Introduction to Earthquake Seismology
Assignment 8

Department of Earth and Atmospheric Sciences **EASA-462**
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Distances on the Earth

Goals:

- Convert geographic coordinates to geocentric coordinates to compute epicentral distances
- Determine differences between computed distances.

Background:

Assignment 7 introduced the computation of epicentral distances, epicenter-to-station azimuth and station-to-epicenter back azimuth for a spherical model of the Earth. These formulas will give incorrect distances for the Earth because of the fact that a polar cross-section of the earth is an ellipse, rather than a circle. Figure 1 illustrates the problem. The geocentric latitude, ϕ_c, is defined is the angle between the line connecting the center of the earth to a point on the surface and the equatorial plane. The geographic or geodetic latitude, ϕ_g, is defined as the angle that the normal to the surface makes with respect to the equatorial plane. At the equator and the poles, these two angles are the same, however the surface distance for a unit change in geocentric latitude is not the same because of the curvature of the surface.

To understand this problem, we will solve some problems in geodesy.

The equation of the spheroid representing the Earth is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1$$

where the x-axis connects the center of the Earth to the latitude-longitude point (0,0) on the surface, the y-axis connects the center of the Earth to the latitude-longitude point (0,90) on the surface and the z-axis connects the center of the Earth to the north pole (90,0). The polar radius, b, and the equatorial radius, a, are used to define the flattening of the spheroid, $f \equiv 1 - b/a$.

To obtain the relation between the geocentric and geographic latitudes, consider the vertical slice through the origin along $y = 0$. The resulting ellipse is shown in Figure 1 and is given by the equation $(x/a)^2 + (z/b)^2 = 1$. For a point, P on this ellipse with coordinates (x, z), we have

$$\tan\phi_c = z/x.$$

To get the geographic latitude, we recall that the slope of the tangent line at a point is dz/dx and that the slope of the normal line at the same point is $-dx/dz$. We thus have

$$\tan\phi_g = -\frac{dx}{dz} = \frac{z}{x}\frac{a^2}{b^2} = \frac{a^2}{b^2}\tan\phi_c$$

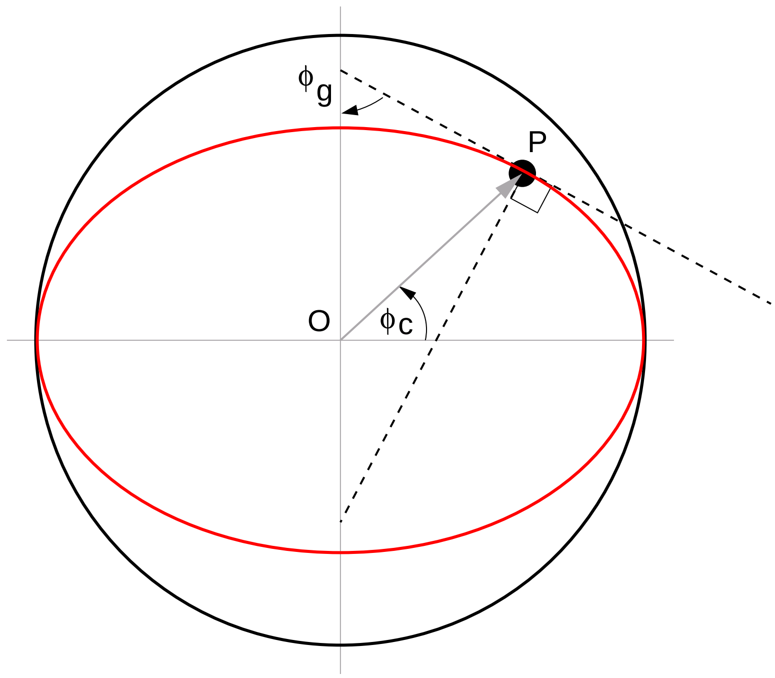

Fig. 1. Cross-sections through the poles and the center of ellipsoid and circumscribing sphere. A the geocentric, ϕ_c, and geodetic, ϕ_g, are indicated.

If we define $(a^2 - b^2) / a^2 = \varepsilon^2 = 2f - f^2$, then $b^2 / a^2 = 1 - \varepsilon^2$, and

$$\tan \phi_c = (1 - \varepsilon^2) \tan \phi_g \tag{1}$$

The object is to determine the azimuth, back-azimuth, arc distance and distance in kilometers between an epicenter, given by the coordinates $(\phi_g, \lambda)_e$ and station, given by $(\phi_g, \lambda)_s$, respectively. We first convert from geographic to geocentric coordinates using (1). We will also compute the *sin* and *cos* of the angles because we need them and because we want (1) to be computationally valid, even in the case $\phi = \pi/2$:

$$\sin \phi_c = \frac{(1 - \varepsilon^2) \sin \phi_g}{\sqrt{((1 - \varepsilon^2)^2 \sin^2 \phi_g + \cos^2 \phi_g)}}$$

$$\cos \phi_c = \frac{\cos \phi_g}{\sqrt{((1 - \varepsilon^2)^2 \sin^2 \phi_g + \cos^2 \phi_g)}}$$

which is constructed to satisfy both (1) and the identity $\sin^2 \phi_c + \cos^2 \phi_c = 1$.

Given the ϕ_c and the λ of the epicenter and station, we compute the direction cosines to define the unit vectors from the center of the spheroid to the two points on the surface as

$$\mathbf{E} = \cos \lambda_e \cos \phi_e \mathbf{i} + \sin \lambda_e \cos \phi_e \mathbf{j} + \sin \phi_e \mathbf{k} = (\alpha_e, \beta_e, \gamma_e)$$

$$\mathbf{S} = \cos \lambda_s \cos \phi_s \mathbf{i} + \sin \lambda_s \cos \phi_s \mathbf{j} + \sin \phi_s \mathbf{k} = (\alpha_s, \beta_s, \gamma_s)$$

where the additional subscript c representing geocentric latitude is dropped for simplicity. The alternate notation of representing a vector by a matrix is also shown, e.g., $\beta_e \rightarrow \sin \lambda_e \cos \phi_e \mathbf{j}$.

The great circle arc, Δ and the *Az* and *Baz* are computed as in Assignment 7. Note that the geocentric latitude seems to be used by seismologists and not geodesists.

The most difficult quantity to compute is the linear distance along the surface on the shortest distance curve connecting E and S. To do is, we first define the normal to the plane define by the points E , S and O, $\mathbf{E}x\mathbf{S}$:

$$\mathbf{y}' = \frac{(\beta_e\gamma_s - \beta_s\gamma_e, \gamma_e\alpha_s - \gamma_s\alpha_e, \alpha_e\beta_s - \beta_e\alpha_s)}{\sqrt{(\beta_e\gamma_s - \beta_s\gamma_e)^2 + (\gamma_e\alpha_s - \gamma_s\alpha_e)^2 + (\alpha_e\beta_s - \beta_e\alpha_s)^2}} \equiv (y_1', y_2', y_3')$$

This plane intersects the ellipsoid as an ellipse (no proof for this assertion) which will be defined by the the coordinates (x', y'). [Note: to avoid a numerical problem when E and S are on the equator, if any unit vector element is $< 10^{-6}$, it can be repalced by 10^{-6} which means introducting a slight error in the results]. We define the horizontal axis as

$$\mathbf{x}' = \frac{(y_2', -y_1', 0)}{\sqrt{((y_1')^2 + (y_2')^2)}} \equiv (x_1', x_2', x_3')$$

and

$$\mathbf{z}' = \frac{(-y_1'y_3', -y_2'y_3', y_1'y_1' + y_2'y_2')}{\sqrt{((y_1')^2 + (y_2')^2)}} \equiv (z_1', z_2', z_3')$$

You will note that the vectors $\mathbf{x}'$, $\mathbf{y}'$ and $\mathbf{z}'$ are unit vectors.

The ellipse created by the intersection of the plane and ellipsoid is defined by the equation

$$\left(\frac{x'}{a'}\right)^2 + \left(\frac{z'}{b'}\right)^2 = 1$$

where x' is a linear distance in the direction $\mathbf{x}'$, and similarly for y'. Because we consider a spheroid, the width of this ellipse in the $\mathbf{x}'$ (horizontal) direction is the equatorial diameter of the spheroid. Hence

$$a' = b'.$$

However the length of this ellipse in the $\mathbf{z}'$ direction depends upon the orientation of the plane defined by E, S and O. If E and S are on the equator, then the intersection of the plane with the spheroid gives a circle, and hence the $b' = a$. On the other hand, if E is on the equator, and O is at the pole, then the intersection is an ellipse with $b' = b$. Since the point $(b'z'_1, b'z'_2, b'z'_3)$ must be on the ellipsoid, we must have

$$b' = \frac{1}{\sqrt{\left(\frac{z'_1}{a}\right)^2 + \left(\frac{z'_2}{a}\right)^2 + \left(\frac{z'_3}{b}\right)^2}}$$

Now that we have defined the ellipse, we must define the position of E and S on the ellipse. This is done by a simple vector dot product that defines the direction of the point with respect to the ellipse coordinate system:

$$E' = (\mathbf{E} \cdot \mathbf{x}', \mathbf{E} \cdot \mathbf{z}')$$

$$S' = (\mathbf{S} \cdot \mathbf{x}', \mathbf{S} \cdot \mathbf{z}')$$

Since we know the ellipse, we do are only interested in the angles with respect to the x'-axis:

$$\theta_E = atan2((\mathbf{E}\cdot\mathbf{z}')/b', (\mathbf{E}\cdot\mathbf{x}')/a')$$

$$\theta_S = atan2((\mathbf{S}\cdot\mathbf{z}')/b', (\mathbf{S}\cdot\mathbf{x}')/a')$$

where we use the ellipse equation $x' = a' \cos\theta$ and $z' = b' \sin\theta$. [Note: for exactness the E' coordinate should be scaled by a scalar multiplier reflecting the true O-E distance in Fig. 1. However this scalar drops out when we form the ratio to get the inverse tangent.]

To get the linear distance along the surface we evaluate the integral

$$\int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dz}{dx}\right)^2}\, dx = \int_{\theta_e}^{\theta_s} \sqrt{(a'^2 \sin^2\theta + b'^2 \cos^2\theta)}\, d\theta = a \int_{\theta_e}^{\theta_s} \sqrt{1 - \varepsilon'^2 \cos^2\theta}\, d\theta \qquad (2)$$

where $\varepsilon'^2 = 1 - (b'/a')^2$. Note that the last integral can be written in terms of the incomplete elliptic integral of the second kind $\int_{\chi} \sqrt{1 - k^2 \sin^2\chi}\, d\chi$ by the transformation $\chi = \frac{\pi}{2} - \theta$.

In evaluating (2) it will be necessary to examine the θ_E and θ_S to ensure that the integration in in the direction of increasing θ and that the integration is over the minor arc.

Computational Concerns

For a spherical and spheroidal model there are a number of inconsistencies. For example. at the poles what is meant by azimuth and back azimuth. For a spheroidal model with both station and epicenter on the equator and 180° apart in longitude, $\Delta = 180°$. For a sphere azimuth can be any value and the back-azimuth depends on the chosen azimuth. For a spheroid representing the Earth, the shortest distance between the two points is across the poles and not along the equator.

Consideration of these special case makes the computer code complex. An alternative for seismology on the Earth, is just to wiggle the coordinates slightly, e.g., replace a latitude of 90° by 89.9999°, a difference in position of roughly 10 meters. If both are on the equator, change the latitudes by 0.0001°.

The Earth

We use the following constants:

a	6378.137 km
b	6356.752 km
f	0.003352810664747481 = 1/298.257223563
e^2	0.006694379990141317

What you must do:

This problem set is identical to that of Assignment 7. Except that you are now to use the geocentric latitude rather than the geodetic latitude. Note: you do not have to evaluate (2) to get the linear distance along the surface.

$\phi_{e_g} = 40$	$\lambda_e = 0$	$\phi_{s_g} = 50$	$\lambda_s = 10$
$\phi_{e_c} =$ _____		$\phi_{s_c} =$ _____	
$\Delta(deg) =$ _____	$Az(deg) =$ _____	$Baz(deg) =$ _____	

$\phi_{e_g} = 60$	$\lambda_e = 0$	$\phi_{s_g} = 70$	$\lambda_s = 10$
$\phi_{e_c} =$ _____		$\phi_{s_c} =$ _____	
$\Delta(deg) =$ _____	$Az(deg) =$ _____	$Baz(deg) =$ _____	

$\phi_{e_g} = 0$	$\lambda_e = -30$	$\phi_{s_g} = 60$	$\lambda_s = 60$
$\phi_{e_c} =$ _____		$\phi_{s_c} =$ _____	
$\Delta(deg) =$ _____	$Az(deg) =$ _____	$Baz(deg) =$ _____	

What you must submit:

Compare the values of Δ with and without the correction to geocentric distances.

References

Bomford, G. (1980). *Geodesy* (4th ed),Clarendon Press, Oxford (§2.15(b), p 120-121). *Appendix A reviews spherical trigonometry and the ellipsoid. §2.15 examines the determination of the mutual distances and azimuths of two points defined by their latitude and longitude. In §2.14(c) the Rudoe technique is characterized as computing the "axes and eccentricity of the ellipse in which the normal section cuts the spheriod." I have not directly compared my derivation with the Rudoe formula.*