1

The layered elastic medium

1.1 Generalized reflection and transmission matrices

The propagator matrix technique introduced in Chapter 3 and used in this chapter provides an easily understandable technique for solving the wave propagation problem. However when the vertical wavenumber, ν, is real, the exponentially increasing terms can lead to numerical problems. The reflection matrix technique of Kennett and Kerry (1979) avoids increasing exponential terms and also provides an insight as to the contribution of individual generalized rays. Chen (1993) introduced a recursive technique that shares the numerical stability of the reflection matrix technique, but which emphasizes computational efficiency. The following reviews the development of Pei et al. (2008, 2009) which adapts Chen (1993) and extends it to handle arbitrary boundary conditions at the top and bottom of the layer stack.

The relation between the displacement stress vector $\mathbf{B}$ and the potential coefficient vector $\mathbf{K}$ is given by

$$\mathbf{B} = \mathbf{E}\Lambda\mathbf{K} \tag{1.1}$$

or

$$\begin{bmatrix} \mathbf{U} \\ \mathbf{T} \end{bmatrix} = \mathbf{E} \begin{bmatrix} \mathbf{e}^{\nu z} & 0 \\ 0 & \mathbf{e}^{-\nu z} \end{bmatrix} \begin{bmatrix} \mathbf{K}^U \\ \mathbf{K}^D \end{bmatrix} \tag{1.2}$$

The notation $\mathbf{e}^{\nu z}$ represents a scalar, or $1x1$ matrix, $e^{\nu z}$ for the SH and fluid problems, or the $2x2$ diagonal matrix with elements $e^{\nu_\alpha z}$ and $e^{\nu_\beta z}$ to represent P- and SV-wavefields, respectively. If the problem to be solved is fully anisotropic, then this would represent a $3x3$ diagonal matrix. We will continue to use the $\mathbf{e}^{\nu z}$ notation in this development to emphasize the generality of the approach.

The useful insight of Chen (1993) and Pei et al. (2008, 2009) is to rewrite (1.2)

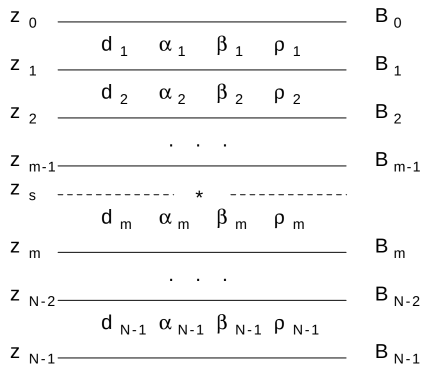

Figure 1.1 Model of layered isotropic medium, showing depths to the interfaces, the stress-displacement vectors at the interfaces, layer thicknesses and the medium parameters within the layers. The source is in layer m at a depth z_s.

for the region $z_{j-1} \le z \le z_j$ as

$$\begin{aligned} \mathbf{B}(z) = \begin{bmatrix} \mathbf{U} \\ \mathbf{T} \end{bmatrix} &= \mathbf{E} \begin{bmatrix} \mathbf{e}^{-\nu_j(z_j - z)} & 0 \\ 0 & \mathbf{e}^{-\nu_j(z - z_{j-1})} \end{bmatrix} \begin{bmatrix} \mathbf{K}^U \mathbf{e}^{\nu_j z_j} \\ \mathbf{K}^D \mathbf{e}^{-\nu_j z_{j-1}} \end{bmatrix} \\ &= \mathbf{E} \begin{bmatrix} \mathbf{e}^{-\nu_j(z_j - z)} & 0 \\ 0 & \mathbf{e}^{-\nu_j(z - z_{j-1})} \end{bmatrix} \begin{bmatrix} \mathbf{C}_U^j \\ \mathbf{C}_D^j \end{bmatrix} \end{aligned} \tag{1.3}$$

Here the ν_j represents the vertical wavenumber for layer j. The result of introducing $\mathbf{C}$ means that it is not necessary to consider the wave delays through a layer and reflection/transmission coefficients as separate stages in the development. This definition also has the advantage that the elements of the diagonal matrix preceding the $\mathbf{C}$'s are always bounded in magnitude by unity for $z_{j-1} \le z \le z_j$ because of our definition of the ν_j.

To extend this new notation to a layered medium, consider Figure 1.1. Further suppose that the boundary conditions at the top, $z = z_0$, and the bottom, $z = z_{N-1}$, have the form

$$\mathbf{B}_0 = \mathbf{H} \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}$$

and

$$\mathbf{B}_{N-1} = \mathbf{G}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{D} \end{bmatrix}$$

As noted previously, the introduction of the $\mathbf{G}$ and $\mathbf{H}$ matrices permits consideration of halfspace, rigid or stress free boundary conditions. The determination of the medium response requires consideration of responses to upward and downward propagating signals.

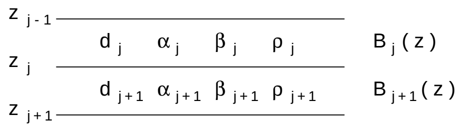

Figure 1.2 Model for development of the generalized R/T coefficients. The layer thicknesses, velocities, densities and displacement-stress functions are indicated.

Bottom-up processing

We first consider the adjacent layers shown in Figure 1.2 and focus first on the boundary conditions at z_j which require continuity of $\mathbf{B}$, e.g., $\mathbf{B}_j(z_j) = \mathbf{B}_{j+1}(z_j)$, where the notation $\mathbf{B}_j(z)$ means the value of $\mathbf{B}$ in layer j at depth z using the $\mathbf{C}^j_{U,D}$ parameters. Using (1.3), we have

$$\mathbf{E}_j \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}^{-\nu_j d_j} \end{bmatrix} \begin{bmatrix} \mathbf{C}^j_U \\ \mathbf{C}^j_D \end{bmatrix} = \mathbf{E}_{j+1} \begin{bmatrix} \mathbf{e}^{-\nu_{j+1} d_{j+1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{C}^{j+1}_U \\ \mathbf{C}^{j+1}_D \end{bmatrix} \tag{1.4}$$

To facilitate discussion, we focus on the response due to a downward propagating signal from the top of layer j onto the boundary at z_j. There will be a reflection, a transmission, and also the effect of signals returned from deeper layers. We further define $\mathbf{C}^j_U = \hat{\mathbf{R}}^j_D \mathbf{C}^j_D$ and $\mathbf{C}^{j+1}_D = \hat{\mathbf{T}}^j_D \mathbf{C}^j_D$. From these we also have $\mathbf{C}^{j+1}_U = \hat{\mathbf{R}}^{j+1}_D \mathbf{C}^{j+1}_D = \hat{\mathbf{R}}^{j+1}_D \hat{\mathbf{T}}^j_D \mathbf{C}^j_D$. The symbols $\hat{\mathbf{R}}_D$ and $\hat{\mathbf{T}}_D$ are called the generalized R/T coefficients, respectively, and represent reflection and transmission responses in a simplified notation, but, unlike the R/T coefficients of Kennett and Kerry (1979) cannot be used as the reflection and transmission coefficients of individual wavefield components at the boundary because they include the delays associated with the ν_j terms.

After factoring the common $\mathbf{C}^j_D$ we have

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}^{-\nu_j d_j} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}}^j_D \\ \mathbf{I} \end{bmatrix} = \mathbf{E}^{-1}_j \mathbf{E}_{j+1} \begin{bmatrix} \mathbf{e}^{-\nu_{j+1} d_{j+1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}}^{j+1}_D \hat{\mathbf{T}}^j_D \\ \hat{\mathbf{T}}^j_D \end{bmatrix}$$

Defining $\mathcal{E} = \mathbf{E}^{-1}_j \mathbf{E}_{j+1}$, and partitioning $\mathcal{E}$ into square sub-matrices, e.g.,

$$\mathcal{E} = \begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{bmatrix}$$

we obtain the following recurrence relations:

$$\begin{aligned} \hat{\mathbf{T}}^j_D &= \left(\mathcal{E}_{21} \mathbf{e}^{-\nu_{j+1} d_{j+1}} \hat{\mathbf{R}}^{j+1}_D + \mathcal{E}_{22} \right)^{-1} \mathbf{e}^{-\nu_j d_j} \\ \hat{\mathbf{R}}^j_D &= \left(\mathcal{E}_{11} \mathbf{e}^{-\nu^{j+1} d_{j+1}} \hat{\mathbf{R}}^{j+1}_D + \mathcal{E}_{12} \right) \hat{\mathbf{T}}^j_D \end{aligned} \tag{1.5}$$

The order of multiplication is important since the individual terms are 2x2 matrices for the isotropic or transverse isotropic P-SV problems.

This now sets the formalism for computation. If we can specify, $\hat{\mathbf{R}}_D^{N-1}$, then we can compute $\hat{\mathbf{T}}_D^{N-2}$ and $\hat{\mathbf{R}}_D^{N-2}$, and continue upward until we have $\hat{\mathbf{T}}_D^1$ and $\hat{\mathbf{R}}_D^1$.

At z_{N-1}, the bottom of the layer $N-1$'st layer, we have

$$\begin{bmatrix}\mathbf{0}\\ \mathbf{D}\end{bmatrix} = \mathbf{G}\mathbf{E}_{N-1}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{e}^{-\nu_{N-1}d_{N-1}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{R}}_D^{N-1}\\ \mathbf{I}\end{bmatrix}\mathbf{C}_D^{N-1} = \mathbf{a}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{e}^{-\nu_{N-1}d_{N-1}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{R}}_D^{N-1}\\ \mathbf{I}\end{bmatrix}\mathbf{C}_D^{N-1} \tag{1.6}$$

from which

$$\mathbf{0} = (\mathbf{a}_{11}\hat{\mathbf{R}}_D^{N-1} + \mathbf{a}_{12}\mathbf{e}^{-\nu_{N-1}d_{N-1}})\mathbf{C}_D^{N-1}$$

and thes we obtain $\hat{\mathbf{R}}_D^{N-1} = -\mathbf{a}_{11}^{-1}\mathbf{a}_{12}\mathbf{e}^{-\nu_{N-1}d_{N-1}}$. The specific condition to use depends on the problem.

Top - down processing

An alternative treatment is to start at (1.4), but now define define $\mathbf{C}_D^{j+1} = \hat{\mathbf{R}}_U^{j+1}\mathbf{C}_U^{j+1}$ and $\mathbf{C}_U^j = \hat{\mathbf{T}}_U^{j+1}\mathbf{C}_U^{j+1}$. From these we also have $\mathbf{C}_D^j = \hat{\mathbf{R}}_U^j\mathbf{C}_U^j = \hat{\mathbf{R}}_U^j\hat{\mathbf{T}}_U^{j+1}\mathbf{C}_U^{j+1}$. Starting again with the continuity of $\mathbf{B}$ at z_j, we have the following, after factoring the common $\mathbf{C}_U^{j+1}$ terms and using the $\hat{\mathbf{R}}_U^{j+1}$ and $\hat{\mathbf{T}}_U^{j+1}$.

$$\mathbf{E}_{j+1}^{-1}\mathbf{E}_j\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{e}^{-\nu_j d_j}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{T}}_U^{j+1}\\ \hat{\mathbf{R}}_U^j\hat{\mathbf{T}}_U^{j+1}\end{bmatrix} = \begin{bmatrix}\mathbf{e}^{-\nu_{j+1}d_{j+1}} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I}\\ \hat{\mathbf{R}}_U^{j+1}\end{bmatrix}$$

Defining $\mathcal{F} = \mathbf{E}_{j+1}^{-1}\mathbf{E}_j$, and partitioning into square submatrices, we have the following recursive relations:

$$\begin{aligned}\hat{\mathbf{T}}_U^{j+1} &= \left(\mathcal{F}_{11} + \mathcal{F}_{12}\mathbf{e}^{-\nu_j d_j}\hat{\mathbf{R}}_U^j\right)^{-1}\mathbf{e}^{-\nu_{j+1}d_{j+1}}\\ \hat{\mathbf{R}}_U^{j+1} &= \left(\mathcal{F}_{21} + \mathcal{F}_{22}\mathbf{e}^{-\nu_j d_j}\hat{\mathbf{R}}_U^j\right)\hat{\mathbf{T}}_U^{j+1}\end{aligned} \tag{1.7}$$

To initiate the recursion, The boundary condition at the top is now written in terms of the $\mathbf{C}_U$. At z_0, the top of the layer stack,

$$\begin{bmatrix}\mathbf{F}\\ \mathbf{0}\end{bmatrix} = \mathbf{H}^{-1}\mathbf{E}_1\begin{bmatrix}\mathbf{e}^{-\nu_1 d_1}\\ \hat{\mathbf{R}}_U^1\end{bmatrix}\mathbf{C}_U^1 = \mathbf{b}\begin{bmatrix}\mathbf{e}^{-\nu_1 d_1}\\ \hat{\mathbf{R}}_U^1\end{bmatrix}\mathbf{C}_U^1$$

from which we have

$$\mathbf{0} = (\mathbf{b}_{21}\mathbf{e}^{-\nu_1 d_1} + \mathbf{b}_{22}\hat{\mathbf{R}}_U^1)\mathbf{C}_U^1$$

which is satisfied by $\hat{\mathbf{R}}_U^1 = -\mathbf{b}_{22}^{-1}\mathbf{b}_{21}\mathbf{e}^{-\nu_1 d_1}$. Thus a recursive scheme can also start at the top boundary and progress downward in the layer stack.

Mixed solid-fluid medium

When the medium consists of a mixture of fluid and solid layers, the solution for

P-SV wave propagation is more complicated. In the special case of the fluid layers at the top or bottom of a stack of elastic layers, an approach involving a modification of propagator matrices was introduced in §**??**. However the layer sequence of alternating fluid and solid regions, such as a floating ice sheet, water and solid earth, could not be handled. Chen and Chen (2002) extended the work of Chen (1993) to address the problem of fluid layers overlying a stack of solid layers. This section shows how this is accomplished using the notation of this section and also extends the development to address the more general problem of intermixed fluid and solid layers.

We will first consider a fluid layer overlying an elastic layer. Thus in Figure 1.2 let layer j be a fluid and $j+1$ be an elastic solid. To facilitate the discussion, we will use the lower case symbols $\mathbf{e}$, $\mathbf{c}_u$ and $\mathbf{c}_d$ when referring to the fluid (1.3) and the upper case symbols $\mathbf{E}$, $\mathbf{C}_U$ and $\mathbf{C}_D$ when referring to the underlying elastic layer. Further $\mathbf{C}_U = [C_{U,\alpha}, C_{U,\beta}]^T$ is a $2x1$ matrix as is $\mathbf{C}_D$. The boundary conditions at z_j are that U_z and T_z are continuous and that $T_r = 0$ in the solid. Expanding (1.4) in detail gives

$$\begin{aligned}
U_z &= e_{11}^j c_u^j + e_{12}^j e^{-\nu_{\alpha_j} d_j} c_d^j \\
&= E_{21}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} C_{U,\alpha}^{j+1} + E_{22}^{j+1} e^{-\nu_{\beta_{j+1}} d_{j+1}} C_{U,\beta}^{j+1} + E_{23}^{j+1} C_{D,\alpha}^{j+1} + E_{24}^{j+1} C_{D,\beta}^{j+1} \\
T_z &= e_{21}^j c_u^j + e_{22}^j e^{-\nu_{\alpha_j} d_j} c_d^j \\
&= E_{31}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} C_{U,\alpha}^{j+1} + E_{32}^{j+1} e^{-\nu_{\beta_{j+1}} d_{j+1}} C_{U,\beta}^{j+1} + E_{33}^{j+1} C_{D,\alpha}^{j+1} + E_{34}^{j+1} C_{D,\beta}^{j+1} \\
T_r &= 0 \\
&= E_{41}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} C_{U,\alpha}^{j+1} + E_{42}^{j+1} e^{-\nu_{\beta_{j+1}} d_{j+1}} C_{U,\beta}^{j+1} + E_{43}^{j+1} C_{D,\alpha}^{j+1} + E_{44}^{j+1} C_{D,\beta}^{j+1}
\end{aligned} \tag{1.8}$$

This notation is general enough to use with transverse isotropic media.

The recursion relations (1.5) and (1.7) permitted the specification of $\hat{\mathbf{T}}_D^j$ and $\hat{\mathbf{R}}_D^j$ in terms of $\hat{\mathbf{R}}_D^{j+1}$ for bottom-up processing and $\hat{\mathbf{T}}_U^{j+1}$ and $\hat{\mathbf{R}}_U^{j+1}$ in terms of $\hat{\mathbf{R}}_U^j$ for top-down processing. For the fluid-solid boundary problem, a somewhat more complicated relation is required to define the $\hat{\mathbf{R}}_{U,D}$ and $\hat{\mathbf{T}}_{U,D}$.

First write (1.8) in matrix form

$$\begin{aligned}
&\begin{bmatrix} e_{11}^j & e_{12}^j \\ e_{21}^j & e_{22}^j \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-\nu_{\alpha_j} d_j} \end{bmatrix} \begin{bmatrix} c_u^j \\ c_d^j \end{bmatrix} \\
&= \begin{bmatrix} E_{21}^{j+1} & E_{22}^{j+1} & E_{23}^{j+1} & E_{24}^{j+1} \\ E_{31}^{j+1} & E_{32}^{j+1} & E_{33}^{j+1} & E_{34}^{j+1} \\ E_{41}^{j+1} & E_{42}^{j+1} & E_{43}^{j+1} & E_{44}^{j+1} \end{bmatrix} \begin{bmatrix} e^{-\nu_{\alpha_{j+1}} d_{j+1}} & 0 & 0 & 0 \\ 0 & e^{-\nu_{\beta_{j+1}} d_{j+1}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{U,\alpha}^{j+1} \\ C_{U,\beta}^{j+1} \\ C_{D,\alpha}^{j+1} \\ C_{D,\beta}^{j+1} \end{bmatrix}
\end{aligned} \tag{1.9}$$

For bottom-up recursion assume that $\hat{\mathbf{R}}_D^{j+1}$ is known. Define $c_u^j = \hat{\mathbf{r}}_d^j c_d^j$, $\hat{\mathbf{C}}_D^{j+1} = \hat{\mathbf{t}}_d^j c_d^j$ and $\hat{\mathbf{C}}_U^{j+1} = \hat{\mathbf{R}}_D^{j+1}\hat{\mathbf{t}}_d^j c_d^j$. Note that $\hat{\mathbf{r}}_d$ is a $1x1$ matrix and $\hat{\mathbf{t}}_d = [t_{d,\alpha}, t_{d,\beta}]^T$ is a $2x1$ matrix. Substituting and then factoring out the common c_d^j gives:

$$\begin{bmatrix} e_{11}^j & e_{12}^j \\ e_{21}^j & e_{22}^j \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_d^j \\ e^{-\nu_{\alpha_j} d_j} \end{bmatrix} = \begin{bmatrix} E_{21}^{j+1} & E_{22}^{j+1} & E_{23}^{j+1} & E_{24}^{j+1} \\ E_{31}^{j+1} & E_{32}^{j+1} & E_{33}^{j+1} & E_{34}^{j+1} \\ E_{41}^{j+1} & E_{42}^{j+1} & E_{43}^{j+1} & E_{44}^{j+1} \end{bmatrix} \begin{bmatrix} e^{-\nu_{\alpha_{j+1}} d_{j+1}} & 0 & 0 & 0 \\ 0 & e^{-\nu_{\beta_{j+1}} d_{j+1}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{D,11}^{j+1} & R_{D,12}^{j+1} \\ R_{D,21}^{j+1} & R_{D,22}^{j+1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_{d,\alpha}^j \\ t_{d,\beta}^j \end{bmatrix} \tag{1.10}$$

Expanding and rearranging the terms yields

$$\begin{bmatrix} e_{12}^j e^{-\nu_{\alpha_j} d_j} \\ e_{22}^j e^{-\nu_{\alpha_j} d_j} \\ 0 \end{bmatrix} = \begin{bmatrix} E_{21}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} R_{D,11}^{j+1} + E_{22}^{j+1} e^{-\nu_{\beta_{j+1}} d_{j+1}} R_{D,21}^{j+1} + E_{23}^{j+1} & E_{21}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} R_{D,12}^{j+1} + E_{22}^{j+1} e^{-\nu_{\beta_{j+1}} d_{j+1}} R_{D,22}^{j+1} + E_{24}^{j+1} & -e_{11}^j \\ E_{31}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} R_{D,11}^{j+1} + E_{32}^{j+1} e^{-\nu_{\beta_{j+1}} d_{j+1}} R_{D,21}^{j+1} + E_{33}^{j+1} & E_{31}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} R_{D,12}^{j+1} + E_{32}^{j+1} e^{-\nu_{\beta_{j+1}} d_{j+1}} R_{D,22}^{j+1} + E_{34}^{j+1} & -e_{21}^j \\ E_{41}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} R_{D,11}^{j+1} + E_{42}^{j+1} e^{-\nu_{\beta_{j+1}} d_{j+1}} R_{D,21}^{j+1} + E_{43}^{j+1} & E_{41}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} R_{D,12}^{j+1} + E_{42}^{j+1} e^{-\nu_{\beta_{j+1}} d_{j+1}} R_{D,22}^{j+1} + E_{44}^{j+1} & 0 \end{bmatrix} \begin{bmatrix} t_{d,\alpha}^j \\ t_{d,\beta}^j \\ r_d^j \end{bmatrix} \tag{1.11}$$

For top-down processing, assume that $\hat{\mathbf{r}}_u^j$ is known. Defining the other constants in terms of $\mathbf{C}_U^{j+1}$, $c_u^j = \hat{\mathbf{T}}_U^{j+1}\mathbf{C}_U^{j+1}$, $c_d^j = \hat{\mathbf{r}}_u^j c_u^j = \hat{\mathbf{r}}_u^j \hat{\mathbf{T}}_U^{j+1}\mathbf{C}_U^{j+1}$, and $\mathbf{C}_D^{j+1} = \hat{\mathbf{R}}_U^{j+1}\mathbf{C}_U^{j+1}$. Here $\hat{\mathbf{r}}_u^j$ is $1x1$, $\hat{\mathbf{R}}_U^{j+1}$ is $2x2$ and $\hat{\mathbf{T}}_U^{j+1}$ is $1x2$.

Placing these into (1.9) gives

$$\begin{bmatrix} e_{11}^j & e_{12}^j \\ e_{21}^j & e_{22}^j \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ e^{-\nu_{\alpha_j} d_j} r_u^j \end{bmatrix} \begin{bmatrix} T_{U,\alpha} & T_{U,\beta} \end{bmatrix} = \begin{bmatrix} E_{21}^{j+1} & E_{22}^{j+1} & E_{23}^{j+1} & E_{24}^{j+1} \\ E_{31}^{j+1} & E_{32}^{j+1} & E_{33}^{j+1} & E_{34}^{j+1} \\ E_{41}^{j+1} & E_{42}^{j+1} & E_{43}^{j+1} & E_{44}^{j+1} \end{bmatrix} \begin{bmatrix} e^{-\nu_{\alpha_{j+1}} d_{j+1}} & 0 & 0 & 0 \\ 0 & e^{-\nu_{\beta_{j+1}} d_{j+1}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ R_{U,11}^{j+1} & R_{U,12}^{j+1} \\ R_{U,21}^{j+1} & R_{U,22}^{j+1} \end{bmatrix} \tag{1.12}$$

Rearranging this leads to the equations

$$
-\begin{bmatrix} E_{21}^{j+1} & E_{22}^{j+1} \\ E_{31}^{j+1} & E_{32}^{j+1} \\ E_{41}^{j+1} & E_{42}^{j+1} \end{bmatrix} \begin{bmatrix} e^{-\nu_{\alpha_{j+1}} d_{j+1}} & 0 \\ 0 & e^{-\nu_{\beta_{j+1}} d_{j+1}} \end{bmatrix} =
\begin{bmatrix} E_{23}^{j+1} & E_{24}^{j+1} & -(e_{11}^j + e_{12}^j e^{-\nu_{\alpha_j} d_j} {}^j r_u^j) \\ E_{33}^{j+1} & E_{34}^{j+1} & -(e_{21}^j + e_{22}^j e^{-\nu_{\alpha_j} d_j} {}^j r_u^j) \\ E_{43}^{j+1} & E_{44}^{j+1} & 0 \end{bmatrix} \begin{bmatrix} R_{U,11}^{j+1} & R_{U,12}^{j+1} \\ R_{U,21}^{j+1} & R_{U,22}^{j+1} \\ T_{U,\alpha}^{j+1} & T_{U,\beta}^{j+1} \end{bmatrix} \tag{1.13}
$$

Note that the solution of (1.11) and (1.13) requires the inverse of a $3x3$ matrix. The beauty of (1.5) and (1.7) is that at most a $2x2$ matrix inverse is required for isotropic or transverse isotropic material. We could also have obtained the desired $\hat{\mathbf{T}}_{U,D}$ and $\hat{\mathbf{R}}_{U,D}$ for the solid-solid problem through an expression similar to (1.11) or (1.13), but then would have had to obtain the inverse of a $4x4$ matrix for the isotropic or transverse isotropic medium.

One must also consider the case for which layer j is elastic and $j+1$ is a fluid. The conditions at z_j are

$$
\begin{aligned}
U_z &= E_{21}^j C_{U,\alpha}^j + E_{22}^j C_{U,\beta}^j + E_{23}^j e^{-\nu_{\alpha_j} d_j} C_{D,\alpha}^j + E_{24}^j e^{-\nu_{\beta_j} d_j} C_{D,\beta}^j \\
&= e_{11}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} c_u^{j+1} + e_{12}^{j+1} c_d^{j+1} \\
T_z &= E_{31}^j C_{U,\alpha}^j + E_{32}^j C_{U,\beta}^j + E_{33}^j e^{-\nu_{\alpha_j} d_j} C_{D,\alpha}^j + E_{34}^j e^{-\nu_{\beta_j} d_j} C_{D,\beta}^j \\
&= e_{21}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} c_u^{j+1} + e_{22}^{j+1} c_d^{j+1} \\
T_r &= E_{41}^j C_{U,\alpha}^j + E_{42}^j C_{U,\beta}^j + E_{43}^j e^{-\nu_{\alpha_j} d_j} C_{D,\alpha}^j + E_{44}^j e^{-\nu_{\beta_j} d_j} C_{D,\beta}^j \\
&= 0
\end{aligned} \tag{1.14}
$$

This can be rewritten as

$$
\begin{bmatrix} E_{21}^j & E_{22}^j & E_{23}^j & E_{24}^j \\ E_{31}^j & E_{32}^j & E_{33}^j & E_{34}^j \\ E_{41}^j & E_{42}^j & E_{43}^j & E_{44}^j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-\nu_{\alpha_j} d_j} & 0 \\ 0 & 0 & 0 & e^{-\nu_{\beta_j} d_j} \end{bmatrix} \begin{bmatrix} C_{U,\alpha}^j \\ C_{U,\beta}^j \\ C_{D,\alpha}^j \\ C_{D,\beta}^j \end{bmatrix}
= \begin{bmatrix} e_{11}^{j+1} & e_{12}^{j+1} \\ e_{21}^{j+1} & e_{22}^{j+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\nu_{\alpha_{j+1}} d_{j+1}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_u^{j+1} \\ c_d^{j+1} \end{bmatrix} \tag{1.15}
$$

For bottom-up processing $\hat{\mathbf{r}}_d^{j+1}$ is known and we define $\mathbf{C}_U^j = \hat{\mathbf{R}}_D^j \mathbf{C}_D^j$, $c_d^{j+1} = \hat{\mathbf{T}}_D^j \mathbf{C}_D^j$ and $\mathbf{c}_u^{j+1} = \hat{\mathbf{r}}_d^{j+1} \hat{\mathbf{T}}_D^j \mathbf{C}_D^j$. Here $\hat{\mathbf{T}}_D^j$ is $1x2$, $\hat{\mathbf{R}}_D^j$ is $2x2$ and $\hat{\mathbf{r}}_d^{j+1}$ is $1x1$. After factoring out the common $\mathbf{C}_D^j$, we have

$$\begin{bmatrix} E_{21}^j & E_{22}^j & E_{23}^j & E_{24}^j \\ E_{31}^j & E_{32}^j & E_{33}^j & E_{34}^j \\ E_{41}^j & E_{42}^j & E_{43}^j & E_{44}^j \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}}_D^j \\ \mathbf{e}^{-\nu_{\alpha_j} d_j} \end{bmatrix} = \begin{bmatrix} e_{11}^{j+1} & e_{12}^{j+1} \\ e_{21}^{j+1} & e_{22}^{j+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\nu_{\alpha_{j+1}} d_{j+1}} \hat{\mathbf{r}}_d^{j+1} \hat{\mathbf{T}}_D^j \\ \hat{\mathbf{T}}_D^j \end{bmatrix} \tag{1.16}$$

Expanding this gives

$$\begin{aligned} &\begin{bmatrix} E_{21}^j & E_{22}^j & -(e_{11}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} r_d^{j+1} + e_{12}^{j+1}) \\ E_{31}^j & E_{32}^j & -(e_{21}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} r_d^{j+1} + e_{22}^{j+1}) \\ E_{41}^j & E_{42}^j & 0 \end{bmatrix} \begin{bmatrix} R_{D,11}^j & R_{D,12}^j \\ R_{D,21}^j & R_{D,22}^j \\ T_{D,\alpha}^j & T_{D,\beta}^j \end{bmatrix} \\ &\qquad = -\begin{bmatrix} E_{23}^j e^{-\nu_{\alpha_j} d_j} & E_{24}^j e^{-\nu_{\beta_j} d_j} \\ E_{33}^j e^{-\nu_{\alpha_j} d_j} & E_{34}^j e^{-\nu_{\beta_j} d_j} \\ E_{43}^j e^{-\nu_{\alpha_j} d_j} & E_{44}^j e^{-\nu_{\beta_j} d_j} \end{bmatrix} \end{aligned} \tag{1.17}$$

For top-down processing, we start with (1.15), $\hat{\mathbf{R}}_U^j$ is known and we define the constants in terms of $\mathbf{c}_u^{j+1}$ as $\mathbf{C}_U^j = \hat{\mathbf{t}}_u^{j+1} \mathbf{c}_u^{j+1}$, $\mathbf{C}_D^j = \hat{\mathbf{R}}_U^j \hat{\mathbf{C}}_U^j = \hat{\mathbf{R}}_U^j \hat{\mathbf{t}}_u^{j+1} \mathbf{c}_u^{j+1}$, and $\mathbf{c}_d^{j+1} = \hat{\mathbf{r}}_u^{j+1} \mathbf{c}_u^{j+1}$. Here $\hat{\mathbf{r}}_u^{j+1}$ is $1x1$, $\hat{\mathbf{R}}_U^j$ is $2x2$ and $\hat{\mathbf{t}}_u^{j+1}$ is $1x2$. From these we obtain

$$\begin{aligned} &\begin{bmatrix} E_{21}^j & E_{22}^j & E_{23}^j & E_{24}^j \\ E_{31}^j & E_{32}^j & E_{33}^j & E_{34}^j \\ E_{41}^j & E_{42}^j & E_{43}^j & E_{44}^j \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e^{-\nu_{\alpha_j} d_j} R_{U,11}^j & e^{-\nu_{\alpha_j} d_j} R_{U,12}^j \\ e^{-\nu_{\beta_j} d_j} R_{U,21}^j & e^{-\nu_{\beta_j} d_j} R_{U,22}^j \end{bmatrix} \begin{bmatrix} t_{u,\alpha}^{j+1} \\ t_{u,\beta}^{j+1} \end{bmatrix} \\ &\qquad = \begin{bmatrix} e_{11}^{j+1} & e_{12}^{j+1} \\ e_{21}^{j+1} & e_{22}^{j+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\nu_{\alpha_{j+1}} d_{j+1}} & 0 \\ 0 & r_u^{j+1} \end{bmatrix} \end{aligned} \tag{1.18}$$

for the layer j fluid and layer $j+1$ solid.

The system of equations to be solved are

$$\begin{aligned} \begin{bmatrix} e_{21}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} \\ e_{22}^{j+1} e^{-\nu_{\alpha_{j+1}} d_{j+1}} \\ 0 \end{bmatrix} = &\left[\begin{matrix} E_{21}^j + E_{23}^j e^{-\nu_{\alpha_j} d_j} R_{U,11}^j + E_{24}^j e^{-\nu_{\beta_j} d_j} R_{U,21}^j \\ E_{31}^j + E_{33}^j e^{-\nu_{\alpha_j} d_j} R_{U,11}^j + E_{34}^j e^{-\nu_{\beta_j} d_j} R_{U,21}^j \\ E_{41}^j + E_{43}^j e^{-\nu_{\alpha_j} d_j} R_{U,11}^j + E_{44}^j e^{-\nu_{\beta_j} d_j} R_{U,21}^j \end{matrix}\right. \\ &\left.\begin{matrix} E_{22}^j + E_{23}^j e^{-\nu_{\alpha_j} d_j} R_{U,12}^j + E_{24}^j e^{-\nu_{\beta_j} d_j} R_{U,22}^j & -e_{12}^{j+1} \\ E_{32}^j + E_{33}^j e^{-\nu_{\alpha_j} d_j} R_{U,12}^j + E_{34}^j e^{-\nu_{\beta_j} d_j} R_{U,22}^j & -e_{22}^{j+1} \\ E_{42}^j + E_{43}^j e^{-\nu_{\alpha_j} d_j} R_{U,12}^j + E_{44}^j e^{-\nu_{\beta_j} d_j} R_{U,22}^j & 0 \end{matrix}\right] \begin{bmatrix} t_{u,\alpha}^{j+1} \\ t_{u,\beta}^{j+1} \\ r_u^{j+1} \end{bmatrix} \end{aligned} \tag{1.19}$$

For a medium with mixed fluid and solid properties, the logic for top-down processing is as follows after identifying each layer as a fluid or solid. For simplicity, we no longer distinguish between the upper and lower case matrices. First define the $\hat{\mathbf{R}}_U^1$ which depends on the boundary condition at z_0. Next examine the boundary at z_1. If the medium type is the same on both sides, then use (1.7) to define the $\hat{\mathbf{T}}_U^2$ and $\hat{\mathbf{R}}_U^2$. If the boundary separates a fluid from a solid, then use (1.12) or (1.19) to get $\hat{\mathbf{T}}_U^2$ and $\hat{\mathbf{R}}_U^2$. After this examine the boundary at z_2, and select the method to get $\hat{\mathbf{T}}_U^3$ and $\hat{\mathbf{R}}_U^3$. For bottom-up processing, the boundary conditions are used to define $\hat{\mathbf{R}}_D^{N-1}$, and then one examines the nature of the boundary at z_{N-2} to select the technique for computing $\hat{\mathbf{T}}_D^{N-1}$ and $\hat{\mathbf{R}}_D^{N-1}$. After this then consider the nature at the next boundary above.

Boundary conditions

The **G** and **H** matrices for the boundary conditions of free surface, rigid surface and halfspace are given in Table 1.1

Table 1.1 *Boundary conditions*

Matrix	Free	Rigid	Halfspace
H	$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$	$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$	$\mathbf{E}_0$
G	$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}$	$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$	$\mathbf{E}_N^{-1}$

Of these matrices the $\mathbf{E}_N^{-1}$ will have problems when the ν function is zero. This arises in part because the solution given in (1.2) is not correct for this case since the concept of upward and downward propagating or attenuating solutions in a region no longer holds. As a temporary expedient, one can let ν be a small number. Of course, if the medium velocities are complex because of anelastic attenuation or if the frequency is complex to avoid singularities on the real wavenumber axis, this problem should not arise.

Trapped modes

For the surface-wave or trapped-mode problems, one searches for the particular combination of frequency and phase velocity that satisfies the boundary conditions at the top and bottom of the layer stack. The bottom-up determination of $\hat{\mathbf{R}}_D^j$ uses the boundary condition at the bottom, while the top-down determination of $\hat{\mathbf{R}}_U^j$ starts with the boundary condition at the surface. Recalling the definitions relating the **C**'s in layer j to the generalized reflection coefficients, we must have

for generality we no long distinguish between c and C or r and R or t and T

$$\mathbf{C}_U^j = \hat{\mathbf{R}}_D^j \mathbf{C}_D^j \qquad \text{and} \quad \mathbf{C}_D^j = \hat{\mathbf{R}}_U^j \mathbf{C}_U^j$$

and which are satisfied by either

$$(\mathbf{I} - \hat{\mathbf{R}}_D^j \hat{\mathbf{R}}_U^j)\mathbf{C}_U^j = \mathbf{0} \qquad \text{or} \qquad (\mathbf{I} - \hat{\mathbf{R}}_U^j \hat{\mathbf{R}}_D^j)\mathbf{C}_D^j = \mathbf{0}$$

does this work for the Rayleigh wave halfspace?

The trapped mode problem is now the solution of

$$\det(\mathbf{I} - \hat{\mathbf{R}}_U^j \hat{\mathbf{R}}_D^j) = 0 \quad \text{or} \quad \det(\mathbf{I} - \hat{\mathbf{R}}_D^j \hat{\mathbf{R}}_U^j) = 0 \tag{1.20}$$

Note that the determinant is a complex quantity and thus one must search for the (ω, c) that makes the determinant zero.

There are several advantages in applying the bottom-up and top-down approaches together. For trapped-mode problems which have a low velocity zone within the model, this approach will provide the correct eigenfunction shapes which may involve oscillatory motion within the low velocity zone and exponentially decaying solutions away from this zone. If the eigenfunctions are used to make synthetics, the layer index j should be chosen such that the source is in that layer. Appendix 2 provides a simple example of the sequence of operations required to obtain the eigenfunctions.

Embedded source

To address the wave propagation problem with a source, it is easiest to modify the model by splitting a source layer so that the source occurs at a layer boundary of the new working model. For the discussion here, we assume that the region above and below the source have layer thicknesses, d, and vertical wavenumber ν with $-$ and $+$ subscripts, respectively. The effect of the source can be described as either a step in the displacement - stress vector or in the potential coefficients:

$$\Delta\mathbf{B} = \mathbf{E}\begin{bmatrix} \mathbf{0} \\ \Sigma^D \end{bmatrix} - \mathbf{E}\begin{bmatrix} -\Sigma^U \\ \mathbf{0} \end{bmatrix}$$

or

$$\Delta\mathbf{K} = \mathbf{E}^{-1}\Delta\mathbf{B} = \begin{bmatrix} \Sigma^U \\ \Sigma^D \end{bmatrix}$$

To relate the potential coefficients $\mathbf{K}(z_s^-)$ and $\mathbf{K}(z_s^+)$ to the $\mathbf{C}$'s above and below the source, we use

$$\mathbf{K}(z_s^-) = \begin{bmatrix} \mathbf{K}^U(z_s^-) \\ \mathbf{K}^D(z_s^-) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}^{-\nu_- d_-} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \hat{\mathbf{R}}_U^- \end{bmatrix} \mathbf{C}_U^-$$

$$\mathbf{K}(z_s^+) = \begin{bmatrix} \mathbf{K}^U(z_s^+) \\ \mathbf{K}^D(z_s^+) \end{bmatrix} = \begin{bmatrix} \mathbf{e}^{-\nu_+ d_+} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}}_D^+ \\ \mathbf{I} \end{bmatrix} \mathbf{C}_D^+$$

Combining we obtain

$$\mathbf{C}_U^- = (\mathbf{I} - \mathbf{e}^{-\nu_+ d_+} \hat{\mathbf{R}}_D^+ \mathbf{e}^{-\nu_- d_-} \hat{\mathbf{R}}_U^-)^{-1} \, (\mathbf{e}^{-\nu_+ d_+} \hat{\mathbf{R}}_D^+ \Sigma^D - \Sigma^U)$$

and

$$\mathbf{C}_D^+ = (\mathbf{I} - \mathbf{e}^{-\nu_- d_-} \hat{\mathbf{R}}_U^- \mathbf{e}^{-\nu_+ d_+} \hat{\mathbf{R}}_D^+)^{-1} (\Sigma^D - \mathbf{e}^{-\nu_- d_-} \hat{\mathbf{R}}_U^- \Sigma^U)$$

For wave propagation above the source, the $\mathbf{C}_D^-$ is obtained from the $\mathbf{C}_U^-$ using the relation $\mathbf{C}_D^- = \hat{\mathbf{R}}_U^- \mathbf{C}_U^-$. Given these, the $\mathbf{B}(z)$ is given by (1.3). The corresponding values for layers above the source are obtained using the $\hat{\mathbf{T}}_U$ and $\hat{\mathbf{R}}_U$ computed from the top-down recursion. Below the source we use $\mathbf{C}_U^+ = \hat{\mathbf{R}}_D^+ \mathbf{C}_D^+$. Although similar to the derivation in Kennett and Kerry (1979), this expression is seemingly more complicated in appearance because of the need to recast the solution in terms of the $\mathbf{C}$'s.

To highlight the computation of the response for a source in a layer, a simple case is discussed in Appendix 2

Exercises

1.1 After some manipulation, it can be show that these equations can be rearranged into the following form:

$$\begin{bmatrix} -E_{21}^{j+1} & -E_{22}^{j+1} & e_{12}^j \\ -E_{31}^{j+1} & -E_{32}^{j+1} & e_{22}^j \\ -E_{41}^{j+1} & -E_{42}^{j+1} & 0 \end{bmatrix} \begin{bmatrix} e^{-\nu_{\alpha_{j+1}} d_{j+1}} & 0 & 0 \\ 0 & e^{-\nu_{\beta_{j+1}} d_{j+1}} & 0 \\ 0 & 0 & e^{-\nu_{\alpha_j} d_j} \end{bmatrix} \begin{bmatrix} C_{U,\alpha}^{j+1} \\ C_{U,\beta}^{j+1} \\ c_d^j \end{bmatrix} = \begin{bmatrix} E_{23}^{j+1} & E_{24}^{j+1} & -e_{11}^j \\ E_{33}^{j+1} & E_{34}^{j+1} & -e_{21}^j \\ E_{43}^{j+1} & E_{44}^{j+1} & 0 \end{bmatrix} \begin{bmatrix} C_{D,\alpha}^{j+1} \\ C_{D,\beta}^{j+1} \\ c_u^j \end{bmatrix} \tag{1.21}$$

To generalize these equations, define

$$\begin{array}{lll} C_{D,\alpha}^{j+1} = t_{d,\alpha}^j c_d^j & C_{D,\alpha}^{j+1} = R_{U,11}^{j+1} C_{U,\alpha} & C_{D,\alpha}^{j+1} = R_{U,12}^{j+1} C_{U,\beta} \\ C_{D,\beta}^{j+1} = t_{d,\beta}^j c_d^j & C_{D,\beta}^{j+1} = R_{U,21}^{j+1} C_{U,\alpha} & C_{D,\beta}^{j+1} = R_{U,22}^{j+1} C_{U,\beta} \\ c_u^j = r_d^j c_d^j & c_u^j = T_{U,\alpha}^{j+1} C_{U,\alpha} & c_u^j = T_{U,\beta}^{j+1} C_{U,\beta} \end{array} \tag{1.22}$$

and then insert into the previous equation to obtain

$$\begin{bmatrix} -E_{21}^{j+1} & -E_{22}^{j+1} & e_{12}^j \\ -E_{31}^{j+1} & -E_{32}^{j+1} & e_{22}^j \\ -E_{41}^{j+1} & -E_{42}^{j+1} & 0 \end{bmatrix} \begin{bmatrix} e^{-\nu_{\alpha_{j+1}} d_{j+1}} & 0 & 0 \\ 0 & e^{-\nu_{\beta_{j+1}} d_{j+1}} & 0 \\ 0 & 0 & e^{-\nu_{\alpha_j} d_j} \end{bmatrix} = \begin{bmatrix} E_{23}^{j+1} & E_{24}^{j+1} & -e_{11}^j \\ E_{33}^{j+1} & E_{34}^{j+1} & -e_{21}^j \\ E_{43}^{j+1} & E_{44}^{j+1} & 0 \end{bmatrix} \begin{bmatrix} R_{U,11}^{j+1} & R_{U,12}^{j+1} & t_{d,\alpha}^j \\ R_{U,21}^{j+1} & R_{U,22}^{j+1} & t_{d,\beta}^j \\ T_{U,\alpha}^{j+1} & T_{U,\beta}^{j+1} & r_d^j \end{bmatrix} \tag{1.23}$$

This is essentially equation (16) of Chen and Chen (2002). The advantage of this formulation is that one obtains $\mathbf{R}_U^{j+1}$, $\mathbf{T}_U^{j+1}$, $\mathbf{r}_d^j$ and $\mathbf{t}_d^j$ in just one operation, but these modified reflection/transmission coefficients are not the generalized reflection/transmission coefficients used in §1.1.

2
Generalized reflection matrices

To better understand the steps to implement the generalized reflection/transmission technique of §1.1, the derivation of the solution for the trapped-mode eigenfunction problem and the embedded source problem are presented in detail for a simple problem.

We revisit the problem of a single fluid layer with a free surface overlying a fluid halfspace that was the focus of Chapter **??**. After defining all needed quantities, they will be combined to form the trapped mode solution of §**??** and the point source problem of (**??**). Figure 2.1 shows the model.

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For the problem of a wave propagation in a fluid, we have

$$\mathbf{E} = \begin{bmatrix} \nu & -\nu \\ -\rho\omega^2 & -\rho\omega^2 \end{bmatrix} \qquad \mathbf{E}^{-1} = \frac{1}{2\rho\nu\omega^2}\begin{bmatrix} \rho\omega^2 & -\nu \\ -\rho\omega^2 & -\nu \end{bmatrix}$$

The source is that used in §**??**

$$\Delta\mathbf{K} = \begin{bmatrix} -2\pi/\nu \\ 2\pi/\nu \end{bmatrix} = \begin{bmatrix} \Sigma^U \\ \Sigma^D \end{bmatrix}$$

To apply the boundary conditions at the $z = H$ we need the matrices $\mathbf{E}_1^{-1}\mathbf{E}_2$ and

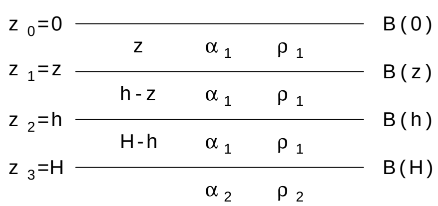

Figure 2.1 Fluid model for testing generalized reflection/transmission matrix technique. The layer is of thickness H, The source is at a depth of h and the response is to be determined at $Z < h$.

$\mathbf{E}_2^{-1}\mathbf{E}_1$

$$\mathbf{E}_1^{-1}\mathbf{E}_2 = \frac{1}{2\rho_1\nu_1}\begin{bmatrix}\rho_2\nu_1 + \rho_1\nu & \rho_2\nu_1 - \rho_1\nu_2 \\ \rho_2\nu_1 - \rho_1\nu & \rho_2\nu_1 + \rho_1\nu_2\end{bmatrix}$$

and

$$\mathbf{E}_2^{-1}\mathbf{E}_1 = \frac{1}{2\rho_2\nu^2}\begin{bmatrix}\rho_1\nu^2 + \rho_2\nu_1 & \rho_1\nu_2 - \rho_2\nu_1 \\ \rho_1\nu^2 - \rho_2\nu_1 & \rho_1\nu_2 + \rho_2\nu_1\end{bmatrix}$$

The initial step is to compute the $\hat{\mathbf{T}}_D$ and $\hat{\mathbf{R}}_D$ by starting at the bottom and moving upward, and also the $\hat{\mathbf{T}}_U$ and $\hat{\mathbf{R}}_U$ by starting at the top and moving down. These are presented in Table 2.1.

Trapped modes

We immediately see that (1.10) is true, and that the condition for the existence of trapped models is

$$\mathbf{I} - \hat{\mathbf{R}}_D^j\hat{\mathbf{R}}_U^j = 1 + \frac{1-P}{1+P}e^{-2\nu^1 H} = 0 \tag{2.1}$$

where we use $P = \rho_1\nu^2/\rho_2\nu^1$ as in Chapter **??**. The zeros of this function are the same as the poles of (**??**). After determining the (c, f) pair that satisfies this equation, the next step it to compute the eigenfunction $\mathbf{B}$ as a function of depth. This is done by first defining the $\mathbf{C}_U^j$ and $\mathbf{C}_D^j$ for each layer, and then using (1.3). Since the solution of the trapped mode eigenfunction problem is invariant to multiplying the eigenfunctions by the same scale, we will start at the top of the layer stack with an arbitrary value for C_D^1:

$$\begin{aligned}\mathbf{C}_D^1 &= 1 \\ \mathbf{C}_U^1 &= \mathbf{R}_D^1\mathbf{C}_D^1 = \left(\frac{1-P}{1+P}\right) \quad e^{-\nu^1(2H-Z)}.\end{aligned}$$

Now use these with (1.3) for $j = 1$ and $z = 0$. For $j = 1$, $z_{j-1} = 0$ and $z_j = Z$. Thus

$$\begin{aligned}\begin{bmatrix}\mathbf{U}\\ \mathbf{T}\end{bmatrix}_{z=0} &= \begin{bmatrix}\mathbf{E}_{11}e^{-\nu_1 Z}\mathbf{C}_D^1 + \mathbf{E}_{12}\mathbf{C}_D^1 \\ \mathbf{E}_{21}e^{-\nu_1 Z}\mathbf{C}_D^1 + \mathbf{E}_{22}\mathbf{C}_D^1\end{bmatrix} \\ &= \begin{bmatrix}\nu_1\left(\frac{1-P}{1+P}\right)e^{-2\nu_1 H} & -\nu_1 \\ -\rho_1\omega^2\left(\frac{1-P}{1+P}\right)e^{-2\nu_1 H} & -\rho_1\omega^2\end{bmatrix}\end{aligned} \tag{2.2}$$

Recall that the boundary condition at $z = 0$ was that of a free surface. Assuming that the (c, f) pair satisfies both boundary conditions, then $C_U^1 = -1$. and thus

$$\begin{bmatrix}\mathbf{U}\\ \mathbf{T}\end{bmatrix}_{z=0} \begin{bmatrix}-2\nu_1 \\ 0\end{bmatrix} \tag{2.3}$$

Now move to the next interface.

$$\mathbf{C}_D^2 = \mathbf{T}_D^1\mathbf{C}_D^1 = e^{-\nu_1 Z}$$
$$\mathbf{C}_U^2 = \mathbf{R}_D^2\mathbf{C}_D^2 = \left(\frac{1-P}{1+P}\right) \quad e^{-\nu_1(2H-h)}$$

from which

$$\begin{aligned}\begin{bmatrix}\mathbf{U}\\ \mathbf{T}\end{bmatrix}_{z=Z} &= \begin{bmatrix}\mathbf{E}_{11}e^{-\nu_1(h-Z)}\mathbf{C}_D^2 + \mathbf{E}_{12}\mathbf{C}_D^2\\ \mathbf{E}_{21}e^{-\nu_1(h-Z)}\mathbf{C}_D^2 + \mathbf{E}_{22}\mathbf{C}_D^2\end{bmatrix}\\ &= \begin{bmatrix}\nu_1\left(\frac{1-P}{1+P}\right)e^{-\nu_1(2H-Z)} & -\nu_1 e^{-\nu_1 Z}\\ -\rho_1\omega^2\left(\frac{1-P}{1+P}\right)e^{-\nu_1 2H-Z} & -\rho_1\omega^2 e^{-\nu_1 Z}\end{bmatrix}\\ &= \begin{bmatrix}-2\nu_1\cosh\nu_1 Z\\ 2\rho_1\omega^2\sinh\nu_1 Z\end{bmatrix}\end{aligned} \tag{2.4}$$

with the last being valid if the (c, f) pair satisfies the boundary conditions.

If continued, this processing sequence would permit the determination of $\mathbf{B}(H)$. There are two ways to obtain $\mathbf{B}(H)$. The first is to evaluate (**??**) at the bottom of layer 3. First define

$$\mathbf{C}_D^3 = \mathbf{T}_D^2\mathbf{C}_D^2 = e^{-\nu_1 h}$$
$$\mathbf{C}_U^3 = \mathbf{R}_D^3\mathbf{C}_D^3 = \left(\frac{1-P}{1+P}\right) \quad e^{-\nu_1(H)}$$

in which case the relation would be

$$\begin{bmatrix}\mathbf{U}\\ \mathbf{T}\end{bmatrix}_{z=H} = \begin{bmatrix}\mathbf{E}_{11}\mathbf{C}_D^3 + \mathbf{E}_{12}e^{-\nu_1(H-h)}\mathbf{C}_D^3\\ \mathbf{E}_{21}\mathbf{C}_D^3 + \mathbf{E}_{22}e^{-\nu_1(H-h)}\mathbf{C}_D^3\end{bmatrix} \tag{2.5}$$

$$= \begin{bmatrix}-2\nu_1\cosh\nu_1 H\\ -2\rho_1\omega^2\sinh\nu_1 H\end{bmatrix} \tag{2.6}$$

$$\tag{2.7}$$

The other method would be to extend the model by adding one more layer with the parameters of the halfspace. The advantage of this is that the same procedure is used to determine the stress-displacement at the top of each layer.

The selected value of $\mathbf{C}_D^1$ was arbitrary. It is often convenient to normalize the $\mathbf{B}$'s such that $U_z(0) = 1$. If we use the propagator matrices of (**??**), we can write

$$\begin{bmatrix}\mathbf{U}\\ \mathbf{T}\end{bmatrix}_{z=0} = \begin{bmatrix}1\\ 0\end{bmatrix} \qquad \begin{bmatrix}\mathbf{U}\\ \mathbf{T}\end{bmatrix}_{z=Z\le H} = \begin{bmatrix}\cosh\nu_1 z\\ -\rho_1\omega^2\sinh\nu_1 z/\nu_1 z\end{bmatrix}$$

As noted in Chapter **??**, the eigenfunctions are required at the source and receiver depths in order to make synthetics by model superposition. To compute the group

Table 2.1 *Generalized reflection and transmission coefficients*

Bottom-up	Top-Down
$\mathbf{a} = \mathbf{E}_1^{-1}\mathbf{E}_2$	$\mathbf{b} = \mathbf{I}$
$\hat{\mathbf{T}}_D^3$ not used	$\hat{\mathbf{T}}_U^1$ not used
$\hat{\mathbf{R}}_D^3 = \frac{\rho_2\nu_1-\rho_1\nu_2}{\rho_2\nu_1+\rho_1\nu_2}e^{-\nu_1(H-h)}$	$\hat{\mathbf{R}}_U^1 = -e^{-\nu_1(Z)}$
......	......
$\mathcal{E} = \mathbf{I}$	$\mathcal{F} = \mathbf{I}$
$\hat{\mathbf{T}}_D^2 = e^{-\nu_1(h-Z)}$	$\hat{\mathbf{T}}_U^2 = e^{-\nu_1(h-Z)}$
$\hat{\mathbf{R}}_D^2 = \frac{\rho_2\nu_1-\rho_1\nu_2}{\rho_2\nu_1+\rho_1\nu_2}e^{-\nu_1(2H-h-Z)}$	$\hat{\mathbf{R}}_U^2 = -e^{-\nu_1(h+Z)}$
......	......
$\mathcal{E} = \mathbf{I}$	$\mathcal{F} = \mathbf{I}$
$\hat{\mathbf{T}}_D^1 = e^{-\nu_1 Z}$	$\hat{\mathbf{T}}_U^3 = e^{-2\nu_1(H-h)}$
$\hat{\mathbf{R}}_D^1 = \frac{\rho_2\nu_1-\rho_1\nu_2}{\rho_2\nu_1+\rho_1\nu_2}e^{-\nu_1(2H-Z)}$	$\hat{\mathbf{R}}_U^3 = -e^{-\nu_1(H+h)}$
......	......

velocity or partial derivatives of the phase velocity, integrals of the eigenfunctions are required.A stable way of evaluating the required integrals was discussed in §**??** in the context of the propagator matrices.

Assuming that the model consists of layers with constant density and velocity, the variational techniques require the evaluation of integrals of the form

$$\int U_z^2 dz \qquad \int \Big(\frac{dU}{dz}\Big)^2 dz \qquad \int U\Big(\frac{dU}{dz}\Big)dz$$

Upper halfspace $z < 0$:

$$\int_{-\infty}^{0} U^2 dz = (E_N)_{11}^2 (K_0^U)^2 / 2\nu_0$$
$$\int_{-\infty}^{0} \left(\frac{dU}{dz}\right)^2 dz = \nu_0^2 (E_N)_{11}^2 (K_0^D)^2 / 2\nu_0$$

In this expression, the K_0^U is obtained from the definition $[K_0^U, K_0^D]^T = E_0^{-1} B(0)$.

Lower halfspace: $z > z_{N-1}$:

$$\int_{z_{N-1}}^{\infty} U^2 dz = (E_N)_{21}^2 (K_N^D)^2 / 2\nu_N$$
$$\int_{z_{N-1}}^{\infty} \left(\frac{dU}{dz}\right)^2 dz = \nu_N^2 (E_N)_{21}^2 (K_N^D)^2 / 2\nu_N$$

In this expression, the K_N^D is obtained from the definition $[K_N^U, K_N^D]^T = E_N^{-1} B(z_{N-1})$.

Layers: $z_{j-1} \le z \le z_j$

$$\begin{aligned}\int_{z_{j-1}}^{z_j} U^2 dz &= (E_j)_{11}^2 {C_U^j}^2 \left[1 - e^{-2\nu_j d_j}\right] / 2\nu_j \\ &+ 2(E_j)_{11}(E_j)_{12} C_U^j C_D^j d_j e^{-\nu_j d_j} \\ &+ (E_j)_{12}^2 (C_D^j)^2 \left[1 - e^{-2\nu_j d_j}\right] / 2_{\nu j} \\ \int_{z_{j-1}}^{z_j} \left(\frac{dU}{dz}\right)^2 dz &= (E_j)_{11}^2 {C_U^j}^2 \left[1 - e^{-2\nu_j d_j}\right] \nu_j / 2 \\ &- 2(E_j)_{11}(E_j)_{12} C_U^j C_D^j d_j e^{-\nu_j d_j} \nu_j^2 \\ &+ (E_j)_{12}^2 (C_D^j)^2 \left[1 - e^{-2\nu_j d_j}\right] \nu_j / 2\end{aligned}$$

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We see that casting of the problem in terms of the C_U and C_D permits a simple, stable determination of the integrals required for determination of the Lagrangian for a trapped mode problem. These same integrals are used for the determination of the group velocity and phase velocity partials with respect to medium parameters.

Embedded source

First consider the displacement above the source. For this model, the source is at the boundary between layers 2 and 3, which is at a depth h in the model. Just above the source we have

$$\begin{aligned} C_U^- = C_U^2 &= \left[1 - e^{-\nu^1 (H-h)} \hat{\mathbf{R}}_D^3 e^{-\nu_1 (h-Z)} \hat{\mathbf{R}}_U^2\right] \left[e^{-\nu_1 (H-h)} \hat{\mathbf{R}}_D^3 \Sigma^D - \Sigma^U\right] \\ &= \frac{2\pi}{\nu_1} \frac{(1-P) e^{-\nu_1 (2H-2h)} + (1+P)}{(1-P) e^{-2\nu_1 H} + (1+P)} \end{aligned}$$

and

$$C_D^2 = \mathbf{R}_U^2 \mathbf{C}_U^2 = -e^{-\nu_1 (h+Z)} C_U^2$$

mention that in this scalar case the exponentials could be combined, but for P-SV the actual order is important since the terms will then all be 2x2 matrices

Now placing these into the expression for the displacement and stress at the top of the layer 2, will give the solution for $Z < h$:

$$\begin{bmatrix} U_z \\ T_z \end{bmatrix} = \begin{bmatrix} \nu_1 & -\nu_1 \\ -\rho_1\omega^2 & \rho_1\omega^2 \end{bmatrix} \begin{bmatrix} e^{-\nu_1(h-Z)} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_U^2 \\ -e^{-\nu_1(h+Z)} C_U^2 \end{bmatrix}$$
$$= \begin{bmatrix} 2\nu_1 e^{-\nu_1 h} \cosh(\nu_1 Z) \\ -2\rho_1\omega^2 e^{-\nu_1 h} \sinh(\nu_1 Z) \end{bmatrix} \mathbf{C}_U^2$$
$$= \begin{bmatrix} 2\nu_1 \cosh(\nu_1 Z) \\ -2\rho_1\omega^2 \sinh(\nu_1 Z) \end{bmatrix} \frac{2\pi}{\nu_1} \frac{\cosh \nu_1(H-h) + P \sinh \nu_1(H-h)}{\cosh \nu_1 H + P \sinh \nu_1 H}$$

The T_z is the same as the **t** of §**??**.

To obtain the solution at the top of layer 1, we use

$$C_U^1 = T_U^2 C_U^2 \qquad\qquad C_D^1 = R_U^1 C_U^1$$

from which

$$\begin{bmatrix} U_z \\ T_z \end{bmatrix} = \begin{bmatrix} \nu_1 & -\nu_1 \\ -\rho_1\omega^2 & \rho_1\omega^2 \end{bmatrix} \begin{bmatrix} e^{-\nu_1 Z} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-\nu_1(h-Z)} C_U^2 \\ -e^{-\nu_1(h)} C_U^2 \end{bmatrix}$$
$$= \begin{bmatrix} 2\nu_1 e^{-\nu_1 h} \\ 0 \end{bmatrix} \mathbf{C}_U^2$$
$$= \begin{bmatrix} 2\nu_1 \\ 0 \end{bmatrix} \frac{2\pi}{\nu_1} \frac{\cosh \nu_1(H-h) + P \sinh \nu_1(H-h)}{\cosh \nu_1 H + P \sinh \nu_1 H}$$

Although this could also have been obtained by setting $z = Z$ in the previous solution, this example illustrates the stops to obtain the solution in the region above the source for multi-layered media.

Below the source we must use the C_D^+.

$$C_D^+ = C_D^3 = \left[1 - e^{-\nu_1(h-Z)} \hat{\mathbf{R}}_U^2 e^{-\nu_1(H-h)} \hat{\mathbf{R}}_D^3\right] \left[\Sigma^D - e^{-\nu_1(H-h)} \hat{\mathbf{R}}_U^2\, \Sigma^U\right]$$
$$= (1+P) \frac{2\pi}{\nu_1} \frac{1 - e^{-2\nu_1 h}}{(1-P)e^{-2\nu_1 H} + (1+P)}$$

and

$$C_U^3 = \mathbf{R}_D^3 \mathbf{C}_D^3 = \frac{1-P}{1+P} e^{-\nu_1(H-h)} C_D^3$$

The solution at the top of layer 3, just beneath the source, is

$$\begin{bmatrix} U_z \\ T_z \end{bmatrix}_{h^+} = \begin{bmatrix} \nu_1 & -\nu_1 \\ -\rho_1\omega^2 & \rho_1\omega^2 \end{bmatrix} \begin{bmatrix} e^{-\nu_1(H-h)} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1-P}{1+P} e^{-\nu_1(H-h)} \\ 1 \end{bmatrix} C_D^3$$
$$= -\begin{bmatrix} \nu_1 (P \cosh \nu_1(H-h) + \sinh \nu_1(H-h)) \\ \rho_1\omega^2 (P \cosh \nu_1(H-h) + \sinh \nu_1(H-h)) \end{bmatrix} \frac{2\pi}{\nu_1} \frac{\sinh \nu_1 h}{\cosh \nu_1 H + P \sinh \nu_1 H}$$

while the solution at the base of layer 3, e.g., $z = H$, is

$$\begin{bmatrix} U_z \\ T_z \end{bmatrix}_H = \begin{bmatrix} \nu_1 & -\nu_1 \\ -\rho_1\omega^2 & \rho_1\omega^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-\nu_1(H-h)} \end{bmatrix} \begin{bmatrix} \frac{1-P}{1+P} e^{-\nu_1(H-h)} \\ 1 \end{bmatrix} C_D^3$$
$$= -\begin{bmatrix} 2P\nu_1 \\ 2\rho_1\omega^2 \end{bmatrix} \frac{2\pi}{\nu_1} \frac{\sinh \nu_1 h}{\cosh \nu_1 H + P \sinh \nu_1 H}$$

If the model had consisted of more layers beneath the source, then given C_D^3 and T_D^3, one would compute

$$C_D^4 = T_D^3 C_D^3 \qquad\qquad C_U^4 = R_D^4 C_D^4$$

with (1.3).

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