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Improved harmonic development of the Earth tide-generating potential

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Abstract. An improved technique is proposed for spectral analysis of an arbitrary function of the Moon/Sun/planet coordinates tabulated over a long period of time. Expansion of the function to Poisson series is made where the amplitudes and arguments of the series' terms are high-degree time polynomials, as opposed to the classical Fourier analysis where the terms' amplitudes are constants, and the arguments are linear functions of time. This leads to an improvement in accuracy of the spectral analysis of the functions over long-term intervals. The technique is applied to produce a harmonic development of the Earth tide-generating potential (TGP) which was preliminarily calculated and tabulated over an interval of 2000 years centered at epoch J2000.0. (The latest NASA/JPL long-term ephemeris DE/LE-406 has been used as the source of the Moon, Sun and planet coordinates.) The final development of the TGP, named KSM03, includes some 27 000 terms of amplitudes down to the level of $1 \times 10^{-8}\ \mathrm{m}^2\,\mathrm{s}^{-2}$. The respective accuracy in calculation of gravity tides at a mid-latitude station is 0.025/0.39 nGal (the root-mean-square/maximum error) when compared with a benchmark gravity tide series numerically computed with the use of the most accurate ephemeris DE-405 at every hour within the interval 1600–2200 (the complete period of time covered by the ephemeris). It exceeds the accuracy of any previously made harmonic development of the TGP in the time domain by a factor of at least 3.

Key words: Tide generating potential – Harmonic development

1 Introduction

The task of harmonic development of the Earth tide generating potential (TGP) has a long history. Doodson (1921) first performed an accurate representation of the TGP by harmonic series. Subsequent expansions were done by Cartwright and Tayler (1971), Cartwright and Edden (1973), Büllesfeld (1985), Xi (1987, 1989), and Tamura (1987, 1995). The latest and to date most accurate harmonic developments of the TGP have been made by Hartmann and Wenzel (1994, 1995) and Roosbeek (1996), who used quite different methods in their studies. Roosbeek's model of the TGP is built by means of an analytical technique. He used analytical series for spherical coordinates of the Moon from ELP2000–85 theory (Chapront-Touzé and Chapront 1988) and those of the major planets from VSOP87 theory (Bretagnon and Francou 1988). Then, after some transformation of the series he obtained the desired harmonic series representing the TGP development. Such an approach separates well terms of close frequencies in the final expansion (but only as far as is done in the original motion theories of the Moon and planets). However, the accuracy of this method is obviously limited by the accuracy of the bodies' coordinates provided by the analytical theories of planetary/lunar motion which have a lower accuracy than numerical ephemerides of DE-series by the Jet Propulsion Laboratory (JPL)/National Aeronautics and Space Administration (NASA) [the latter are recommended by the International Earth Rotation Service (IERS) Conventions (McCarthy and Petit 2003) for precise calculations]. Hartmann and Wenzel (1994, 1995) chose another method for developing their model of the TGP. As a source of the Moon/planet coordinates they used the then most accurate numerical ephemeris DE-200 (Standish and Williams 1981). They calculated numerical values of relevant functions of the attracting bodies' coordinates contained in the TGP formulation over a 300-year interval (1850–2150) with a small sampling step and made a Fourier analysis of the obtained data. A disadvantage of this approach is the non-proper separation of close frequencies when making a Fourier analysis of data sampled over a relatively short period of time. Nevertheless, Hartmann and Wenzel were able to build an accurate harmonic development of the TGP which was one of the best at the time of their study.

In our work we followed the latter approach. The following major modifications have been introduced by us to further increase the accuracy of the TGP expansion.

1. We developed and used an improved technique of spectral analysis to obtain expansions of the relevant functions to Poisson series with the terms' amplitudes and arguments being high-degree polynomials of time (as opposed to classical Fourier analysis where the terms' amplitudes are constants and the arguments are linear functions of time).
2. The most up-to-date planetary and lunar ephemerides DE/LE-405, -406 (Standish 1998a) were used.
3. In order to decrease the effect of close frequencies the improved spectral analysis was done over a 2000-year interval of time, 1000–3000.

As a result, we can reach an accuracy of the harmonic development of the TGP in the time domain that improves on the accuracy of any previously developed model. Respective computation of gravity tides at a mid-latitude station can be done at sub-nGal level and is proven to be stable over the period 1600–2200 (the complete time span covered by the ephemeris DE/LE-405). This allows use of the present TGP development in treatment of advanced gravimeter data in the future and in high-accuracy prediction of Earth tides.

2 Formulation of TGP expansion problem

The classical representation of the Earth TGP generated by external attracting bodies (the Moon, Sun, planets) at an arbitrary point P on the Earth's surface at epoch t is

$$V(t) = \sum_j \mu_j \sum_{n=2}^{\infty} \frac{r^n}{r_j^{n+1}(t)} P_n(\cos\psi_j(t)) \tag{1}$$

where V is the value of the TGP at P; r is the geocentric distance to P; μ_j and r_j are, respectively, the gravitational parameter and geocentric distance to the jth body; ψ_j is the angle between P and the jth body as seen from the Earth's center; and P_n is the Legendre polynomial of degree n.

Equation (1) is expanded in our study as

$$\begin{aligned} V(t) &= \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{r}{R_E}\right)^n \bar{P}_{nm}(\sin\varphi')\Big[C_{nm}(t)\cos m\theta^{(A)}(t) \\ &\quad + S_{nm}(t)\sin m\theta^{(A)}(t)\Big] \\ &\equiv \sum_{n=2}^{\infty}\sum_{m=0}^{n} V_{nm}(t) \end{aligned} \tag{2}$$

where

$$\begin{aligned} C_{nm}(t) &= \frac{1}{2n+1}\sum_j \frac{\mu_j}{R_E}\left(\frac{R_E}{r_j(t)}\right)^{n+1} \\ &\quad \times \bar{P}_{nm}(\sin\delta_j(t))\cos m\alpha_j^{(A)}(t) \end{aligned} \tag{3}$$

$$\begin{aligned} S_{nm}(t) &= \frac{1}{2n+1}\sum_j \frac{\mu_j}{R_E}\left(\frac{R_E}{r_j(t)}\right)^{n+1} \\ &\quad \times \bar{P}_{nm}(\sin\delta_j(t))\sin m\alpha_j^{(A)}(t) \end{aligned} \tag{4}$$

and R_E is the mean Earth equatorial radius; $\alpha_j^{(A)}(t)$ and $\delta_j(t)$ are, respectively, the instantaneous right ascension and declination of the jth body referred to the true geoequator of epoch t with an origin at point A—that being the projection of the mean equinox of date (see Fig. 1); $\theta^{(A)}(t)$ is the local mean sidereal time at P reckoned from the same point A—so that it is related to the Earth fixed east longitude (from Greenwich) λ of P simply as

$$\theta^{(A)}(t) = \lambda + \text{GMST} \tag{5}$$

[GMST is Greenwich mean sidereal time defined by the well-known expression by Aoki et al. (1982)]; φ' is the geocentric latitude of the point P; and $\bar{P}_{nm}$ is the normalized associated Legendre function related to the unnormalized one (P_{nm}) as

$$\bar{P}_{nm} = N_{nm}P_{nm} \tag{6}$$

where

$$N_{nm} = \sqrt{\frac{\delta_m(2n+1)(n-m)!}{(n+m)!}} \tag{7}$$

and

$$\delta_m = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{if } m \neq 0 \end{cases} \tag{8}$$

The classical expression for the TGP [Eq. (1)] is completed by some additional terms reflecting the main effect of Earth's flattening (Wilhelm 1983; Dahlen 1993; Hartmann and Wenzel 1995; Roosbeek 1996), which can be re-written as follows:

$$\begin{aligned} V^{fl}(t) = \frac{r}{R_E}\Big\{&\bar{P}_{10}(\sin\varphi')C_{10}(t) + \bar{P}_{11}(\sin\varphi') \\ &\times \left[C_{11}(t)\cos\theta^{(A)}(t) + S_{11}(t)\sin\theta^{(A)}(t)\right]\Big\} \end{aligned} \tag{9}$$

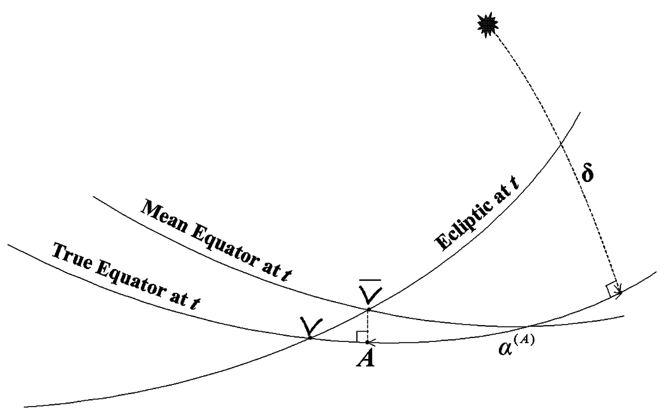

Fig. 1. The spherical coordinates used when developing the TGP

where

$$C_{10}(t) = \sqrt{\frac{15}{7}} \frac{\bar{J}_2}{R_E} \sum_j \mu_j \left(\frac{R_E}{r_j(t)}\right)^4 \bar{P}_{30}(\sin \delta_j(t)) \tag{10}$$

$$C_{11}(t) = \sqrt{\frac{10}{7}} \frac{\bar{J}_2}{R_E} \sum_j \mu_j \left(\frac{R_E}{r_j(t)}\right)^4 \times \bar{P}_{31}(\sin \delta_j(t)) \cos \alpha_j^{(A)}(t) \tag{11}$$

$$S_{11}(t) = \sqrt{\frac{10}{7}} \frac{\bar{J}_2}{R_E} \sum_j \mu_j \left(\frac{R_E}{r_j(t)}\right)^4 \times \bar{P}_{31}(\sin \delta_j(t)) \sin \alpha_j^{(A)}(t) \tag{12}$$

$\bar{J}_2$ is the normalized value for the dynamical form factor of the Earth ($\bar{J}_2 = J_2/N_{20}$).

The coefficients $C_{nm}(t)$ and $S_{nm}(t)$ contain information about instantaneous positions of the attracting bodies at every epoch t at which we calculate the TGP value $V(t) + V^{fl}(t)$. Angles $\alpha_j^{(A)}(t)$ and $\delta_j(t)$ in Eqs. (3), (4), and (10)–(12) are reckoned along and from the true geoequator of the epoch t, so the relevant values for the coefficients $C_{nm}(t)$ and $S_{nm}(t)$ fully take into account the effects of both precision and nutation in obliquity. When computing the TGP at a certain point P on the Earth's surface at epoch t we need to know just differences $\theta^{(A)}(t) - \alpha_j^{(A)}(t)$ between the local sidereal time at P and the right ascension of the jth attracting body. Because of the choice for the origin point A (see Fig. 1), nutation in longitude does not affect values for $\alpha_j^{(A)}(t)$ [and consequently values for $C_{nm}(t)$ and $S_{nm}(t)$]. Therefore, we do not have to take into account nutation in longitude when calculating $\theta^{(A)}(t)$ as well, i.e. the GMST has to be used in Eq. (5).

Having harmonic expansions for $C_{nm}(t)$ and $S_{nm}(t)$ we can further calculate the time-dependent values of the TGP at an arbitrary point $P(r, \varphi', \lambda)$ on the Earth's surface by using the relations of Eqs. (2), (5), and (9). The tidal acceleration along the Earth radius (or 'the gravity tide') is obtained as the radial derivative of the TGP

$$g(t) \equiv \frac{\partial(V(t) + V^{fl}(t))}{\partial r} = \sum_{n=1}^{\infty} \frac{n}{r} \sum_{m=0}^{n} V_{nm}(t) + \frac{1}{r} V^{fl}(t) \tag{13}$$

In our work the coefficients $C_{nm}(t)$ and $S_{nm}(t)$ have been expanded to precision Poisson series by an improved method of spectral analysis made over 2000 years, 1000–3000. A description of the method and the results obtained are presented in the following sections.

3 Improved technique of spectral analysis

Let $f(t)$ be an arbitrary function tabulated by its numerical values over an interval of time $[-T, T]$ with a small sampling step.

Over the same interval we will build an analytical representation of the function by a finite h-order Poisson series of the following form:

$$f(t) \approx \sum_{k=1}^{N} \left\{ \left[A_{k0}^c + A_{k1}^c t + \cdots + A_{kh}^c t^h\right] \cos \omega_k(t) + \left[A_{k0}^s + A_{k1}^s t + \cdots + A_{kh}^s t^h\right] \sin \omega_k(t) \right\} \tag{14}$$

where $A_{k0}^c, A_{k1}^c, \ldots, A_{kh}^s$ are constants and $\omega_k(t)$ are some pre-defined arguments which are assumed to be q-degree polynomials of an independent variable (e.g. of time)

$$\omega_k(t) = \nu_k t + \nu_{k2} t^2 + \cdots + \nu_{kq} t^q \tag{15}$$

For this we find the projections of $f(t)$ on a basis generated by functions

$$\mathbf{c}_{kl}(t) \equiv t^l \cos \omega_k(t), \quad \mathbf{s}_{kl}(t) \equiv t^l \sin \omega_k(t) \quad (k = 1, 2, \ldots, N; \ l = 0, 1, \ldots, h) \tag{16}$$

through numerical computation of the following scalar products:

$$A_{kl}^c = \langle f, \mathbf{c}_{kl} \rangle \equiv \frac{1}{2T} \int_{-T}^{T} f(t) t^l \cos \omega_k(t) \chi(t)\, \mathrm{d}t \tag{17}$$

$$A_{kl}^s = \langle f, \mathbf{s}_{kl} \rangle \equiv \frac{1}{2T} \int_{-T}^{T} f(t) t^l \sin \omega_k(t) \chi(t)\, \mathrm{d}t \tag{18}$$

by using the definition

$$\langle f, g \rangle \equiv \frac{1}{2T} \int_{-T}^{T} f(t) \bar{g}(t) \chi(t)\, \mathrm{d}t \tag{19}$$

($\bar{g}$ is the complex conjugate to the g function); $\chi(t) = 1 + \cos \frac{\pi}{T} t$ is the Hanning filter chosen as the weight function.

The proper choice of arguments $\omega_k(t)$ depends on the specific task (e.g. they can be multipliers of Delaunay arguments and/or planetary mean longitudes, etc.). However, the basis functions $\mathbf{c}_{k_1 l_1}(t)$, $\mathbf{s}_{k_1 l_1}(t)$, $\mathbf{c}_{k_2 l_2}(t)$, $\mathbf{s}_{k_2 l_2}(t)$, … are not usually orthogonal. So, we have to perform an orthogonalization process over the expansion coefficients in order to improve the quality of representation of Eq. (14) and avoid superfluous terms. For this procedure we used the algorithm developed by Šidlichovský and Nesvorný (1997). Equations (20)–(27) present the algorithm which we have generalized as indicated below.

Let $f(t)$ be a tabulated complex function and let $\{\mathbf{e}_i\}_{i=1,2,\ldots,M}$ be a set of M basis functions [in our study equal to the complete set of $\mathbf{c}_{kl}(t)$, $\mathbf{s}_{kl}(t)$ so that $M = 2 \times N \times (h+1)$]. The function $f(t)$ is developed on the basis $\{\mathbf{e}\}$ as

$$f(t) = \sum_{i=1}^{M} A_i^{(M)} \mathbf{e}_i + f_M(t) \tag{20}$$

where $A_i^{(M)}$ is a coefficient at $\mathbf{e}_i$ after expanding $f(t)$ over M basis functions, and $f_M(t)$, the difference between the original function and its approximation by M terms, proves to be minimal. Let us define the projections $F_i \equiv \langle f_{i-1}, \mathbf{e}_i \rangle$ and $Q_{ij} \equiv \langle \mathbf{e}_i, \mathbf{e}_j \rangle$. The original algorithm by Šidlichovský and Nesvorný (1997) employs a certain normalized basis $\{\mathbf{e}\}$ where the latter scalar product (Q_{ij}) is always a real-valued function. We here expand their result to the case of an arbitrary non-normalized basis $\{\mathbf{e}\}$ where Q_{ij} can take complex values as well.

Thus coefficients $A_i^{(M)}$ are iteratively calculated as follows. At the first step

$$\alpha_{11} = \frac{1}{\sqrt{Q_{11}}}; \; A_1^{(1)} = \alpha_{11}^2 F_1; \; f_1(t) = f_0(t) - A_1^{(1)} \mathbf{e}_1 \tag{21}$$

where $f_0(t) \equiv f(t)$ and α_{ij} are hereafter some calculated complex constants.

At the mth step, for every $j = 1, 2, \ldots, m-1$ we compute the following complex coefficients:

$$B_j^{(m)} = -\sum_{s=1}^{j} \bar{\alpha}_{js} Q_{ms} \tag{22}$$

$$\alpha_{mm} = \left(Q_{mm} - \sum_{s=1}^{m-1} \bar{B}_s^{(m)} B_s^{(m)} \right)^{-1/2} \tag{23}$$

(by construction the coefficient α_{mm} take on a real value for any m)

$$\alpha_{mj} = \alpha_{mm} \sum_{s=j}^{m-1} B_s^{(m)} \alpha_{sj} \tag{24}$$

$$A_m^{(m)} = \alpha_{mm}^2 F_m \tag{25}$$

$$A_j^{(m)} = A_j^{(m-1)} + \alpha_{mm} \alpha_{mj} F_m \tag{26}$$

$$f_m(t) = f_{m-1}(t) - \alpha_{mm} F_m \sum_{i=1}^{m} \alpha_{mi} \mathbf{e}_i \tag{27}$$

where $\bar{\alpha}_{js}$ and $\bar{B}_s^{(m)}$ are complex conjugate values of the relevant quantities.

For the selected basis of Eq. (16), the projections $F_i \equiv \langle f_{i-1}, \mathbf{e}_i \rangle$ are numerically calculated according to Eqs. (17) and (18). The values for scalar products of the basis functions $Q_{ij} \equiv \langle \mathbf{e}_i, \mathbf{e}_j \rangle$ can be found analytically through the following steps.

Step 1. As far as trigonometric functions can be represented in exponential form, we shall further deal with definite integrals of the form

$$I_n(v) \equiv \frac{1}{2T} \int_{-T}^{T} t^n e^{ivt} \left(1 + \cos\frac{\pi}{T} t\right) \mathrm{d}t = I_n^a(v) + I_n^b(v) + I_n^c(v) \tag{28}$$

where $i \equiv \sqrt{-1}$ and

$$I_n^a(v) \equiv \frac{1}{2T} \int_{-T}^{T} t^n e^{ivt} \, \mathrm{d}t \tag{29}$$

$$I_n^b(v) \equiv \frac{1}{4T} \int_{-T}^{T} t^n e^{i(v+\frac{\pi}{T})t} \, \mathrm{d}t \tag{30}$$

$$I_n^c(v) \equiv \frac{1}{4T} \int_{-T}^{T} t^n e^{i(v-\frac{\pi}{T})t} \, \mathrm{d}t \tag{31}$$

It is easy to find

$$I_n^a(0) = 2I_n^b\left(-\frac{\pi}{T}\right) = 2I_n^c\left(\frac{\pi}{T}\right) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{T^n}{n+1} & \text{if } n \text{ is even} \end{cases} \tag{32}$$

Otherwise, if $n = 0$

$$I_0^a(v) = \frac{\sin(vT)}{vT} \tag{33}$$

$$I_0^b(v) = -\frac{1}{2} \frac{\sin(vT)}{vT + \pi} \tag{34}$$

$$I_0^c(v) = -\frac{1}{2} \frac{\sin(vT)}{vT - \pi} \tag{35}$$

If $n \geq 1$ we calculate the integrals iteratively

$$I_n^a(v) = \frac{i}{vT} \left(TnI_{n-1}^a - T^n \psi(v) \right) \tag{36}$$

$$I_n^b(v) = \frac{i}{vT + \pi} \left(TnI_{n-1}^b + \frac{1}{2} T^n \psi(v) \right) \tag{37}$$

$$I_n^c(v) = \frac{i}{vT - \pi} \left(TnI_{n-1}^c + \frac{1}{2} T^n \psi(v) \right) \tag{38}$$

where

$$\psi(v) = \begin{cases} \cos vT & \text{if } n \text{ is odd} \\ i \sin vT & \text{if } n \text{ is even} \end{cases} \tag{39}$$

Step 2. We partially expand the exponential function of the argument of Eq. (15) to a power series of t by assuming smallness of the second and further items in the right-hand side of Eq. (15) with respect to the first term. This task can easily be performed by means of a computer algebra system (we have used the MAPLE V program package by Waterloo Maple Software). The result is as follows (where the maximal degree of the polynomial argument, q, has been restricted to a value of 4)

$$\mathrm{e}^{i\left(vt + v_2 t^2 + v_3 t^3 + v_4 t^4\right)} = \mathrm{e}^{ivt} \left[1 + iv_2 t^2 + iv_3 t^3. + \left(iv_4 - \frac{1}{2} v_2^2 \right) t^4 - v_2 v_3 t^5 + \cdots \right] \tag{40}$$

In our study we obtain an expansion [Eq. (40)] up to terms proportional to t^{24}. Then we define a new integral function of a polynomial argument $\omega(t)$ as

$$J_n(\omega) \equiv \frac{1}{2T}\int_{-T}^{T} t^n e^{i\left(vt+v_2t^2+v_3t^3+v_4t^4\right)}\left(1+\cos\frac{\pi}{T}t\right)\mathrm{d}t \qquad (41)$$

By combining Eqs. (28) and (40) we can write

$$J_n(\omega) = I_n(v) + iv_2 I_{n+2}(v) + iv_3 I_{n+3}(v) + \left(iv_4 - \frac{1}{2}v_2^2\right)I_{n+4}(v) - v_2v_3 I_{n+5}(v) + \cdots \qquad (42)$$

Step 3. We expand trigonometric functions to exponential ones, and then the required scalar products can be calculated as

$$\begin{aligned} &< \mathbf{c}_{k_1l_1}, \mathbf{s}_{k_2l_2} > \\ &= \frac{1}{2T}\int_{-T}^{T} t^{l_1+l_2}\left(\frac{e^{i\omega_{k_1}} + e^{-i\omega_{k_1}}}{2}\right) \\ &\quad \times \left(\frac{e^{i\omega_{k_2}} - e^{-i\omega_{k_2}}}{2i}\right)\left(1+\cos\frac{\pi}{T}t\right)\mathrm{d}t \\ &= \frac{1}{4i}\Big\{ - J_{l_1+l_2}(\omega_{k_1} - \omega_{k_2}) - J_{l_1+l_2}(-\omega_{k_1} - \omega_{k_2}) \\ &\quad + J_{l_1+l_2}(\omega_{k_1} + \omega_{k_2}) + J_{l_1+l_2}(-\omega_{k_1} + \omega_{k_2})\Big\} \end{aligned} \qquad (43)$$

and analogously

$$\begin{aligned} &< \mathbf{s}_{k_1l_1}, \mathbf{c}_{k_2l_2} > \\ &= \frac{1}{4i}\Big\{ J_{l_1+l_2}(\omega_{k_1} - \omega_{k_2}) - J_{l_1+l_2}(-\omega_{k_1} - \omega_{k_2}) \\ &\quad + J_{l_1+l_2}(\omega_{k_1} + \omega_{k_2}) - J_{l_1+l_2}(-\omega_{k_1} + \omega_{k_2})\Big\} \end{aligned} \qquad (44)$$

$$\begin{aligned} &< \mathbf{s}_{k_1l_1}, \mathbf{s}_{k_2l_2} > \\ &= \frac{1}{4}\Big\{ J_{l_1+l_2}(\omega_{k_1} - \omega_{k_2}) - J_{l_1+l_2}(-\omega_{k_1} - \omega_{k_2}) \\ &\quad - J_{l_1+l_2}(\omega_{k_1} + \omega_{k_2}) + J_{l_1+l_2}(-\omega_{k_1} + \omega_{k_2})\Big\} \end{aligned} \qquad (45)$$

$$\begin{aligned} &< \mathbf{c}_{k_1l_1}, \mathbf{c}_{k_2l_2} > \\ &= \frac{1}{4}\Big\{ J_{l_1+l_2}(\omega_{k_1} - \omega_{k_2}) + J_{l_1+l_2}(-\omega_{k_1} - \omega_{k_2}) \\ &\quad + J_{l_1+l_2}(\omega_{k_1} + \omega_{k_2}) + J_{l_1+l_2}(-\omega_{k_1} + \omega_{k_2})\Big\} \end{aligned} \qquad (46)$$

This is the technique of spectral analysis allowing expansion of an arbitrary tabulated function of the Moon/Sun/planet coordinates to Poisson series, with the arguments being high-degree polynomials of time.

4 Harmonic development of the TGP by the new technique

The formalism described in the previous section has been applied to accurate expansion of the Earth TGP over 2000 years, 1000–3000. At every six hours within that interval of time we calculated numerical values for the coefficients C_{nm} and S_{nm} of the TGP expansion according to Eqs. (3), (4), and (10)–(12). The latest JPL long-term ephemeris DE/LE-406 was employed as a source of the Moon, Sun and planet coordinates. Note that positions of the attracting bodies are calculated by us not in a rapidly rotating Earth-fixed coordinate system [as is done, for example, in the study of Hartmann and Wenzel (1995)] but in a coordinate system that rotates much more slowly with time (see Fig. 1). This allows separation of high-frequency terms in the TGP expansion due to the Earth's rotation from the terms just caused by the Moon, Sun and planet motion. It has been proven that the sampling step of six hours' duration is small enough for detecting all essential waves in the spectrum of C_{nm} and S_{nm}.

When calculating the coefficients we used values for the planetary gravitational parameters from Standish (1998a) and values for $\bar{J}_2$ and R_E from the IERS Conventions (McCarthy and Petit 2003). [The value for the latter constant, which further has to be used in Eqs. (2) and (9) along with expansions of the coefficients, is 6378136.3 m.]

The arguments of Eq. (15) in the expansion of Eq. (14) were selected as follows. From Simon et al. (1994) we took complete fourth-order polynomial expressions for mean longitude of the ascending node of the Moon Ω, for Delaunay variables D, l', l, and F (mean elongation of the Moon from the Sun, mean anomaly of the Sun, mean anomaly of the Moon, and mean longitude of the Moon subtracted by Ω, respectively), and for mean longitudes of Venus, Jupiter, Mars, Saturn, and Mercury. [As Hartmann and Wenzel (1994) showed, the attraction of other planets is negligible when calculating the TGP.] The set of arguments of the Moon, Sun and planet motion referred to the mean ecliptic and equinox of date was chosen using the latest value of the precession constant. Then we preliminarily evaluated a spectrum of the tabulated numerical values of C_{nm} and S_{nm} at numerous combinations of multipliers of the arguments' frequencies. For that a classical Fourier analysis (fast Fourier transform, FFT) of the data arrays at frequencies specially defined by the FFT was first made. Then the approximate amplitude of a spectrum's wave at every combination of multipliers of the frequencies (cut to linear functions at this stage) was found through interpolating the results of FFT. After that, for all waves in the C_{nm} and S_{nm} spectrum which had the preliminary amplitude exceeding or equal to a pre-set minimal level ($10^{-8}\,\mathrm{m}^2\,\mathrm{s}^{-2}$ in our study) we carried out the improved harmonic analysis of the original data arrays by using Eqs. (20)–(46) in order to account for the high-degree polynomial form [Eq. (15)] of the arguments (where q is equal to 4 as the maxi-

Table 1. Number of terms in expansion of the Earth TGP coefficients C_{nm}/S_{nm}

n	m						
	0	1	2	3	4	5	6
1	111/-	107/103					
2	3181/-	3293/3689	3862/3632				
3	780/-	791/819	937/884	862/899			
4	233/-	242/215	288/259	276/296	238/230		
5	36/-	38/39	48/44	50/52	53/52	45/45	
6	1/-	1/1	2/1	2/2	2/2	3/3	2/2

mum degree of polynomials for Delaunay variables). The maximal order h of the polynomials representing amplitudes in Poisson series [Eq. (14)] for the Earth TGP is equal to 2. Numerical tests proved that further increasing the order h does not improve the solution quality at a limited interval of time, 1000–3000. Thus, the amplitudes of coefficients in our development are not constants, but slowly changing variables described by second-order time polynomials. Such a form of representation of terms' amplitudes and arguments is very similar to that used in modern analytical models of lunar/planetary motion and nutation.

The final expansions for coefficients C_{nm} and S_{nm} include waves of amplitude not less than $10^{-8}\,\mathrm{m}^2\,\mathrm{s}^{-2}$ only. The number N of such terms in the development of Eq. (14) for every coefficient C_{nm} and S_{nm} of degree n and order m is presented in Table 1. The total number of waves included in the final spectrum of the TGP, named KSM03, is the sum of all N and equal to 26 753.

Convergence of the series for every coefficient was checked through evaluation of the maximum and RMS values for approximation error $f_M(t)$ in Eq. (20) over the entire time interval, 1000–3000. At every increment in number of included waves the error was calculated as the difference between the coefficient value $f(t)$ (at every tabulated epoch) and its approximation given by the current series of M terms. Table 2 shows the final maximum errors of the harmonic development of coefficients C_{nm} and S_{nm}.

The major terms of the expansion are presented in Table 3.

The complete set of coefficients of KSM03 development of the Earth TGP can be found at http://lnfm1.sai.msu.ru/neb/images/ksm/tgp/coeff.zip. (The description of the data format is done in file http://lnfm1.sai.msu.ru/neb/images/ksm/tgp/readme.pdf)

5 Comparisons and tests of KSM03

We made a comparison between the KSM03 series and the most recent developments of the Earth TGP: HW95 (Hartmann and Wenzel 1995) and RATGP95 (Roosbeek 1996). It should be remembered that, unlike the latter expansions, wave frequencies of KSM03 series are done not in an Earth-fixed reference frame but in a celestial reference frame defined by the true geoequator of date and the projection of the mean equinox of date as the origin of right ascensions (Fig. 1). The major advantage of such a choice for the series format is that it explicitly separates two different time arguments to be used in calculating the wave frequencies in an Earth-fixed frame. One of these is T_{eph}, the time argument for JPL planetary/lunar ephemerides (Standish 1998b), which is close to Barycentric Dynamical Time (TDB). It is used when calculating the components of the wave frequencies caused by the Moon, Sun and planet motion, and is exactly the time argument in KSM03 series for coefficients $C_{nm}(t)$ and $S_{nm}(t)$ in Eqs. (3), (4), and (10–12). The second time argument is UT1, which is used for calculating another component of the final frequencies—the sidereal time (or Earth rotation) rate (Aoki et al. 1982). In our expansion of the TGP the latter time argument is used in Eq. (5) only. The UT1 time scale is much less stable than T_{eph}, and UT1 time is difficult to predict for long to sufficient accuracy—so it is preferable when coefficients $C_{nm}(t)$ and $S_{nm}(t)$ are both calculated and expanded without any use of that time argument. When calculating the TGP and gravity tide at a certain epoch the relevant value for UT1 can be taken from, for example, IERS publications (or determined from observations) and used in Eq. (5) for calculating just terms $m\theta^{(A)}$ in Eq. (2) and (9).

However, for the purpose of comparison with the previous TGP developments in the frequency domain the major waves from KSM03 expansion were re-calculated to an Earth-fixed reference frame. (The Earth rotation rate was assumed to be uniform.) Table 4 gives values for the amplitudes of the principal waves obtained in HW95, RATGP95, and KSM03 developments. The wave frequencies shown in the table are linear combinations of frequencies of Doodson variables: τ mean local lunar time; s mean lunar longitude; h mean solar longitude; p mean longitude of lunar perigee; N' negative mean longitude of the lunar ascending node; and p_s mean longitude of solar perigee.

Table 2. Maximum approximation error for C_{nm}/S_{nm} coefficients over the time interval 1000–3000 [$\mathrm{m}^2\,\mathrm{s}^{-2}\times 10^5$]

n	m						
	0	1	2	3	4	5	6
1	0.027/-	0.030/0.034					
2	1.041/-	0.936/1.155	1.437/1.219				
3	0.145/-	0.192/0.177	0.193/0.192	0.216/0.191			
4	0.044/-	0.059/0.048	0.055/0.059	0.070 /0.064	0.060/0.060		
5	0.015/-	0.018/0.016	0.016 /0.015	0.018/0.017	0.020/0.021	0.016/0.017	
6	0.003/-	0.004/0.004	0.003/0.004	0.004/0.004	0.004/0.004	0.003/0.004	0.004/0.004

Table 3. Amplitudes of the major terms in KSM03 expansion of the Earth TGP (m^2/s^{-2})

Coefficient	Multipliers of					Period (days)	A_0^c	$A_1^c \times 10^3$ years	$A_2^c \times (10^3$ years$)^2$	A_0^s	$A_1^s \times 10^3$ years	$A_2^s \times (10^3$ years$)^2$
	Ω	D	l'	l	F							
C_{22}	2	0	0	0	2	13.7	0.2861	0.0003	0.0005	−1.2039	−0.0011	0.0001
S_{22}	2	0	0	0	2	13.7	1.1994	0.0012	−0.0001	0.2850	0.0003	0.0005
C_{20}	0	0	0	0	0	–	−0.8696	−0.0028	0	0.	0.	0
S_{21}	0	0	0	0	0	–	0.7206	−0.0031	0	0.	0.	0
C_{22}	2	−2	0	0	2	182.6	−0.5377	−0.0005	0	0.2057	0.0002	0.0001
S_{22}	2	−2	0	0	2	182.6	−0.2049	−0.0002	−0.0001	−0.5358	−0.0005	0.
C_{21}	2	0	0	0	2	13.7	0.5200	−0.0027	−0.0001	0.1235	−0.0006	0.0002
S_{21}	2	0	0	0	2	13.7	−0.1133	0.0005	−0.0002	0.4771	−0.0020	−0.0001
C_{21}	2	−2	0	0	2	182.6	−0.0888	0.0005	0	−0.2323	0.0012	0
C_{22}	2	0	0	1	2	9.1	−0.2018	−0.0002	−0.0001	0.1241	0.0001	−0.0001
S_{22}	2	0	0	1	2	9.1	−0.1237	−0.0001	0.0001	−0.2010	−0.0002	−0.0001
S_{21}	2	−2	0	0	2	182.6	0.2134	−0.0009	−0.0001	−0.0815	0.0005	0
C_{20}	2	0	0	0	2	13.7	−0.0426	0.0004	−0.0001	0.1791	−0.0019	0
C_{22}	0	0	0	0	0	–	0.1562	−0.0016	0.	0	0.	0.
C_{21}	1	0	0	0	0	−6798.4	−0.0917	−0.0001	0	−0.0643	0.0001	0
C_{21}	1	0	0	0	2	13.6	−0.0829	−0.0001	0	0.0735	0.0001	0
C_{21}	2	0	0	1	2	9.1	−0.0536	0.0003	0.0001	−0.0872	0.0005	0
C_{20}	0	0	0	1	0	27.6	0.0687	0.0002	0	0.0688	0.0002	0
S_{21}	2	0	0	1	2	9.1	0.0800	−0.0003	0	−0.0499	0.0002	0.0001
C_{20}	2	−2	0	0	2	182.6	0.0797	−0.0008	0.0001	−0.0306	0.0001	0

Figure 2 shows differences between amplitudes of the major waves in the three developments of the TGP.

As our analysis shows, the main differences in amplitudes of the major waves in HW95, RATGP95, and KSM03 arise from their either aliasing with or separating from other small waves of very close frequencies (differing by $\pm 2\dot{p}_s$). However, if we merge the waves of such close frequencies, the amplitudes of combined waves will be almost identical in the three developments under comparison. Dehant and Bretagnon (1998) also strongly recommend merging the terms of frequencies differing by $\pm 2\dot{p}_s$ in the Earth TGP developments and prove this procedure only leads to a polynomial form of the combined terms' amplitudes. (This is what KSM03 development has: the terms of such close frequencies are merged and the terms' amplitudes are second-degree time polynomials). Aliasing or separating terms of frequency p_s does not change the result of calculation of the Earth tides (Dehant 1997), although it can be important in certain other applications, where a preference can be given to an analytical solution, such as RATGP95. At the same time, the latter does not include many of the terms of small amplitudes presented in both HW95 and KSM03.

In the time domain the accuracy of KSM03 expansion of the Earth TGP has been checked by computation of the gravity tide values [Eq. (13)] at a mid-latitude station. For this we choose the Black Forest Observatory (BFO) Schiltach ($r = 6\,366\,836.9$ m, $\varphi = 48.3306°$N, $\lambda = 8.3300°$E) at which Hartmann and Wenzel (1995) and Roosbeek (1996) also computed the tidal gravity by using their expansions of the TGP. First, we calculated the total tidal gravity at that station by means of strict expressions of Eqs. (2)–(4) and (9)–(13) where the Moon, Sun and planet spherical coordinates were computed using the most precise JPL ephemeris DE/LE-405. The gravity tides at BFO were calculated at every hour within the whole time span covered by that ephemeris, 1600–2200. Then we calculated the gravity tides at the same point and at the same set of epochs by using KSM03 expansion of the TGP and compared the results with the exact values. The maximal deviation between the two sets of data at any epoch within the whole time span of six hundred years' length does not exceed 0.39 nGal (1 nGal$= 10^{-11}$ m s^{-2}). The corresponding RMS difference between the data over the same interval is less than 0.025 nGal.

Figure 3 shows how the accuracy of calculation of the gravity tide at BFO depends on the number of terms taken from KSM03 series. It is also interesting to estimate how much KSM03 series can be truncated in order to ensure the accuracy of both RATGP95 and HW95 solution (approximately, because KSM03 series have a different format). Thus, the maximum residual of 1.23 nGal (the accuracy of HW95, which includes 12 935 terms) is reached when taking 12 770 terms from KSM03, and the maximum residual of 5 nGal (obtained in RATGP95, 6499 terms) is ensured by some 5800 terms from KSM03. (Let us remember that residuals in KSM03 solution are estimated over the period 1600–2200, while those in HW95 and RATGP95 are done over 1850–2150 and 1987–1993, respectively.)

6 Conclusions

1. A new method of harmonic development of an arbitrary function of the Moon/Sun/planet coordinates tabulated over a long interval of time is proposed. Unlike classical Fourier analysis, it allows us to obtain the expansion directly as Poisson series with terms of non-linear amplitudes and frequencies

Table 4. Comparison of the principal waves in different TGPs in Earth-fixed reference frame

Wave name	Doodson no.	Multipliers of						Frequency (degree/hour)	Wave amplitude (m^2/s^{-2}) as given by		
		τ	s	h	p	N'	p_s		HW95	RATGP95	KSM03
MS_0	055, 555	0	0	0	0	0	0	0.00000000	0.8695488	0.8695776	0.8695547
	055, 565	0	0	0	0	1	0	0.00220641	0.0771964	0.0771945	0.0771994
S_a	056, 554	0	0	1	0	0	−1	0.04106668	0.0136032	0.0136030	0.0142511[a]
S_{sa}	057, 555	0	0	2	0	0	0	0.08213728	0.0856534	0.0856663	0.0853588[b]
M_{sm}	063, 655	0	1	−2	1	0	0	0.47152105	0.0185961	0.0185992	0.0185984
M_m	065, 455	0	1	0	−1	0	0	0.54437471	0.0972501	0.0972513	0.0972478
	065, 555[c]	0	1	0	0	0	0	0.54901652	0.0103721	0.0103720	0.0103724
M_{sf}	073, 555	0	2	−2	0	0	0	1.01589576	0.0161328	0.0161356	0.0161293
M_f	075, 555	0	2	0	0	0	0	1.09803304	0.1841040	0.1840650	0.1840904
	075, 565	0	2	0	0	1	0	1.10023945	0.0763324	0.0763260	0.0763297
M_{tm}	085, 455	0	3	0	−1	0	0	1.64240775	0.0352501	0.0352416	0.0352463
	085, 465	0	3	0	−1	1	0	1.64461415	0.0146096	0.0146100	0.0146104
$2Q_1$	125, 755	1	−3	0	2	0	0	12.85428619	0.0129857	0.0129809	0.0129816
σ_1	127, 555	1	−3	2	0	0	0	12.92713984	0.0156593	0.0156655	0.0156662
	135, 645	1	−2	0	1	−1	0	13.39645449	0.0185064	0.0185024	0.0185025
Q_1	135, 655	1	−2	0	1	0	0	13.39866089	0.0981306	0.0980967	0.0981023
ρ_1	137, 455	1	−2	2	−1	0	0	13.47151455	0.0186261	0.0186319	0.0186334
	145, 545	1	−1	0	0	−1	0	13.94082919	0.0966889	0.0966709	0.0966697
O_1	145, 555	1	−1	0	0	0	0	13.94303560	0.5125257	0.5123571	0.5123859
	155, 455	1	0	0	−1	0	0	14.48741031	0.0144896	0.0144840	0.0144888
M_1	155, 655	1	0	0	1	0	0	14.49669393	0.0402872	0.0402931	0.0402965
π_1	162, 556	1	1	−3	0	0	1	14.91786468	0.0139377	0.0139380	0.0139413
P_1	163, 555	1	1	−2	0	0	0	14.95893136	0.2384361	0.2384226	0.2385504[d]
	165, 545	1	1	0	0	−1	0	15.03886223	0.0142682	0.0142594	0.0142694
K_1	165, 555	1	1	0	0	0	0	15.04106864	0.7205113[e]	0.7206175	0.7206445
	165, 565	1	1	0	0	1	0	15.04327505	0.0977846	0.0977803	0.0977636
ϕ_1	167, 555	1	1	2	0	0	0	15.12320592	0.0102599	0.0102607	0.0101341[f]
J_1	175, 455	1	2	0	−1	0	0	15.58544335	0.0403017	0.0402939	0.0402959
OO_1	185, 555	1	3	0	0	0	0	16.13910168	0.0220445	0.0220424	0.0220476
	185, 565	1	3	0	0	1	0	16.14130809	0.0141252	0.0141215	0.0141231
$2N_2$	235, 755	2	−2	0	2	0	0	27.89535483	0.0313070[g]	0.0312918	0.0312918
μ_2	237, 555	2	−2	2	0	0	0	27.96820848	0.0377851[h]	0.0377628	0.0377628
N_2	245, 655	2	−1	0	1	0	0	28.43972953	0.2365822[I]	0.2364763	0.2364737
v_2	247, 455	2	−1	2	−1	0	0	28.51258319	0.0449405[j]	0.0449157	0.0449156
	255, 545	2	0	0	0	−1	0	28.98189783	0.0461038[k]	0.0460873	0.0460814
M_2	255, 555	2	0	0	0	0	0	28.98410424	1.2356349[l]	1.2351162	1.2351037
L_2	265, 455	2	1	0	−1	0	0	29.52847895	0.0349289	0.0349159	0.0349140
T_2	272, 556	2	2	−3	0	0	1	29.95893332	0.0336007	0.0335945	0.0335941
S_2	273, 555	2	2	−2	0	0	0	30.00000000	0.5748299[m]	0.5746685	0.5746403
K_2	275, 555	2	2	0	0	0	0	30.08213728	0.1561924	0.1561929	0.1562081
	275, 565	2	2	0	0	1	0	30.08434369	0.0465509	0.0465733	0.0465689
M_3	355, 555[c]	3	0	0	0	0	0	43.47615636	0.0149687[n]	0.0149591	0.0149588

[a]Result of aliasing the wave with another wave of close frequency 056, 556 ($+2p_s$)
[b]Result of aliasing the wave with another wave of close frequency 057, 553 ($-2p_s$)
[c]For this wave the degree n of the spherical harmonic development is 3
[d]Result of aliasing the wave with another wave of close frequency 163, 557 ($+2p_s$)
[e]Effect of separating the wave from another wave of close frequency 165, 553 ($-2p_s$)
[f]Result of aliasing the wave with another wave of close frequency 167, 553 ($-2p_s$)
[g]Effect of separating the wave from another wave of close frequency 235, 757 ($+2p_s$)
[h]Effect of separating the wave from another wave of close frequency 237, 557 ($+2p_s$)
[i]Effect of separating the wave from another wave of close frequency 245, 657 ($+2p_s$)
[j]Effect of separating the wave from another wave of close frequency 247, 457 ($+2p_s$)
[k]Effect of separating the wave from another wave of close frequency 255, 547 ($+2p_s$)
[l]Effect of separating the wave from another wave of close frequency 255, 557 ($+2p_s$)
[m]Effect of separating the wave from another wave of close frequency 273, 557 ($+2p_s$)
[n]Effect of separating the wave from another wave of close frequency 355, 557 ($+2p_s$)

(i.e. in the form that the modern analytical theories of lunar/planetary motion and nutation models have).

2. The method is applied to produce a new harmonic development of the Earth TGP over the period 1000–3000 (KSM03 solution), including 26 753 terms of amplitudes down to the level of $1 \times 10^{-8}\ m^2\ s^{-2}$.
3. The accuracy of KSM03 in the frequency domain is close to that of HW95 and RATGP95. Having merged the terms of the close frequencies (differing by $\pm\ 2\dot{p}_s$), we obtain amplitudes of combined waves that

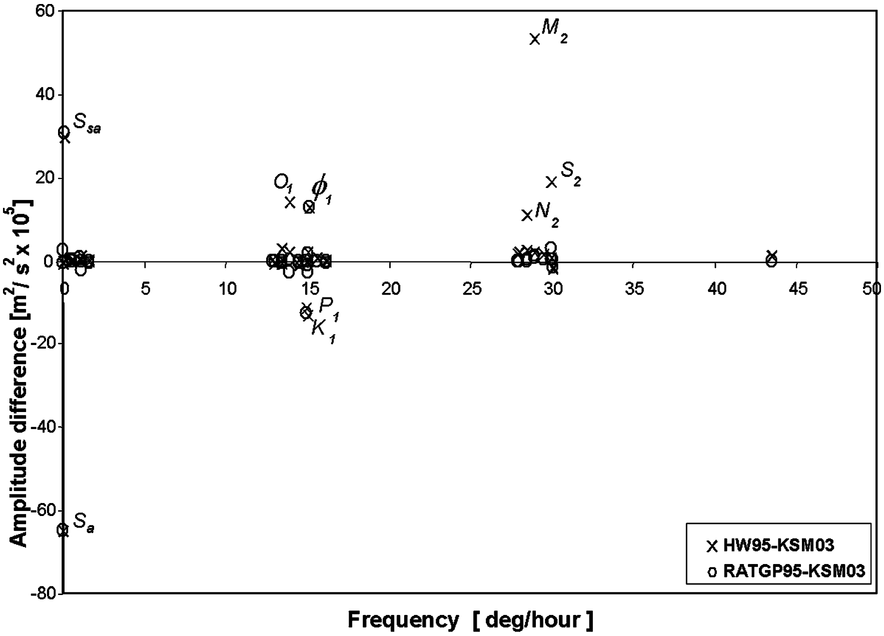

Fig. 2. Differences between amplitudes of major waves in HW95/RATGP95 and KSM03

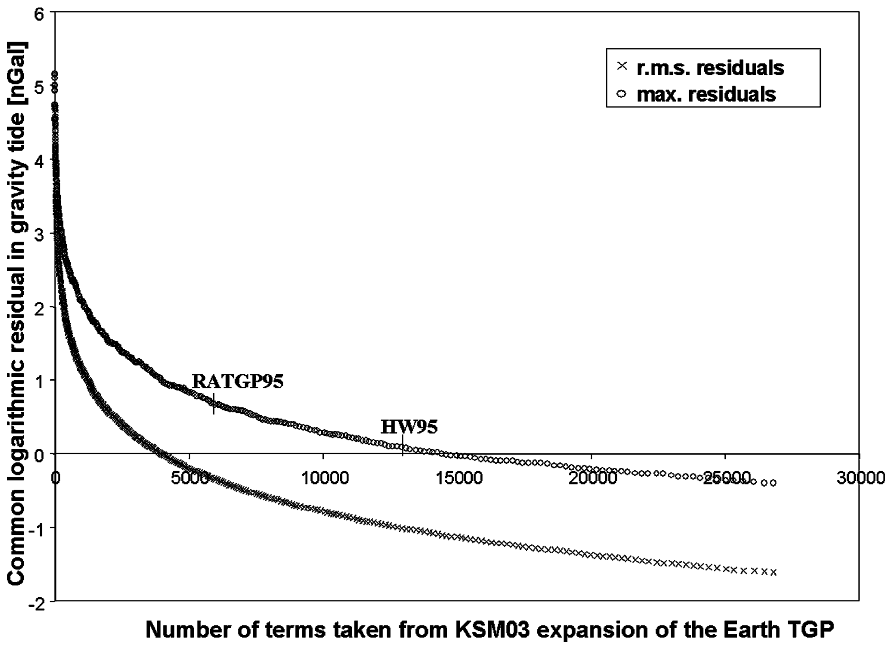

Fig. 3. Residuals between gravity tides calculated by KSM03 expansion of the Earth TGP and those obtained from DE/LE 405 ephemerides (at BFO station over time interval 1600–2200)

are almost identical in all three developments. Present analytical expansions of the TGP (like RATGP95) do not include many of the waves of small amplitudes presented in both HW95 and KSM03.

4. The accuracy of KSM03 in the time domain is estimated to be 0.025/0.39 nGal (the RMS/maximum error) when calculating the gravity tides at a mid-latitude station over 600 years, 1600–2200. It is better, in the time domain, than the respective accuracy of any TGP development made previously by a factor of at least 3, and is valid over a twice longer interval of time.

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