotation tensor, e.g.,

$$\epsilon = \frac{1}{2}\left(\mathbf{G} + \mathbf{G}^T\right) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{A.2.4}$$

and

$$\mathbf{\Omega} = \frac{1}{2}\left(\mathbf{G} - \mathbf{G}^T\right) = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix}. \tag{A.2.5}$$

The differential displacment can also be written as

$$\delta\mathbf{u} = \epsilon\,\delta\mathbf{x} + \mathbf{\Omega}\,\delta\mathbf{x} = \epsilon\,\delta\mathbf{x} + \overrightarrow{\omega} \times \delta\mathbf{x} \tag{A.2.6}$$

where the rotation vector is defined as

$$\overrightarrow{\omega} = \frac{1}{2}\nabla \times \mathbf{u} = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \\ \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \end{bmatrix} \tag{A.2.7}$$

This vector notation shows that the ω_{12} element of Ω is associated with the $\mathbf{e}_3$ vector component of $\overrightarrow{\omega}$, indicating a rotation in the $1 - 2$ plane.

A.2.1 Cartesian coordinate system rotation

Now consider a primed Cartesian coordinate system rleated to the unprimed system through the transformation matrix, such that

$$\begin{bmatrix} x_{1'} \\ x_{2'} \\ x_{3'} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{T}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{A.2.8}$$

where $\mathbf{T}$ is an orthonormal matrix, e.g., $\mathbf{T}^T = \mathbf{T}^{-1}$. In addition, the displacments in the two coordinate systems are related by $\mathbf{u}' = \mathbf{T}^T\mathbf{u}$. By the chain rule of differentiation,

$$\begin{aligned} \frac{\partial()}{\partial x_1} &= \frac{\partial()}{\partial x_{1'}}\frac{\partial x_{1'}}{\partial x_1} + \frac{\partial()}{\partial x_{2'}}\frac{\partial x_{2'}}{\partial x_1} + \frac{\partial()}{\partial x_{3'}}\frac{\partial x_{3'}}{\partial x_1} \\ &= \frac{\partial()}{\partial x_{1'}}a_{11} + \frac{\partial()}{\partial x_{2'}}a_{21} + \frac{\partial()}{\partial x_{3'}}a_{31} \end{aligned} \tag{A.2.9}$$

The partials with respect to x_2 and x_3 are similarly defined. The deformation in this new coordinate system is defined as

$$\delta\mathbf{u}' = \epsilon'\,\delta\mathbf{x}' + \mathbf{\Omega}'\,\delta\mathbf{x}' \tag{A.2.10}$$

where

$$\epsilon' = \begin{bmatrix} \frac{\partial u_{1'}}{\partial x_{1'}} & \frac{1}{2}\left(\frac{\partial u_{1'}}{\partial x_{2'}} + \frac{\partial u_{2'}}{\partial x_{1'}}\right) & \frac{1}{2}\left(\frac{\partial u_{1'}}{\partial x_{3'}} + \frac{\partial u_{3'}}{\partial x_{1'}}\right) \\ \frac{1}{2}\left(\frac{\partial u_{2'}}{\partial x_{1'}} + \frac{\partial u_{1'}}{\partial x_{2'}}\right) & \frac{\partial u_{2'}}{\partial x_{2'}} & \frac{1}{2}\left(\frac{\partial u_{2'}}{\partial x_{3'}} + \frac{\partial u_{3'}}{\partial x_{2'}}\right) \\ \frac{1}{2}\left(\frac{\partial u_{3'}}{\partial x_{1'}} + \frac{\partial u_{1'}}{\partial x_{3'}}\right) & \frac{1}{2}\left(\frac{\partial u_{3'}}{\partial x_{2'}} + \frac{\partial u_{2'}}{\partial x_{3'}}\right) & \frac{\partial u_{3'}}{\partial x_{3'}} \end{bmatrix} \tag{A.2.11}$$

and

$$\mathbf{\Omega}' = \begin{bmatrix} 0 & \frac{1}{2}\Big(\frac{\partial u_{1'}}{\partial x_{2'}} - \frac{\partial u_{2'}}{\partial x_{1'}}\Big) & \frac{1}{2}\Big(\frac{\partial u_{1'}}{\partial x_{3'}} - \frac{\partial u_{3'}}{\partial x_{1'}}\Big) \\ \frac{1}{2}\Big(\frac{\partial u_{2'}}{\partial x_{1'}} - \frac{\partial u_{1'}}{\partial x_{2'}}\Big) & 0 & \frac{1}{2}\Big(\frac{\partial u_{2'}}{\partial x_{3'}} - \frac{\partial u_{3'}}{\partial x_{2'}}\Big) \\ \frac{1}{2}\Big(\frac{\partial u_{3'}}{\partial x_{1'}} - \frac{\partial u_{1'}}{\partial x_{3'}}\Big) & \frac{1}{2}\Big(\frac{\partial u_{3'}}{\partial x_{2'}} - \frac{\partial u_{2'}}{\partial x_{3'}}\Big) & 0 \end{bmatrix}. \tag{A.2.12}$$

Since $u_1 = a_{11}u_{1'} + a_{21}u_{2'} + a_{31}u_{3'}$, we have, for example,

$$\begin{aligned} e_{11} &= a_{11}^2 \frac{\partial u_{1'}}{\partial x_{1'}} a_{11}a_{21}\Big(\frac{\partial u_{1'}}{\partial x_{2'}} + \frac{\partial u_{2'}}{\partial x_{1'}}\Big) + \\ & a_{11}a_{31}\Big(\frac{\partial u_{1'}}{\partial x_{3'}} + \frac{\partial u_{3'}}{\partial x_{1'}}\Big) + a_{21}^2 \frac{\partial u_{2'}}{\partial x_{2'}} \\ & a_{12}a_{31}\Big(\frac{\partial u_{2'}}{\partial x_{3'}} + \frac{\partial u_{3'}}{\partial x_{2'}}\Big) + a_{31}^2 \frac{\partial u_{3'}}{\partial x_{3'}} \\ &= \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ ... & ... & ... \\ ... & ... & ... \end{bmatrix} \begin{bmatrix} e'_{11} & e'_{12} & e'_{13} \\ e'_{21} & e'_{22} & e'_{23} \\ e'_{31} & e'_{32} & e'_{33} \end{bmatrix} \begin{bmatrix} a_{11} & ... & ... \\ a_{21} & ... & ... \\ a_{31} & ... & ... \end{bmatrix} \end{aligned} \tag{A.2.13}$$

or

$$\epsilon' = \mathbf{T}^T \epsilon \mathbf{T} \tag{A.2.14}$$

which is similar to the effect of a coordinate system rotation on the moment tensor. A similar derivation would show that

$$\mathbf{\Omega}' = \mathbf{T}^T \mathbf{\Omega} \mathbf{T} \tag{A.2.15}$$

Because $\mathbf{T}$ is orthogonal, it follows that

$$\epsilon = \mathbf{T} \epsilon' \mathbf{T}^T \tag{A.2.16}$$

and

$$\mathbf{\Omega} = \mathbf{T} \mathbf{\Omega}' \mathbf{T}^T \tag{A.2.17}$$

A.3 Cylindrical coordinate systems

Fung (1994) provided a simple derivation to express strain in a cylindrical coordinate system. First define the primed coordinate system as one that arises from a simple rotation matrix e.g.,

$$\mathbf{T}^T = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A.3.1}$$

Such that $\mathbf{x}' = \mathbf{T}^T\mathbf{x}$. In this new coordinate system, we define $u_r = u_{1'}$, $u_\phi = u_{2'}$ and $u_z = u_{3'}$. Thus

$$\begin{aligned} u_1 &= u_r \cos\phi - u_\phi \sin\phi \\ u_2 &= u_r \sin\phi + u_\phi \cos\phi. \\ u_3 &= u_z \end{aligned} \tag{A.3.2}$$

If the strain matrix in the primed-coordinate system is defined as

$$\begin{bmatrix} e_{rr} & e_{r\phi} & e_{rz} \\ e_{\phi r} & e_{\phi\phi} & e_{\phi z} \\ e_{zr} & e_{\phi z} & e_{zz} \end{bmatrix} = \begin{bmatrix} e'_{11} & e'_{12} & e'_{13} \\ e'_{21} & e'_{22} & e'_{23} \\ e'_{31} & e'_{32} & e'_{33} \end{bmatrix} \tag{A.3.3}$$

then the one can easily show that

$$\begin{aligned} e_{rr} &= e_{11}\cos^2\phi + e_{22}\sin^2\phi + e_{12}\sin 2\phi \\ e_{\phi\phi} &= e_{11}\sin^2\phi + e_{22}\cos^2\phi - e_{12}\sin 2\phi \\ e_{r\phi} &= (e_{22} - e_{11})\cos\phi\sin\phi + e_{12}(\cos^2\phi - \sin^2\phi) \\ e_{rz} &= e_{13}\cos\phi + e_{23}\sin\phi \\ e_{\phi z} &= -e_{13}\sin\phi + e_{23}\cos\phi \\ e_{zz} &= e_{33} \end{aligned} \tag{A.3.4}$$

Similarly if the rotation matrix is defined as

$$\begin{bmatrix} 0 & \omega_{r\phi} & \omega_{rz} \\ -\omega_{r\phi} & 0 & \omega_{\phi z} \\ -\omega_{rz} & -\omega_{\phi z} & 0 \end{bmatrix} = \begin{bmatrix} 0' & \omega'_{12} & \omega'_{13} \\ -\omega'_{21} & 0' & \omega'_{23} \\ -\omega'_{31} & -\omega'_{32} & 0' \end{bmatrix}, \tag{A.3.5}$$

then the one can easily show that

$$\begin{aligned} \omega_{r\phi} &= \omega_{12} \\ \omega_{rz} &= \omega_{13}\cos\phi + \omega_{23}\sin\phi \\ \omega_{\phi z} &= -\omega_{13}\sin\phi + \omega_{23}\cos\phi \end{aligned} \tag{A.3.6}$$

The next step is to express the e_{ij} and ω_{ij} in terms of cylindrical coordinates and then substitute these into these expressions. To express the $\frac{\partial u_i}{\partial x_j}$ in terms of the cylindrical coordinates, we need the operators

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} \tag{A.3.7}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi} \tag{A.3.8}$$

Using these we have

$$
\begin{aligned}
\frac{\partial u_1}{\partial x_1} &= \Big(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\Big)(u_r\cos\phi - u_\phi\sin\phi) \\
&= \cos^2\phi\frac{\partial u_r}{\partial r} + \sin^2\phi\Big(\frac{u_r}{r} + \frac{\partial u_\phi}{r\partial\phi}\Big) - \cos\phi\sin\phi\Big(\frac{\partial u_\phi}{\partial r} + \frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r}\Big) \\
\frac{\partial u_2}{\partial x_1} &= \Big(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\Big)(u_r\sin\phi + u_\phi\cos\phi) \\
&= \sin\phi\cos\phi\Big(\frac{\partial u_r}{\partial r} - \frac{u_r}{r} - \frac{\partial u_\phi}{r\partial\phi}\Big) + \cos^2\phi\frac{\partial u_\phi}{\partial r} - \sin^2\phi\Big(\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r}\Big) \\
\frac{\partial u_3}{\partial x_1} &= \Big(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\Big)u_z \\
&= \cos\phi\frac{\partial u_z}{\partial r} - \frac{\sin\phi}{r}\frac{\partial u_z}{\partial\phi} \\
\frac{\partial u_1}{\partial x_2} &= \Big(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\Big)(u_r\cos\phi - u_\phi\sin\phi) \\
&= -\sin^2\phi\frac{\partial u_\phi}{\partial r} + \cos^2\phi\Big(\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r}\Big) + \cos\phi\sin\phi\Big(\frac{\partial u_r}{\partial r} - \frac{\partial u_\phi}{r\partial\phi} - \frac{u_r}{r}\Big) \\
\frac{\partial u_2}{\partial x_2} &= \Big(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\Big)(u_r\sin\phi + u_\phi\cos\phi) \qquad \text{(A.3.9)} \\
&= \sin^2\phi\frac{\partial u_r}{\partial r} + \cos^2\phi\Big(\frac{u_r}{r} + \frac{\partial u_\phi}{r\partial\phi}\Big) + \cos\phi\sin\phi\Big(\frac{\partial u_\phi}{\partial r} + \frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r}\Big) \\
\frac{\partial u_3}{\partial x_2} &= \Big(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\Big)u_z \\
&= \sin\phi\frac{\partial u_z}{\partial r} + \frac{\cos\phi}{r}\frac{\partial u_z}{\partial\phi} \\
\frac{\partial u_1}{\partial x_3} &= \Big(\frac{\partial}{\partial z}\Big)(u_r\cos\phi - u_\phi\sin\phi) \\
&= \cos\phi\frac{\partial u_r}{\partial z} - \sin\phi\frac{\partial u_\phi}{\partial z} \\
\frac{\partial u_2}{\partial x_3} &= \Big(\frac{\partial}{\partial z}\Big)(u_r\sin\phi + u_\phi\cos\phi) \\
&= \sin\phi\frac{\partial u_r}{\partial z} + \cos\phi\frac{\partial u_\phi}{\partial z} \\
\frac{\partial u_3}{\partial x_3} &= \frac{\partial u_z}{\partial z}
\end{aligned}
$$

Substituting these into the definitions for e_{ij} and ω_{ij}, one then arrives at the folowing expressions for strain and rotation in cylindrical coordinates:

$$
\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r} & e_{r\phi} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) \\
e_{\phi\phi} &= \frac{1}{r}\left(\frac{\partial u_\phi}{\partial \phi} + u_r\right) & e_{rz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \\
e_{zz} &= \frac{\partial u_z}{\partial z} & e_{\phi z} &= \frac{1}{2}\left(\frac{\partial u_\phi}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \phi}\right)
\end{aligned}
\tag{A.3.10}
$$

and

$$
\begin{aligned}
\omega_{r\phi} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi} - \frac{\partial u_\phi}{\partial r} - \frac{1}{r}u_\phi\right) \\
\omega_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) \\
\omega_{\phi z} &= \frac{1}{2}\left(\frac{\partial u_\phi}{\partial z} - \frac{1}{r}\frac{\partial u_z}{\partial \phi}\right)
\end{aligned}
$$

and the dilatation Δ is given by

$$
\Delta = \nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = e_{rr} + e_{\phi\phi} + e_{zz}. \tag{A.3.11}
$$

which is as expected for a divergence in cylindrical coordinates.

A.4 Computation of Cartesian strains, stresses and rotations

The effort in the previous section showed how to compute the stresses and rotations in a cylindrical coordinate system in terms of the cylindrical displacements. This is a requirement before modifying existing code to also compute these new quantities. However it will be necessary to transform these quantities to a local cartesian system to compare with observations.

Thus far the (r, ϕ, z) refer to the position of the observation point with respect to the source of the signal. Now consider the local coordinate system in Figure A.1. Here the $\mathbf{e}_r$ and $\mathbf{e}_\phi$ indicate the radial and transverse directions with respec to the source. A new coordinate system x-y is imposed at the observation point, O. The x vector $\mathbf{e}_R$ makes an angle θ with respect to the $x-$axis. The local coorinate system axes do not have to be oriented N-S or E-W, respectively. In the case of DAS measurements, the $x-$axis may be aligned with respect to the direction of the fiber cable.

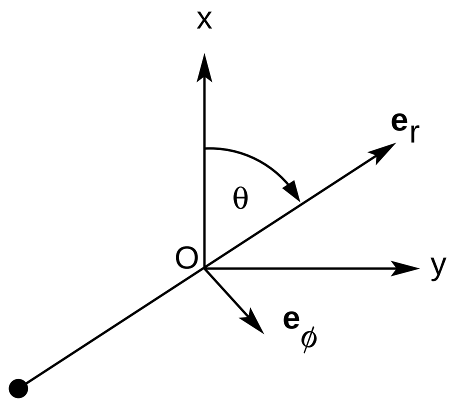

Figure A.1 Sketch of local coordinate system at observation point.

These coordinate systems are related by the transformation

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_r \\ u_\phi \\ u_z \end{bmatrix} \tag{A.4.1}$$

where the square matrix is defined as **T**.

Applying the tensor rotation rules, we have

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \mathbf{T} \begin{bmatrix} e_{rr} & e_{r\phi} & e_{rz} \\ e_{\phi r} & e_{\phi\phi} & e_{\phi z} \\ e_{zr} & e_{\phi z} & e_{zz} \end{bmatrix} \mathbf{T}^T \tag{A.4.2}$$

In a similar manner the rotated stresses are

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \mathbf{T} \begin{bmatrix} \sigma_{rr} & \sigma_{r\phi} & \sigma_{rz} \\ \sigma_{\phi r} & \sigma_{\phi\phi} & \sigma_{\phi z} \\ \sigma_{zr} & \sigma_{\phi z} & \sigma_{zz} \end{bmatrix} \mathbf{T}^T \tag{A.4.3}$$

and the transformed rotatations are

$$\begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{21} & 0 & \omega_{23} \\ -\omega_{31} & -\omega_{32} & 0 \end{bmatrix} = \mathbf{T} \begin{bmatrix} 0 & \omega_{r\phi} & \omega_{rz} \\ -\omega_{\phi r} & 0 & \omega_{\phi z} \\ -\omega_{zr} & -\omega_{\phi z} & 0 \end{bmatrix} \mathbf{T}^T \tag{A.4.4}$$

Finally the dilatation is

$$\Delta = e_{11} + e_{22} + e_{33} = e_{rr} + e_{\phi\phi} + e_{zz} \tag{A.4.5}$$

Bibliography

Aki, K., and Richards, P. G. 2002. *Quantitative Seismology*. Sausalito: University Science Books.

Bower, A. F. 2010. *Applied mechanics of solids*. CRC Press.

Fung, Y. C. 1994. *A First Course in Continuum Mechanics: For Physical and Biological Scientists and Engineers*. 3 edn. Prentice Hall.

Hughes, William F., and Gaylord, Eber W. 1964. *Basic Equations of Engineering (Schaum's Outline Series)*. McGraw-Hill.

Keilis-Borok, V. I. (ed). 1989. *Seismic Surface Waves in a Laterally Inhomogeneous Earth*. Dordrecht: Kluwer Academic Publishers Hill.

Levshin, A. L., and Yanson, Z. A. 1971. Поверхностные волны в вертикально и радиально неоднородных средах. Pages 147–177 of: Алгоритмы интерпретации сейсмических данных. Бычислительная Сейсмология, vol. 5. Академия Наук СССР.

Love, A. E. H. 1944. *The Mathematical Theory of Elasticity*. Dover Publications.

Sollberger, David, Igel, Heiner, Schmelzbach, Cedric, Edme, Pascal, van Manen, Dirk-Jan, Bernauer, Felix, Yuan, Shihao, Wassermann, Joachim, Schreiber, Ulrich, and Robertsson, Johan O. A. 2020. Seismological Processing of Six Degree-of-Freedom Ground-Motion Data. *Sensors*, **20**(23), 6904.

Takeuchi, H., and Saito, M. 1972. Seismic surface waves. Pages 217–295 of: *Methods in Computational Physics, Volume 11: Seismology: Surface Waves and Earth Oscillations*. Academic Press.