This document reviews multiple filter analysis, and the adaptation to that processing technique to estimate phase velocities through the cross-correlation of recorded noise.

Multiple Filter Analysis

The following discussion of multiple filter analysis follows Herrmann (1973).

Let the dispersed surface wave be represented by the relation

$$f(t,r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega, r) \exp(i\omega t) d\omega \tag{1}$$

where

$$F(\omega, r) = A(\omega, r) \exp(-ikr + \phi) \tag{2}$$

and ϕ is the source phase and k is the wavenumber, which is related to the phase velocity through the definition $\omega = kc$.

The processing starts with the application of a narrow bandpass Gaussian filter about a center frequency ω_0 by the filter $H(\omega - \omega_0)$ where the function H is defined as

$$H(\omega) = \begin{cases} \exp(-\alpha\omega^2/\omega_0^2) & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases} \tag{3}$$

Under the condition that $(\alpha/\omega_0^2)^2 \gg (r/2d^2k/d\omega_0^2)^2$, the filtered signal is

$$g(t,r) = \frac{1}{2\pi} A(\omega_0)\omega_0 \sqrt{\frac{\pi}{\alpha}} \exp[i(\omega_0 t - k_0 r + \phi)] \exp\left[-\frac{\omega_0^2}{4\alpha} (t - r/U_0)^2 \right] \tag{4}$$

The last term defines the envelope, which is a maximum at a time corresponding to a group velocity arrival. The group velocity, U, is defined as $U = d\omega/dk$. This expression indicates that the narrow band-pass filtered signal can be used to estimate the group velocity by using the time of envelope maximum and the spectral amplitude A at $\omega = \omega_0$, through the envelope amplitude, e.g.,

$$A = (2\pi/\omega_0)\sqrt{(\alpha/\pi)}\ |g(r/U_0, r)| \tag{5}$$

The phase term can be used to estimate the phase velocity if the source term in known. The phase at the group velocity arrival, e.g., $t = r/U_0$, is

$$\Phi = \tan^{-1}\left[\mathrm{Im}\, g(r/U_0, r) / \mathrm{Re}\, g(r/U_0, r) \right] = r\omega_0/U - r\omega_0/c + \phi + N2\pi \tag{6}$$

The $N2\pi$ term arises because of the periodicity of the $\tan^{-1}$ function.

The source phase term can be eliminated if a two-station technique is used, e.g., if two stations are used along the same azimuth from the source. In this case the difference in the Φ's for each trace would be interpreted as

$$\Phi_2 - \Phi_1 = (r_2 - r_1)\omega_0(1/U - 1/c) + (N_2 - N_1)2\pi$$

Fourier Transform

The development here follows Lin et al (2008) who used the results of Snieder (2004). The significant difference between the development in those papers and that used here is in the definition of the Fourier transform pair. The Computer Programs in Seismology codes use the convention

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \exp(-i\omega t) dt$$

and, for the inverse transform,

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \exp(+i\omega t) dt$$

With this definition, one can show that the Fourier transform pair for cross-correlation is

$$C_{12}(t) = \int_{-\infty}^{\infty} x_2(\tau) x_1(\tau + t) d\tau \quad \leftrightarrow \quad X_2^*(\omega) X_1(\omega)$$

Stationary Phase Approximation to an Integral In addition recall that the method of stationary phase can be used to approximate an integral

$$I = \int_{-\infty}^{\infty} g(k) e^{if(k)} dk \approx \sqrt{2\pi}\, g(k_0) e^{if(k_0)} e^{\pm i \frac{\pi}{4}} \left| \frac{d^2 f}{dk^2} \right|_{k=k_0}^{-1/2} \qquad (7)$$

where $k = k_0$ is that value of k that makes $df/dk = 0$ and the $\pm$ sign is taken according to whether the sign of $\left(\frac{d^2 f}{dk^2} \right)$ is $\pm$ is $\pm$.

Point Force Green's Function

Since we will focus on surface waves, a review of the point source Green's functions is appropriate. The 3-component displacements for in impulsive source pooint force observed at an azimuth ϕ are

$$u_z = (F_1 \cos \phi + F_2 \sin \phi)\, ZHF + F_3\, ZVF$$

$$u_r = (F_1 \cos \phi + F_2 \sin \phi)\, RHF + F_3\, RVF$$

$$u_\phi = (F_1 \sin \phi - F_2 \cos \phi)\, THF + F_3$$

where the forces, F_1, F_2 and F_3, are in the north, east and downward directions, respectively, and ϕ is the azimuth from the source to the observation point measured in a direction east of north. (Note this ϕ is not the same as that of (6)).

For fundamental mode surface waves in the far field, the expressions for the functions in this expression are

$$ZVF = \frac{1}{\sqrt{2\pi}} A_R U_z(h) U_z(z) \frac{1}{\sqrt{kr}} e^{-i(kr + \frac{5}{4}\pi)} = \left[\frac{1}{\sqrt{2\pi}} A_R U_z(h) U_z(z) \frac{1}{\sqrt{kr}} e^{-i\frac{\pi}{2}} \right] e^{-i(kr - \frac{\pi}{4})}$$

$$RVF = \frac{1}{\sqrt{2\pi}} A_R U_z(h) U_r(z) \frac{1}{\sqrt{kr}} e^{-i(kr + 3\frac{\pi}{4})} = \left[\frac{1}{\sqrt{2\pi}} A_R U_z(h) U_r(z) \frac{1}{\sqrt{kr}} e^{-i\pi} \right] e^{-i(kr - \frac{\pi}{4})}$$

$$ZHF = \frac{1}{\sqrt{2\pi}} A_R U_r(h) U_z(z) \frac{1}{\sqrt{kr}} e^{-i(kr - \frac{\pi}{4})} = \left[\frac{1}{\sqrt{2\pi}} A_R U_r(h) U_z(z) \frac{1}{\sqrt{kr}} \right] e^{-i(kr - \frac{\pi}{4})} \quad (8)$$

$$RHF = \frac{1}{\sqrt{2\pi}} A_R U_r(h) U_r(z) \frac{1}{\sqrt{kr}} e^{-i(kr + \frac{\pi}{4})} = \left[\frac{1}{\sqrt{2\pi}} A_R U_r(h) U_r(z) \frac{1}{\sqrt{kr}} e^{-i\frac{\pi}{2}} \right] e^{-i(kr - \frac{\pi}{4})}$$

$$THF = \frac{1}{\sqrt{2\pi}} A_L U_\phi(h) U_\phi(z) \frac{1}{\sqrt{kr}} e^{-i(kr - 3\frac{\pi}{4})} = \left[\frac{1}{\sqrt{2\pi}} A_L U_\phi(h) U_\phi(z) \frac{1}{\sqrt{kr}} e^{i\frac{\pi}{2}} \right] e^{-i(kr - \frac{\pi}{4})}$$

where $A_L = 1/2 c_L U_L I_{0L}$ for the Love wave and $A_R = 1/2 c_R U_R I_{0R}$ for the Rayleigh wave. The eigenfunctions U_z, U_r and U_ϕ are solutions of differential equations for P-SV and SH waves with the boundary conditions of zero stress at $z = 0$ and expnentially derease as $z \rightarrow \infty$. For the fundamental mode Rayleign wave, the ellipticity U_r / U_z is positive at $z = 0$. Finally z represents the receiver depth and h the source depth in the halfspace. The energy integrals are defined in terms of integrals fof the eigenfunctions as $A_L = \int_0^\infty \rho {U_\phi}^2 dz$ and $A_R = \int_0^\infty \rho [{U_z}^2 + {U_r}^2] dz$ and

The second expression for each Green's functions rearranges the complex part of the solution into a form that will appear later when considering random souces of scattering.

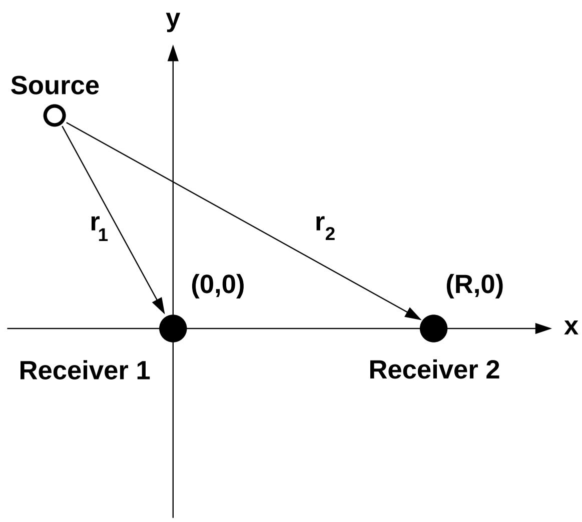
Given the back azimuth, ϕ_b, from the observation point to the source, the Cartesian displacements are given by a simple transformation:

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} -\cos\phi_b & \sin\phi_b & 0 \\ -\sin\phi_b & -\cos\phi_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_r \\ u_\phi \\ u_z \end{bmatrix}$$

Scattering and Coda waves

To understand the result of cross-correlating the noise recordings at two locations, we follow the exposition by Snieder (2004). The significant difference between our derivation and that of Snieder arises from the definition of the Fourier transform. For R and k positive, $\exp(-ikR)$ represents a wave propagation in the $+R$ direction as time increases in our formulation, but as $\exp(ikR)$ in that of Snieder (2004).

Consider Figure 1, which is adapted from Snieder (2004). The source of the observed signal is generated at the scattering source.



The Fourier transformed vector displacement, in cartesian coordinates) at the two observation points can be written as

$$\mathbf{U}_{1,2}(\omega) = \sum_s \sum_m S^{(s,m)} \left[\mathbf{p}^m(h, z_{1,2}, \phi_{1,2}) \frac{1}{\sqrt{k_m X^s_{1,2}}} \right] e^{-i(k_m X^s_{1,2} - \frac{\pi}{4})} \tag{9}$$

The indices, 1 or 2, refer to the specific observation point, m to the mode (and wave type corresponding to each displacement component, s to the scattering source, h to the souce depth, z to the receiver depth, and X to the horizontal distance from the source to the receiver. The concept of mode m is generalized to combine the concepts of eigenfunction mode and wavetype.

While Sneider considered that the point s would be the source of scattered waves, we see that (9) can also be ionterpreted as the superposition of point forces applied at s, This is the reason that the square brackets are use to emphasize the similarity to (8).

Snieder (2004) now assumes a distribution of scatterers in the x-y plane that permits replacing the summation over the scatterers to an integral in the $x - y$ plane. If n is the density of scatterers per unit area, then the cross-correlation between the recordings on component i at receiver 1 an component j at receiver two is

$$C_{ij}(t) = \int_{-\infty}^{\infty} u_{1j}(\tau) u_{2i}(t+\tau) d\tau$$

or

$$C_{ij}(\omega) = \sum_{m,m'} \int n(x,y)\, p_i^m(h, z_2, \phi_2)\, p_j^{m'} * (h, z_1, \phi_1) S^m(\omega) S^{m'}(\omega) * \qquad \hat{}$$

$$\cdot \frac{e^{-i(k_m X_2 - k_{m'} X_1)}}{\sqrt{k_m k_{m'} X_2 X_1}}\, dx\, dy$$

Now integrate over the y-coordinate, using the method of stationary phase. The condition for stationary phase requires the $y = 0$. Using (7), the cross-correlation becomes

$$C_{ij}(\omega) = \sum_m \int_{-\infty}^{\infty} \sqrt{2\pi}\, p_i^m(h, z_2, \phi_2)\ p_j^{m'}(h, z_1, \phi_1) S^m(\omega) S^{m'}(\omega)\, *$$

$$\frac{1}{\sqrt{k_m k_{m'}}} \frac{1}{\sqrt{|k_m|R-x| - k_{m'}|x||}}\ e^{-i(k_m|R-x| - k_{m'}|x| - \eta\frac{\pi}{4})}\ dx \qquad (10)$$

where η is ± 1 depending on the sign of $-(k_m/|R-x| - k_{m'}/|x|)$, e.g., +1 for $x < 0$ and -1 for $x > R$.

Snieder(2004) now argues that the integral over x has a non-zero contributions when the exponential is not oscillatory, which occurs only for $x < 0$ and for $x > R$. The reason for this statement is that if $x < 0$, the exponential if of the form $-i(kR - \eta\,\frac{\pi}{4})$ and $-i(-kR - \eta\,\frac{\pi}{4})$ for $x > R$. For $0 < x < R$, the exponential depends on x and is thus does not contribute to the integral. Also to avoid an osciallating integral, the k_m must equal $k_{m'}$. This latter point means that for isotropic or transversely isotropic media, that the only non-zero cross-correlations will be those between u_z at receivers 1 and 2, and similarly between the u_y's and the u_x's, which will the receivers, u_y's which will involve Rayleigh-wave motion for the first two and Love for the last. In addition, the gross correlations between the u_z and the u_y will be non-zero because both will record the Rayleigh wave. In the latter case the there will be a $\frac{\pi}{2}$ phase difference between the u_{z_1} - u_{z_2} and the u_{z_1} - u_{z_2} cross-correlations.

As a result of these considerations (eqn 23 in Sneider, 2004),

$$C_{ij}(\omega) = \sqrt{2\pi} \sum_m \left\{ \frac{c_m}{\omega} \frac{1}{\sqrt{k_m R}}\, e^{-i(k_m R - \frac{\pi}{4})} \int_{-\infty}^{0} p_i^m(h, z_2, 0) p_j^m(h, z_1, 0) * n\, dx \right.$$

$$\left. + \frac{c_m}{\omega} \frac{1}{\sqrt{k_m R}}\, e^{+i(k_m R - \frac{\pi}{4})} \int_0^{\infty} p_i^m(h, z_2, \pi) p_j^m(h, z_1, \pi) * n\, dx \right\} \overline{|S^m(\omega)|}^2 \qquad (11)$$

The first term represents signals generated in the region $x < -0$ propagating in the positive x- direction , while the second represents signals generated in the region $x > R$ and propagating in the negative x-direction.

For cross-correlations between the same components at each station, the integrands will be real. Thus the wave propagation

For purposes of relating the signals to the point force Green's functions, we note that the integrand is real for the cross-correlation of the same components. This means that the phase term in (2) is $\pi/4$, and thus a phase velocity can be obtained as part of the multiple filter processing.

After further consideration, Sbeider (2004) gives

$$C_{ij}(\omega) = \pi \sum_m c_m \left\{ \frac{G_{ij}^m(\mathbf{r}_2, \mathbf{r}_1)}{i\omega} \int_{-\infty}^{0} n dx + \left(\frac{G_{ij}^m(\mathbf{r}_1, \mathbf{r}_2)}{i\omega} \right)^{\dagger} \int_R^{\infty} n dx \right\} \overline{|S^m(\omega)|}^2 \qquad (12)$$

where the † denotes the Hermitian conjugate.

This neans that the cross-correlation of uniformly distributed noise sources will give a symmetric function about zero time lag. Recall that this expression arises from the convolution of the displacement series. Thus the cross-correlation is prpoprtional to the integral of the Grene's function.

The estimation of phase velocity focuses on the phase term. What happens if cross-correlates ground velocity rather than ground displacment. The result is that (12) will have a leading ω^2 term due to the fact that in the frequency domain, cross-correlation invlved multiplying the spectrum at one location by the complex conjugate of the spectrum at the other site, thus $\omega usp2 = (i\omega)(i\omega^*)$. Thus the phase term of the cross-correlation is not changes, just the amlitude spectrum.

Thus for cross-correlating the Z components at the two stations, or the E or N components, equation (6) can be rearranged to solve for the phase velocity:

$$c = \frac{\omega_0 r}{-\Phi + \pi/4 + \omega_0 r/U_0 + N2\pi} \tag{8}$$

Note that this expression differs from Equation (7) of Lin et al (2008) in the sign of the $\pi/4$ term. The difference is assumed to be due to the definition of the Fourier transform used.

While discussion the cross-correlation, it is also useful to consider the circumstances under which (8) could be used with synthetics. Specifically can one use the ZVF, RHF and THF Green's functions as surrogrates for the cross-correlation. Note that each of these contains a term such $e^{i\pm\pi/2}$ in the square brackets on the right. Rather than performing a Hilbert transform on the synthetic, or since the primary interest is in preserving the pahse term, a multiplication by $-1/i\omega$, which is an integration and a polarity reversal will adjust the phase term from ZVF and RHF. The THF requires an additional multiplication by -1 in order to have the phase term agree with the phase term for waves propagating in the $+x$ direction in (7).

It is simple using the *mt* command of **gsac** to perform these operation. The only point to recall is that the **spulse96** command to compute a recorded velocities for a step source, the default procedure gives a time history that is equivalent to the ground displacement for an impulsive source. The assertion is easily seen from Fourier transforms.

The specific command sequence is given in the section "Generating the proper synthetic" which mentions the script in EMPIRICAL_GREEN/DIST/EXAMPLE.GRN/DOIT.

Summary

This document reviewed multiple filter analysis and shoed how the output can be used to determine phase velocity from inter-station empirical Green's functions. This required a review and adaptation of the paper bu Snieder (2004). Finally a way was determined to use synthetics from point forces get phase velocities.

References

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