Comparison between the Transfer Functions
of three Superconducting Gravimeters

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Abstract

The transfer functions of the superconducting gravimeters OSG-CT40, SG-C021 and OSG-050 operating in Walferdange (Luxembourg), Membach (Belgium), and Pecný (Czech Republic), respectively, have been experimentally determined by injecting known voltages into the feedback loop of the control electronics. The transfer function is expressed in terms of either its Laplace transforms or by zeros and poles. The latter is widely used in seismology. In particular in the high frequency seismic band, the full transfer function of the Superconducting Gravimeter (SG) is required for data analysis. The results for these three SGs are different enough that the transfer function cannot be calculated theoretically or assumed to be the same for all the SGs. An accurate and precise determination has to be performed for each SG.

1. Introduction

In geophysics, Superconducting Gravimeters (SGs) are used to continuously monitor relative gravity changes. They are the most precise instruments to study of solid earth tides: for instance, it is possible to measure tidal amplitudes in the diurnal and semi-diurnal bands with a precision of about 0.1-0.2 nm/s^2 for integration periods of 2-3 years. Their instrumental drift is extremely low (typically around 10 nm/s^2 per year) and smooth [*Van Camp and Francis, 2007*]. SGs observations are used to monitor the ocean loading effects, to validate the global ocean tides models, to record gravity changes due to the atmosphere, as air mass redistribution and pressure changes related to meteorological events [*Goodkind, 1999*], and to monitor the water storage changes [*Goodkind, 1999, Creutzfeld et al., 2010*]. At higher frequencies, SGs record normal modes of the Earth excited after big Earthquakes [*van Camp, 1999]*.

Some of these applications require a precise determination of the instrumental drift of the SGs. Simultaneous measurements of the SGs side-by-side with an absolute gravimeter has been proved to be very efficient not only to estimate SGs long term drift but also to calibrate the relative SGs. In addition, bad AG values due to malfunctioning of the absolute gravimeter can be detected from regular comparisons with the continuous SG time series.

The knowledge of the transfer function of SGs is essential to fully exploit their observations [*Van Camp, 1998; Van camp et al., 2000*]. Besides analysis of their observations, the transfer functions play an important role when comparing or combining data sets from multiple SGs or other instrumentations as absolute gravimeters or seismometers. To reach optimal performance of SGs in tidal research, where SGs can be considered as world most sensitive instruments, those instruments should be calibrated with an accuracy of 0.1% in amplitude and 0.01 second in phase [*Hinderer et al., 1991, Baker and Bos, 2003*]. Van Camp et al. [2000] were the first to determine experimentally the transfer function of a SG. Step and sine waves voltages are injected into the feedback circuit of the control electronics of the gravimeter and the system response is recorded.

This method was applied to determine transfer function of the cryogenic gravimeter SG-C021 operating in Membach (Belgium), in a series of experiences (1996-2005) for different outputs (depending on the analog filter) and different data acquisition systems. A precision better

than 0.01 second in the phase response (time lag) was obtained. In 2007, the same method was applied to determine the transfer function of the OSG-CT40 operating in Walferdange (Luxembourg), as well as of the OSG-050 operating in Pecný (Czech Republic) (OSG meaning Observatory Superconducting Gravimeter).

In this paper, the calibration experiment carried out in Walferdange is described. The transfer functions obtained for the three gravimeters are then compared. They are represented in terms of Laplace transforms [*Scherbaum, 2001*]. In seismology, this is the standard formulation used by the seismic Incorporated Research Institutions in Seismology (IRIS)) data base [*http://www.iris.edu*], where some SGs data are archived. In such a Data Base, the information on the transfer function (being considered as important as the observations themselves) is mandatory.

2. Functioning principle of the superconducting gravimeter

In superconducting gravimeters, a hollow superconducting niobium sphere is in equilibrium under the combined action of the gravity force on the sphere and a vertical upward directed levitation force. This force is provided by the magnetic field generated by a pair of superconducting niobium coils with persistent current [*Goodkind, 1999*]. Two coils - their configuration respect to the sphere and the ratios of currents in the coils - allow one to independently adjusting the total levitating force and the force gradient in such a way that a small change in gravity can induce a large variation in the sphere vertical position. This variation is detected by an electrostatic device (a capacitance bridge constituted by three capacitor plates and the levitating sphere) and a feedback magnetic force (generated by a feedback coil) brings the sphere back to its initial position. The feedback integrator voltage is linearly proportional to changes in the acceleration of gravity. To allow the stable levitation to occur, the gravity sensor (Figure 1) must be maintained in a condition of superconductivity (niobium is superconducting below 9.3 K). This is realized by placing the gravity sensor inside a Dewar filled with liquid helium (4.2 K boiling point).
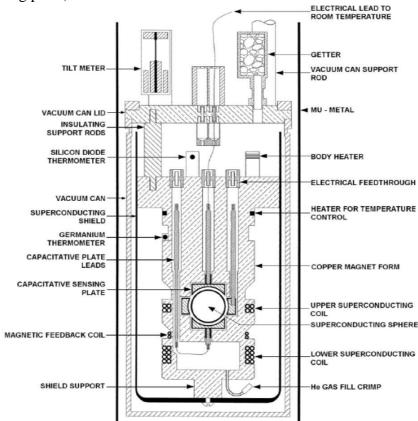

Figure 1. Gravity sensor unit of the Superconducting Gravimeter (Figure from the GWR Manual).

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The unique characteristics of SGs lie in the continuity of the gravity signal registration, the linearity and stability of feedback system, the very high sensitivity (5 (nm/s²)²/Hz corresponding to a precision of 0.2 nm/s² (or 0.02 µGal) at a period of 100 s [*Van camp et al., 2005; Rosat et al, 2009*] and a low instrumental drift of a few µGal/year [*Van Camp and Francis*, 2007].

The instrument calibration can be obtained by fitting the SG observed signal to known signals (i.e. Earth tides or modeled inertial effects). It can be further improved by comparing the SG observations to simultaneous absolute gravity measurements in a nearby location. Francis et al. (1998) showed that accuracy on the amplitude calibration factor of 0.1% can be achieved within 4 days of observations during high tides. However, this method does not provide a reliable phase calibration. This latter requires another type of experiment to determine the transfer function.

The feedback voltage is the output signal from the SG gravity control card (Figure 2, upper part). On the card, an analog low-pass filter is provided as an anti-aliasing filter for digitizing the gravity signal. The card and the filter significantly affect the transfer function of SGs.

For the SG-C021, the feedback voltage was provided with the use of three different cards [*Van Camp et al. 2000, Van Camp et al. 2008*]. Until 1997, the electronics was provided with a 6-pole Butterworth tide filter and a card with a 2-pole Butterworth Gravity Signal (GS) with cutoff periods at 72 second and 1 second, respectively. Since 1997, in order to fulfill the Global Geodynamics Project (GGP) requirements [*Crossley et al., 1999*] and to improve the quality of the electronics with up to date components, a card with an 8-pole Butterworth low-pass filter (GGP-1) and cutoff period at 16 second replaced the old version.

Currently, most of the superconducting gravimeters are equipped with a GGP-1 filter or, alternatively, a GGP-2 filter, having cutoff period at 32 second.

The transfer function determinations for the SG-C021 were also conducted for different data acquisition systems, i.e. K2000 voltmeters and a Quanterra 330 data logger. Van Camp et al. [2008] concluded that both the low-pass filter characteristics and the data acquisition system characteristics have an effect on the instrument response.
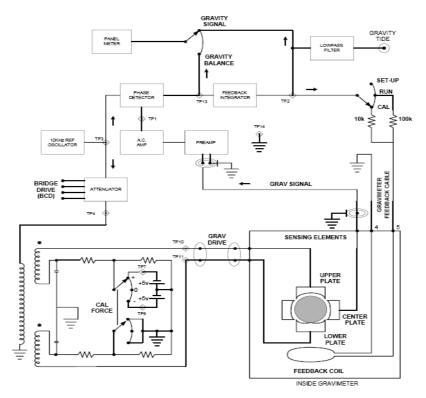

Figure 2. Scheme of the superconducting gravimeter control electronics. The gravity control card includes the feedback integrator and the low-pass filter (Figure from the GWR Manual).

3. The Laplace transform and the transfer function

The Laplace transform represents a powerful differential instrument for the analysis of Linear Time Invariant systems (LTI), such as electronic circuits [*Bertoni et al., 2003, Ambardar, 1995, Beerends et al., 2003*]. The Laplace operator acts on functions in the time domain, transforming them into functions in the frequency domain. The system input and output are functions of the complex angular frequency or Laplace variable, usually denoted *s*, expressed in radians per unit of time.

If *f(t)* represents a real function of time defined for positive values of the time variable *t*, the Laplace Transform of *f(t)* is defined as

$$\text{L}\left[f(t)\right] = F(s) = \lim_{\substack{T \to \infty \\ \varepsilon \to 0}} \int_{\varepsilon}^{T} f(t) \cdot e^{-st} dt = \int_{0+}^{\infty} f(t) \cdot e^{-st} dt \qquad\qquad 0 < \varepsilon < T \qquad\qquad (1)$$

where *s* is a complex variable defined by $s = \sigma + i \cdot \omega$.

The differential Laplace operator is a linear operator. The Laplace Transform of the time derivative of a function *f(t)* having *F(s)* as Laplace transform, is expressed as

$$\text{L}\frac{df(t)}{dt} = s \cdot F(s) - f(0^{+}) \qquad\qquad (2)$$

The Laplace Transform of the time integral of a function *f(t)* having *F(s)* as Laplace transform, is expressed as

$$\text{L}\left[\int_{0}^{t} f(\tau) d\tau\right] = \frac{F(s)}{s} \qquad\qquad (3)$$

Laplace transform provides solutions to the differential equations characterizing LTI systems, reducing them to more easily solvable algebraic relations.

4. Transfer function and frequency response for LTI systems

A transfer function (or network function) for a LTI system is a mathematical relationship (in the spatial or temporal frequency domain) between the model output and input [*Di Stefano et al., 2004*]. In the case of continuous input signal *x(t)* and output signal *y(t) in time domain*, the transfer function of a LTI system can be expressed as the ratio between the output Laplace transform Y(s) and the input Laplace transform X(s):

$$H(s) = \frac{Y(s)}{X(s)} \qquad\qquad (4)$$

where *X(s)*= L*[x(t)]* and *Y(s)*= L*[y(t)]*. The transfer function also corresponds to the Laplace transform of the system's impulse response.

For LTI systems, because of the previously underlined properties of the Laplace transform, the transfer function is generally represented by the ratio of two polynomials of the Laplace complex variable *s*:

$$H(s) = \frac{\sum\limits_{j=0}^{m} b_j s^j}{\sum\limits_{j=0}^{n} a_j s^j} \qquad\qquad (5)$$

The poles/zeros are defined as the values of s for which the denominator/numerator of the transfer function is equal to zero [*Scherbaum, 2001*]. In the time domain, each pole is associated with a response mode of the system. The impulse response of the system is a linear combination of the different response modes. Thus, the transfer function completely defines the system response.

If the input of a LTI system is a sinusoidal signal with frequency ω (rad/s), it can be represented in complex form:

$$x(t) = |X| \cdot e^{i(\omega t + \varphi_X)} = |X| \cdot e^{i\varphi_X} \cdot e^{i\omega t} = X \cdot e^{i\omega t} \qquad\qquad (6)$$

where $|X|$ is the input amplitude, φ_X the input phase and i represents the imaginary number. The corresponding system output is also a sinusoidal signal having the same frequency ω but generally a different phase and amplitude:

$$y(t) = |Y| \cdot e^{i(\omega t + \varphi_Y)} = |Y| \cdot e^{i\varphi_Y} \cdot e^{i\omega t} = Y \cdot e^{i\omega t} \qquad\qquad (7)$$

where $|Y|$ is the output amplitude and φ_Y the output phase. The amplitude frequency response represents the ratio between the output and input amplitudes as a function of the frequency ω, and is defined as the gain:

$$G(\omega) = \frac{|Y(\omega)|}{|X(\omega)|} \qquad\qquad (8)$$

The phase frequency response represents the difference between the output and input phases as a function of the frequency ω:

$$\phi(\omega) = \varphi_Y(\omega) - \varphi_X(\omega) \qquad\qquad (9)$$

For a discrete frequencies sample the frequency response in complex form is

$$R(\omega_n) = \frac{|Y_n|}{|X_n|} \cdot e^{i \cdot \phi_n} = G_n \cdot e^{i \cdot \phi_n} = \frac{Y_n}{X_n} \, , \qquad \textit{n=1: frequency sample length} \qquad (10)$$

where, for an input signal at frequency ω_n, G_n and $\varPhi_n$ represent the gain and the phase shift, respectively.

Least-squares fit algorithms allow one to determine the polynomial coefficients of the transfer function H_s (Eq. 5) from the experimental frequency response in complex form (Eq. 9), determined on a limited chosen frequencies. Conversely, the frequency response in complex form, and consequently the gain and phase lag, can be derived from the transfer function H_s where the variable s is replaced with the variable $(\omega \cdot i)$ for positive values of ω.
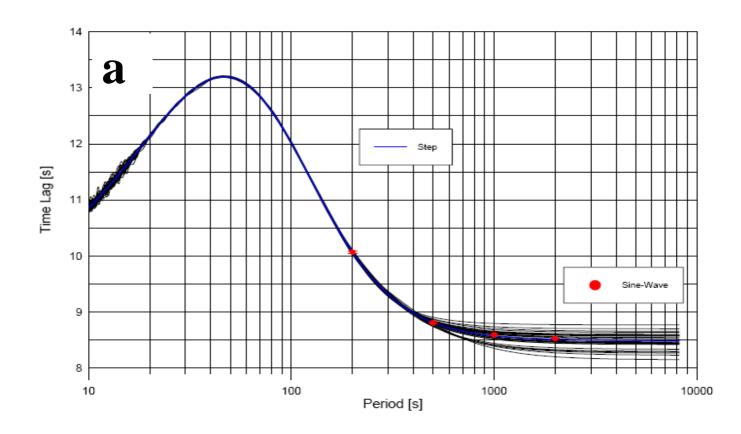
5. Experimental determination of the frequency response for the OSG-CT40

The frequency response for the OSG-CT40 operating in Walferdange (Luxembourg) was determined using the *Van Camp et al.* [2000] procedure. The frequency response was experimentally obtained by injecting step functions and sine waves (input signal) at defined voltages into the feedback loop of the gravimeter. The output signal was taken from the GGP-1 low pass filter.
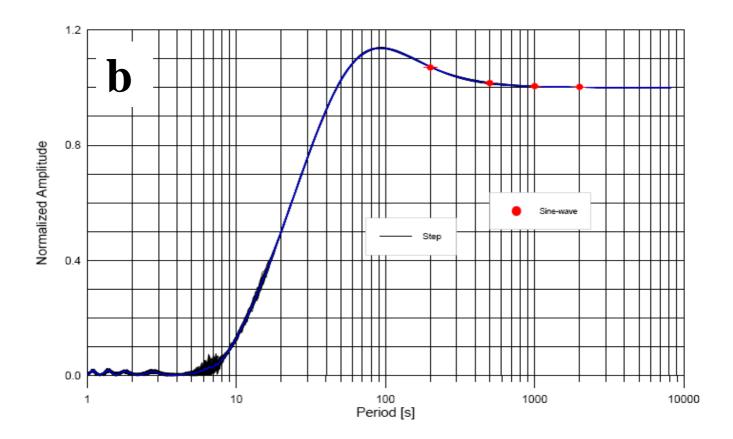
In the step function method [*Richter and Wenzel, 1991; Wenzel, 1994; Van Camp et al., 2000*], the step response function is differentiated to obtain the impulse response function. The Fourier spectrum of the impulse response function corresponds to the transfer function of the system [*Bloomfeld, 1976, Van Camp et al., 2000*]. In the sine wave method, the transfer function is obtained by fitting both the input signal (waves injected at different frequencies) and the output signal (instrument response) with a sinusoidal function. The amplitude ratios and phase differences as a function of the input frequencies correspond to the instrumental frequency response (Eq.7 and Eq. 9).

The superconducting gravimeter can be considered as a Linear Time Invariant system [*Goodkind, 1999*]. It means that both sine waves and step functions should provide the same transfer function. The comparison of results from the two methods gives the opportunity to assess their accuracy.

29 time steps and sine waves, with 4 Volt amplitude, at four different periods (200 second, 500 second, 1000 second and 2000 second) were injected into the feedback loop of the control electronics of the SG. The instrument frequency responses obtained with the sine wave and the step function methods are given in Table 1 and displayed in Figures 3. As expected and found previously by Van Camp et al. (2000), both methods give similar results consistent within their uncertainties.

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Figures 3. Frequency response of the OSG-CT40 obtained by injecting sine waves (red dots) and step functions (continuous line) into the instrument electronics: a. Phase as a function of period represented in terms of time lag (s); b. Normalized amplitudes a function of period.
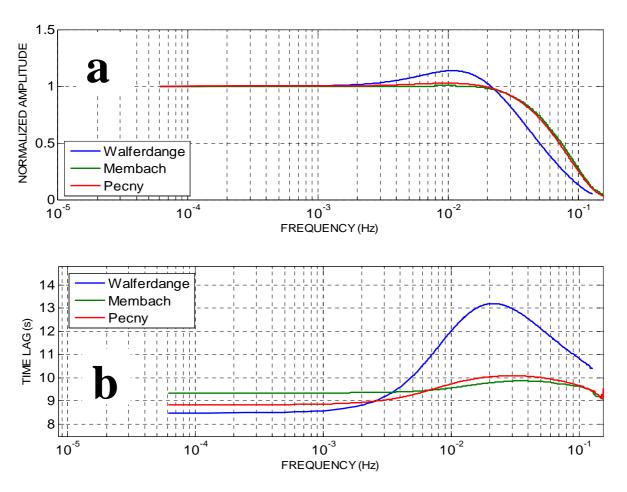
Table 1. Time lags and normalized amplitudes of the OSG-CT40 obtained, for four different periods, using the sine waves and the step functions methods.

Period/ second	Sine Waves Time lag/second	Step functions Time lag/second	Sine Waves Amplitude	Step functions Amplitude
200	9.818+-0.011	9.823+-0.017	1.044374+-0.000353	1.0469+-0.0004
500	8.571+-0.011	8.554+-0.042	0.990787+-0.000138	0.9892+-0.0130
1000	8.343+-0.003	8.323+-0.111	0.980199+-0.000022	0.9787+-0.0005
2000	8.281+-0.020	8.256+-0.136	0.977218+-0.000006	0.9759+-0.0001

6. Comparison between the transfer functions of the three gravimeters

In this section, we compare the frequency responses of the OSG-CT40 in Walferdange (experiment of 2007) with the ones of the SG-C021 in Membach (experiment of 2005) and of the OSG-050 in Pecný (experiment of 2007). The transfer function of the OSG-050 was determined using only step functions with voltages of 10 Volt and 15 Volt.

In the three experiments, the GGP-1 filter output was used. For the SG-C021, the output data were acquired with a Quanterra 330 data logger [*Van Camp et al., 2008*]. The amplitude and phase responses of the three instruments are displayed in Figures 4.

Figures 4. Normalized amplitude responses (a) and time lag (b) experimentally determined for the OSG-CT40 (blue), the SG-C021with the Quanterra 330 data logger (green) and the OSG-050(red).

The frequency responses, especially in phase, are significantly different in shape, for frequencies higher than 10^{-3} Hz. Differences up to 30% for the phase and up to 10% for the amplitude are observed.

The polynomial coefficients of the transfer functions are obtained from the complex experimental frequency response with a least-squares fit. The form of the transfer function is defined by the ratio of two polynomials of the complex Laplace variable s (Eq. 5). For the three gravimeters, the numerator and denominator of the transfer function are best modeled as 6th order polynomials of the Laplace variable s. Lowest orders are not sufficient to match the experimental transfer functions while highest orders do not improve the fit. For the OSG-CT40, the average difference between the modeled and observed values of the frequency response is $3 \cdot 10^{-5}$ in amplitude and $6 \cdot 10^{-5}$ second in phase. Similar results are obtained for the other gravimeters.

We stress that the order of the denominator must be equal or superior to the order of the numerator, otherwise the gain would be unbounded for increasing frequencies.

From the transfer function $H(s)$, the instrument frequency response (amplitude and phase) is calculated by replacing in equations (11), (12) and (13) the variable s with the variable $(\omega \cdot i)$. The transfer functions for the three SGs are:

- OSG-CT040

$$H(s) = \frac{-0.03897s^6 + 0.08883s^5 - 0.1268s^4 + 0.1159s^3 - 0.06664s^2 + 0.01835s + 0.0010281}{s^6 + 1.744s^5 + 1.6s^4 + 0.8269s^3 + 0.2292s^2 - 0.0271s + 0.001028} \quad (11)$$

- SG-C021

$$H(s) = \frac{0.02815s^6 + 0.02244s^5 - 0.01175s^4 + 0.02098s^3 - 0.02972s^2 + 0.01456s + 0.002007}{s^6 + 1.324s^5 + 1.215s^4 + 0.628s^3 + 0.202s^2 - 0.03342s + 0.002007}$$
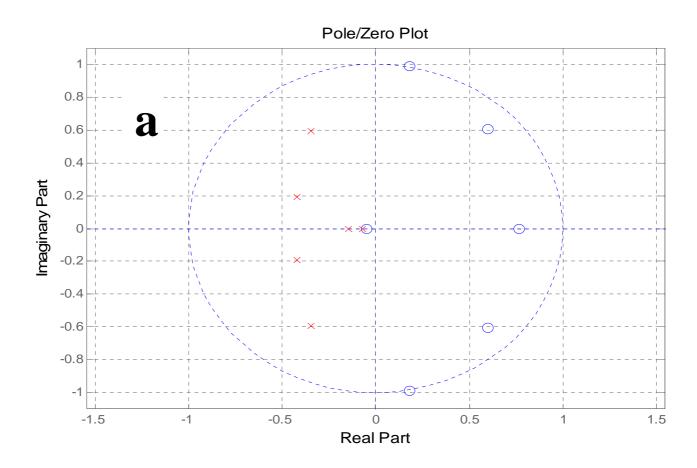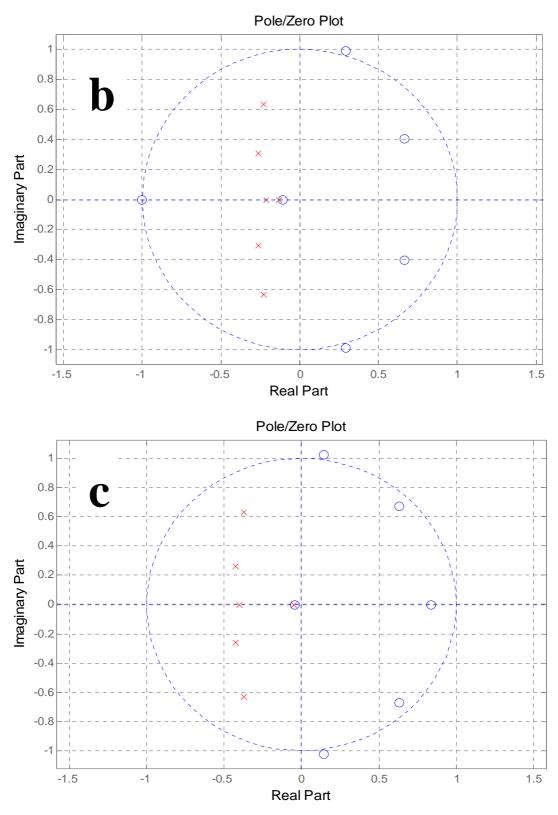
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- OSG-050

$$H(s) = \frac{-0.07454s^6 + 0.1751s^5 - 0.2599s^4 + 0.2502s^3 - 0.1558s^2 + 0.04905s + 0.002634}{s^6 + 2.043s^5 + 2.149s^4 + 1.305s^3 + 0.4478s^2 - 0.07236s + 0.002634}$$

(13)

In Figures 5, the poles and zeros of the transfer functions for the three gravimeters are shown. The poles are the values of s that make the denominator of the transfer function equal to zero leading to the divergence (i.e. instability) of the transfer function. As previously pointed out, each pole is associated, in the time domain, to a mode of the instrument response, which is expressed by:

$$y_h(t) = \sum_{i=1}^{n} C_i \cdot e^{p_i \cdot t}$$

(12)

where p represents the poles in complex form and C are constants depending on the initial conditions.

Figures 5. Pole-Zero plots: representation in the complex coordinate system of the points corresponding to the poles and zeros of the transfer function, for the OSG-CT040 (a), the SG-C021 (b) and the OSG-050 (c). The poles are represented with red crosses; the zeros are represented with blue circles.

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From the pole-zero diagrams (Figures 5), some qualitative observations on the systems responses can be drawn. For the three instruments, the poles are all located in the left half of the s plan (real part <0). This implies that all the response components tend to 0 for t tending to infinite, and consequently the systems stability.

The response is qualitatively similar for the three gravimeters. The three systems are characterized by two pairs of conjugate complex poles, corresponding to two sinusoidal decaying response components, and two real poles, corresponding to two exponentially decaying response components. The differences lie in the decay rates (defined by the poles real part) and the frequency of the oscillations of the sinusoidal modes (defined by the poles imaginary part). The nearest the pole is to the imaginary axis, the slowest is the decay rate. The nearest the pole is to the real axis, the lowest is the oscillation frequency. The response component with the slowest decay rate represents the dominant response mode.

For the three gravimeters, the dominant modes are exponential terms with different decay rates. The dominant terms for the OSG-CT040, the SG-C021 and OSG-050 persist approximately for 56 s, 30 s and 80 s, respectively.

Conclusions

The frequency response (amplitude and time lag) of the OSG-CT40 from Walferdange in Luxembourg has been experimentally determined using the procedure of Van Camp et al. (2000). The same precision and accuracy as this previous study were obtained.

The transfer functions from three SGs were also compared. The differences can reach 10% in amplitude and 30% in phase in the seismic band at frequencies higher than 10^{-3} Hz.

For a complete and accurate calibration of SGs, we recommend to the SG operators to carry out the same procedure. The transfer function is definitively unique for each superconducting gravimeter (including the gravity control card and the data acquisition system). We also encourage expressing the transfer function in terms of Laplace Transforms, which is widely used in seismology. It provides a compact and efficient way to express the transfer function. Its determination is essential to analyze and interpret the SGs' observations especially in the seismic frequency band.

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